Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 12, No. 1-2/2020

ON CONTROLLABILITY FOR A FRACTIONAL DIFFERENTIAL INCLUSION OF CAPUTO-FABRIZIO TYPE*

Aurelian Cernea†

Dedicated to Dr. Vasile Drăgan on the occasion of his 70^{th} anniversary

Abstract

We consider a fractional differential inclusion involving Caputo-Fabrizio fractional derivative and we obtain a sufficient condition for h-local controllability along a reference trajectory. To derive this result we use convex linearizations of the fractional differential inclusion. More precisely, we show that the fractional differential inclusion is h-locally controlable around a solution z if a certain linearized inclusion is λ-locally controlable around the null solution for every $\lambda \in \partial h(z(T))$, where ∂h denotes Clarke's generalized Jacobian of the locally Lipschitz function h.
MSC: 34A60, 26A33, 26A42, 34B15.

keywords: fractional derivative, differential inclusion, local controllability

*Accepted for publication in revised form on March 30, 2020
†acernea@fmi.unibuc.ro Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, 010014 Bucharest and Academy of Romanian Scientists, Splaiul Independenţei 54, 050094 Bucharest, Romania

1 Introduction

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([3, 10, 11, 12, 13] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena. In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [4] allows to use Cauchy conditions which have physical meanings.

Recently, a new fractional order derivative with regular kernel has been introduced by Caputo and Fabrizio [5]. The Caputo-Fabrizio operator is useful for modeling several classes of problems with the dynamics having the exponential decay law. This new definition is able to describe better heterogeneousness, systems with different scales with memory effects, the wave movement on surface of shallow water, the heat transfer model, mass-spring-damper model ([15]) etc.. Another good property of this new definition is that using Laplace transform of the fractional derivative the fractional differential equation turns into a classical differential equation of integer order. Properties of this definition have been studied in [1, 5, 6, 15] etc.. Some recent papers are devoted to qualitative results for fractional differential equations defined by Caputo-Fabrizio fractional derivative [16, 17, 20] etc..

In this paper we study the following problem

$$D^{\sigma}_{CF}x(t) \in F(t, x(t)) \quad a.e. \ ([0, T]), \quad x(0) \in X_0, \quad x'(0) \in X_1, \tag{1.1}$$

where $F(.,.) : [0, T] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, D^{σ}_{CF} denotes Caputo-Fabrizio's fractional derivative of order $\sigma \in (1, 2)$ and $X_0, X_1 \subset \mathbf{R}$ are closed sets.

Consider S_F the set of all solutions of (1.1) and let $R_F(T)$ be the reachable set of (1.1) at moment T. For a solution $z(.) \in S_F$ and for a locally Lipschitz function $h : \mathbf{R} \to \mathbf{R}^m$ we say that the differential inclusion (1.1) is *h-locally controllable* around $z(.)$ if $h(z(T)) \in \text{int}(h(R_F(T)))$. In particular, if h is the identity mapping and $m = 1$ the above definitions reduces to the usual concept of local controllability of systems around a solution.

The goal of the present paper is to obtain a sufficient condition for h-local controllability of inclusion (1.1). This result is derived using a technique developed by Tuan for classical differential inclusions ([18]). More exactly, we show that inclusion (1.1) is h-locally controlable around the solution $z(.)$ if a certain linearized fractional differential inclusion is λ-locally controlable around the null solution for every $\lambda \in \partial h(z(T))$, where $\partial h(.)$

denotes Clarke's generalized Jacobian matrix of the locally Lipschitz function h. The main tools in our approach is a continuous version of Filippov's theorem for solutions of problem (1.1) obtained recently in [8] and a certain generalization of the classical open mapping principle in [19]. We note that such kind of results exists in the literature for other classes of differential inclusions (e.g., [7]).

The paper is organized as follows: in Section 2 we present the notations and the preliminary results to be used in the sequel and in Section 3 we provide our main results.

2 Preliminaries

Let $T > 0$, $I := [0, T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Denote by $\mathcal{P}(\mathbf{R})$ the family of all nonempty subsets of $\mathbf{R}$ and by $\mathcal{B}(\mathbf{R})$ the family of all Borel subsets of $\mathbf{R}$.

As usual, we denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I, \mathbf{R})$ the Banach space of all integrable functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|\mathrm{d}t$.

In [5] the following notions were introduced.

Definition 2.1. a) Caputo-Fabrizio integral of order $\alpha \in (0, 1)$ of a function $f \in AC_{loc}([0, \infty), \mathbf{R})$ (which means that $f'(.)$ is integrable on $[0, T]$ for any $T > 0$) is defined by

$$I^{\alpha}_{CF} f(t) = (1 - \alpha)f(t) + \alpha \int_0^t f(s)ds.$$

b) Caputo-Fabrizio fractional derivative of order $\alpha \in (0, 1)$ of f is defined for $t \geq 0$ by

$$D^{\alpha}_{CF} f(t) = \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f'(s)ds.$$

c) Caputo-Fabrizio fractional derivative of order $\sigma = \alpha + n$, $\alpha \in (0, 1)$ $n \in \mathbf{N}$ of f is defined by

$$D^{\sigma}_{CF} f(t) = D^{\alpha}_{CF}(D^{n}_{CF} f(t)).$$

In particular, if $\sigma = \alpha + 1$, $\alpha \in (0, 1)$ $D^{\sigma}_{CF} f(t) = \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f''(s)ds$.

Definition 2.2. A mapping $x(.) \in AC(I, \mathbf{R})$ is called a *solution* of problem (1.1) if there exists a function $f(.) \in L^1(I, \mathbf{R})$ such that $f(t) \in F(t, x(t))$ *a.e.* (I), $D^{\alpha}_{CF} x(t) = f(t)$, $t \in I$ and $x(0) = x_0 \in X_0, x'(0) = x_1 \in X_1$.

In this case we say that $(x(.), f(.))$ is a *trajectory-selection pair* of (1.1).

Hypothesis 2.3. (i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ *measurable.*

(ii) *There exists* $L(.) \in L^1(I, (0, \infty))$ *such that, for almost all* $t \in I$, $F(t, .)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t, x), F(t, y)) \le L(t)|x - y| \quad \forall x, y \in \mathbf{R},$$

where $d_H(.,.)$ *is the Hausdorff distance*

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\}.$$

Hypothesis 2.4. i) S is a separable metric space and $a(.), b(.) : S \to \mathbf{R}$, $c(.) : S \to (0, \infty)$ are continuous mappings.

ii) There exists the continuous mappings $y(.) : S \to AC(I, \mathbf{R})$ and $p(.) : S \to \mathbf{R}$ such that

$$d(D(y(s))^\sigma_{CF}(t), F(t, y(s)(t)) \le p(s)(t) \quad a.e. \ (I), \ \forall \ s \in S.$$

We use next the notations

$$\xi(s)(t) = Me^{Mm(t)}[tc(s) + |a(s) - y(s)(0)| + T|b(s) - (y(s))'(0)|] \\ + \int_0^t p(s)(u)e^{M(m(t)-m(u))}du, \quad M = (1 - \alpha) + \alpha T, \quad m(t) = \int_0^t L(s)ds.$$

The main tool in characterizing regular tangent cones to reachable sets of our fractional differential inclusion is a certain version of Filippov's theorem for fractional differential inclusion (1.1) in [8].

Theorem 2.5. *Assume that Hypotheses 2.5 and 2.6 are satisfied.*

Then there exist a continuous mapping $x(.) : S \to C(I, \mathbf{R})$*, such that for any* $s \in S$*,* $x(s)(.)$ *is a solution of problem*

$$D^\sigma_{CF} z(t) \in F(t, z(t)), \quad x(0) = a(s), \quad x'(0) = b(s)$$

and

$$|x(s)(t) - y(s)(t)| \le \xi(s)(t) \quad \forall (t, s) \in I \times S.$$

In what follows $X \subset \mathbf{R}^n$ and $B \subset \mathbf{R}^n$ is the closed unit ball.

Definition 2.6. ([14]) A closed convex cone $C \subset \mathbf{R}^n$ is said to be a *regular tangent cone* to the set X at $x \in X$ if there exists continuous mappings $q_\lambda : C \cap B \to \mathbf{R}^n$, $\forall \lambda > 0$ satisfying

$$\lim_{\lambda \to 0+} \max_{v \in C \cap B} \frac{||q_\lambda(v)||}{\lambda} = 0,$$

$$x + \lambda v + q_\lambda(v) \in X \quad \forall \lambda > 0, v \in C \cap B.$$

From the multitude of the intrinsic tangent cones in the literature (e.g. [2]) the *contingent*, the *quasitangent* and *Clarke's tangent cones*, defined, respectively, by

$$K_x X = \{v \in \mathbf{R}^n;\ \exists\, s_m \to 0+,\ x_m \in X :\ \tfrac{x_m - x}{s_m} \to v\},$$
$$Q_x X = \{v \in \mathbf{R}^n;\quad \forall\, s_m \to 0+, \exists x_m \in X :\ \tfrac{x_m - x}{s_m} \to v\},$$
$$C_x X = \{v \in \mathbf{R}^n;\ \forall\, (x_m, s_m) \to (x, 0+),\ x_m \in X, \exists\, y_m \in X :\ \tfrac{y_m - x_m}{s_m} \to v\}$$

seem to be among the most oftenly used in the study of different problems involving nonsmooth sets and mappings. We recall that, in contrast with $K_x X, Q_x X$, the cone $C_x X$ is convex and one has $C_x X \subset Q_x X \subset K_x X$.

The results in the next section will be expressed, in the case when the mapping $g(.) : X \subset \mathbf{R}^n \to \mathbf{R}^m$ is locally Lipschitz at x, in terms of the Clarke generalized Jacobian matrix, defined in [9] by

$$\partial g(x) = \mathrm{co}\{\lim_{i\to\infty} g'(x_i); \quad x_i \to x, \quad x_i \in X \backslash \Omega_g\},$$

where $\mathrm{co}\{M\}$ denotes the convex hull of a set M and Ω_g is the set of points at which g is not differentiable.

Corresponding to each type of tangent cone, say $\tau_x X$ one may introduce (e.g., [2]) a *set-valued directional derivative* of a multifunction $G(.) : X \subset \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ (in particular of a single-valued mapping) at a point $(x, y) \in Graph(G)$ as follows

$$\tau_y G(x; v) = \{w \in \mathbf{R}^n; \quad (v, w) \in \tau_{(x,y)} Graph(G)\},$$

$v \in \tau_x X$.

We recall that a set-valued map, $A(.) : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ is said to be a *convex* (respectively, closed convex) *process* if $Graph(A(.)) \subset \mathbf{R}^n \times \mathbf{R}^n$ is a convex (respectively, closed convex) cone. For the basic properties of convex processes we refer to [2], but we shall use here only the above definition.

Hypothesis 2.7. i) *Hypothesis 2.3 is satisfied and $X_0, X_1 \subset \mathbf{R}$ are closed sets.*

ii) $(z(.), f(.)) \in C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ *is a trajectory-selection pair of (1.1) and a family $A(t, .) : \mathbf{R} \to \mathcal{P}(\mathbf{R})$, $t \in I$ of convex processes satisfying the condition*

$$A(t, u) \subset Q_{f(t)} F(t, .)(z(t); u) \quad \forall\, u \in dom(A(t, .)),\ a.e.\, t \in I \qquad (2.1)$$

is assumed to be given and defines the variational inclusion

$$D^{\sigma}_{CF}w(t) \in A(t, w(t)). \tag{2.2}$$

Remark 2.8. We mention that for any set-valued map $F(.,.)$, one may find an infinite number of families of convex process $A(t,.)$, $t \in I$, satisfying condition (2.1); in fact any family of closed convex subcones of the quasitangent cones, $\overline{A}(t) \subset Q_{(z(t),f(t))}graph(F(t,.))$, defines the family of closed convex process

$$A(t,u) = \{v \in \mathbf{R};\ (u,v) \in \overline{A}(t)\}, \quad u, v \in \mathbf{R},\ t \in I$$

that satisfy condition (2.1). For example, we may take an "intrinsic" family of such closed convex process; namely, Clarke's convex-valued directional derivatives $C_{f(t)}F(t,.)(z(t);.)$.

When $F(t,.)$ is assumed to be Lipschitz a.e. on I an alternative characterization of the quasitangent directional derivative is (e.g., [2])

$$Q_{f(t)}F(t,.)((z(t);u)) = \{w \in \mathbf{R};\ \lim_{\theta \to 0+} \frac{1}{\theta} d(f(t) + \theta w, F(t, z(t) + \theta u)) = 0\}. \tag{2.3}$$

In what follows $B_{\mathbf{R}^n}$ denotes the closed unit ball in $\mathbf{R}^n$ and 0_n denotes the null element in $\mathbf{R}^n$.

Consider $h : \mathbf{R} \to \mathbf{R}^m$ an arbitrary given function.

Definition 2.9. Inclusion (1.1) is said to be *h-locally controllable* around $z(.)$ if $h(z(T)) \in \text{int}(h(R_F(T)))$.

Inclusion (1.1) is said to be *locally controllable* around the solution $z(.)$ if $z(T) \in \text{int}(R_F(T))$.

Finally, a key tool in the proof of our results is the following generalization of the classical open mapping principle due to Warga ([19]).

For $k \in \mathbf{N}$ we define

$$\Sigma_k := \{\gamma = (\gamma_1, ..., \gamma_k) \in \mathbf{R}^k;\quad \sum_{i=1}^{k} \gamma_i \le 1,\ \gamma_i \ge 0,\ i = \overline{1,k}\}.$$

Lemma 2.10. ([19]) *Let $\delta \le 1$, let $g(.) : \mathbf{R}^k \to \mathbf{R}^m$ be a mapping that is C^1 in a neighborhood of 0_k containing $\delta B_{\mathbf{R}^k}$. Assume that there exists $\beta > 0$ such that for every $\theta \in \delta\Sigma_k$, $\beta B_{\mathbf{R}^m} \subset g'(\theta)\Sigma_k$. Then, for any continuous mapping $\psi : \delta\Sigma_k \to \mathbf{R}^m$ that satisfies $\sup_{\theta \in \delta\Sigma_k} ||g(\theta) - \psi(\theta)|| \le \frac{\delta\beta}{32}$ we have $\psi(0_k) + \frac{\delta\beta}{16} B_{\mathbf{R}^m} \subset \psi(\delta\Sigma_k)$.*

3 The main result

In what follows C_0 is a regular tangent cone to X_0 at $z(0)$, C_1 is a regular tangent cone to X_1 at $z'(0)$. Denote by S_A the set of all solutions of the differential inclusion

$$D^{\sigma}_{CF}v(t) \in A(t, v(t)), \quad v(0) \in C_0, \quad v'(0) \in C_1$$

and by $R_A(T) = \{x(T);\ x(.) \in S_A\}$ its reachable set at time T.

Theorem 3.1. *Assume that Hypothesis 2.7 is satisfied and let* $h : \mathbf{R} \to \mathbf{R}^m$ *be a Lipschitz function with Lipschitz constant* $l > 0$.

Then inclusion (1.1) is h-local controllable around the solution $z(.)$ *if*

$$0_m \in \text{int}(\lambda R_A(T)) \quad \forall \lambda \in \partial h(z(T)). \tag{3.1}$$

Proof. By (3.1), since $\lambda R_A(T)$ is a convex cone, it follows that $\lambda R_A(T) = \mathbf{R}^m$ $\forall \lambda \in \partial h(z(T))$. Therefore using the compactness of $\partial h(z(T))$ (e.g., [9]), we have that for every $\beta > 0$ there exist $k \in \mathbf{N}$ and $u_j \in R_A(T)$ $j = 1, 2, ..., k$ such that

$$\beta B_{\mathbf{R}^m} \subset \lambda(u(\Sigma_k)) \quad \forall \lambda \in \partial h(z(T)), \tag{3.2}$$

where

$$u(\Sigma_k) = \{u(\gamma) := \sum_{j=1}^{k} \gamma_j u_j, \quad \gamma = (\gamma_1, ..., \gamma_k) \in \Sigma_k\}.$$

Using an usual separation theorem we deduce the existence of $\beta_1, \rho_1 > 0$ such that for all $\lambda \in L(\mathbf{R}, \mathbf{R}^m)$ with $d(\lambda, \partial h(z(T))) \leq \rho_1$ we have

$$\beta_1 B_{\mathbf{R}^m} \subset \lambda(u(\Sigma_k)). \tag{3.3}$$

Since $u_j \in R_A(T)$, $j = 1, ..., k$, there exist $(w_j(.), g_j(.))$, $j = 1, ..., k$ trajectory-selection pairs of (2.2) such that $u_j = w_j(T)$, $j = 1, ..., k$. We note that $\beta > 0$ can be take small enough such that $|w_j(0)| \leq 1$, $j = 1, ..., k$.

For $s = (s_1, ..., s_k) \in \mathbf{R}^k$ we define

$$w(t, s) = \sum_{j=1}^{k} s_j w_j(t), \quad \overline{g}(t, s) = \sum_{j=1}^{k} s_j g_j(t).$$

Obviously, $w(., s) \in S_A$, $\forall s \in \Sigma_k$.

Taking into account the definition of C_0 and C_1, for every $\varepsilon > 0$ there exists a continuous mapping $o_\varepsilon : \Sigma_k \to \mathbf{R}^n$ such that

$$z(0) + \varepsilon w(0,s) + o_\varepsilon(s) \in X_0, z'(0) + \varepsilon \frac{\partial w}{\partial t}(0,s) + o_\varepsilon(s) \in X_1 \tag{3.4}$$

$$\lim_{\varepsilon \to 0+} \max_{s \in \Sigma_k} \frac{|o_\varepsilon(s)|}{\varepsilon} = 0. \tag{3.5}$$

Recall that $(z(.), f(.)) \in C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ is a trajectory-selection pair of (1.1) and define

$$p_\varepsilon(s)(t) := \frac{1}{\varepsilon} d(\overline{g}(t,s), F(t, z(t) + \varepsilon w(t,s)) - f(t)),$$

$$q(t) := \sum_{j=1}^{k} [|g_j(t)| + L(t)|w_j(t)|], \quad t \in I.$$

Then, for every $s \in \Sigma_k$ one has

$p_\varepsilon(s)(t) \le |\overline{g}(t,s)| + \frac{1}{\varepsilon} d_H(0_n, F(t, z(t) + \varepsilon w(t,s)) - f(t)) \le |\overline{g}(t,s)| +$
$\frac{1}{\varepsilon} \mathrm{d}_H(F(t, z(t)), F(t, z(t) + \varepsilon w(t,s))) \le |\overline{g}(t,s)| + L(t)|w(t,s)| \le q(t).$

Next, if $s_1, s_2 \in \Sigma_k$ one has

$|p_\varepsilon(s_1)(t) - p_\varepsilon(s_2)(t)| \le |\overline{g}(t,s_1) - \overline{g}(t,s_2)| + \frac{1}{\varepsilon} \mathrm{d}_H(F(t, z(t) + \varepsilon w(t,s_1)),$
$F(t, z(t) + \varepsilon w(t,s_2))) \le ||s_1 - s_2||. \max_{j=\overline{1,k}} [|g_j(t)| + L(t)|w_j(t)|],$

thus $p_\varepsilon(.)(t)$ is Lipschitz with a Lipschitz constant not depending on ε.

On the other hand, from (2.3) it follows that

$$\lim_{\varepsilon \to 0} p_\varepsilon(s)(t) = 0 \quad a.e. \,(I), \quad \forall s \in \Sigma_k$$

and hence

$$\lim_{\varepsilon \to 0+} \max_{s \in \Sigma_k} p_\varepsilon(s)(t) = 0 \quad a.e. \,(I). \tag{3.7}$$

Therefore, from (3.6), (3.7) and Lebesgue dominated convergence theorem we obtain

$$\lim_{\varepsilon \to 0+} \int_0^T \max_{s \in \Sigma_k} p_\varepsilon(s)(t) \mathrm{d}t = 0. \tag{3.8}$$

By (3.4), (3.5), (3.8) and the upper semicontinuity of the Clarke generalized Jacobian we can find $\varepsilon_0, e_0 > 0$ such that

$$(1+T) M e^{Mm(T)} \max_{s \in \Sigma_k} \frac{|o_{\varepsilon_0}(s)|}{\varepsilon_0} + MT e^{Mm(T)} \int_0^T \max_{s \in \Sigma_k} p_{\varepsilon_0}(s)(t) \mathrm{d}t \le \frac{\beta_1}{2^8 l^2}, \tag{3.9}$$

$$\varepsilon_0 w(T,s) \leq \frac{e_0}{2} \quad \forall s \in \Sigma_k. \tag{3.10}$$

If we define

$$y(s)(t) := z(t) + \varepsilon_0 w(t,s),$$
$$g(s)(t) := f(t) + \varepsilon_0 \overline{g}(t,s),$$
$$a(s) := z(0) + \varepsilon_0 w(0,s) + o_{\varepsilon_0}(s),$$
$$b(s) := z'(0) + \varepsilon_0 \frac{\partial w}{\partial t}(0,s) + o_{\varepsilon_0}(s),\ s \in \mathbf{R}^k,$$

then we apply Theorem 2.5 and we find that there exists the continuous function $x(.) : \Sigma_k \to C(I,\mathbf{R})$ such that for any $s \in \Sigma_k$ the function $x(s)(.)$ is solution of the differential inclusion

$$D_c^{\alpha,\rho} x(t) \in F(t,x(t)), \quad x(s)(0) = a_0(s),\ (x(s))'(0) = b(s)$$

and one has

$$|x(s)(T) - y(s)(T)| \leq \frac{\varepsilon_0 \beta_1}{2^6 l} \quad \forall s \in \Sigma_k. \tag{3.11}$$

We define

$$h_0(x) := \int_{\mathbf{R}} h(x - by)\chi(y)\mathrm{d}y, \quad x \in \mathbf{R},$$
$$\phi(s) := h_0(z(T) + \varepsilon_0 w(T,s)),$$

where $\chi(.) : \mathbf{R} \to [0,1]$ is a C^∞ function with the support contained in B that satisfies $\int_{\mathbf{R}} \chi(y)\mathrm{d}y = 1$ and $b = \min\{\frac{e_0}{2}, \frac{\varepsilon_0\beta_1}{2^6 l}\}$.

Therefore $h_0(.)$ is of class C^∞ and verifies

$$|h(x) - h_0(x)| \leq lb, \tag{3.12}$$

$$h_0'(x) = \int_{\mathbf{R}} h'(x - by)\chi(y)\mathrm{d}y. \tag{3.13}$$

In particular,

$$h_0'(x) \in \overline{\mathrm{co}}\{h'(u); \quad |u - x| \leq b, \quad h'(u) \text{ exists}\},$$

$$\phi'(s)\mu = h_0'(z(T) + \varepsilon_0 w(T,\mu)) \quad \forall \mu \in \Sigma_k.$$

Using again the upper semicontinuity of Clarke's generalized Jacobian we obtain

$$\begin{aligned} &d(h_0'(z(T) + \varepsilon_0 w(T,s)), \partial h(z(T))) \leq \sup\{d(h_0'(u), \partial h(z(T)));\ |u - z(T)| \\ &\leq |u - (z(T) + \varepsilon_0 w(T,s))| + |\varepsilon_0 w(t,s)| \leq e_0, \quad h'(u)\ \text{exists}\} < \rho_1. \end{aligned}$$

The last inequality with (3.3) gives

$$\varepsilon_0\beta_1 B_{\mathbf{R}^m} \subset \phi'(s)\Sigma_k \quad \forall s \in \Sigma_k.$$

Finally, for $s \in \Sigma_k$, we put $\psi(s) = h(x(s)(T))$.

Obviously, $\psi(.)$ is continuous and from (3.11), (3.12), (3.13) one has

$||\psi(s) - \phi(s)|| = ||h(x(s)(T)) - h_0(y(s)(T))|| \leq ||h(x(s)(T)) - h(y(s)(T))||$ $+||h(y(s)(T)) - h_0(y(s)(T))|| \leq l|x(s)(T) - y(s)(T)| + lb \leq \frac{\varepsilon_0\beta_1}{64} + \frac{\varepsilon_0\beta_1}{64} = \frac{\varepsilon_0\beta_1}{32}$.

We apply Lemma 2.10 and we find that

$$h(x(0_k)(T)) + \frac{\varepsilon_0\beta_1}{16} B_{R^m} \subset \psi(\Sigma_k) \subset h(R_F(T)).$$

On the other hand, $||h(z(T)) - h(x(0_k)(T))|| \leq \frac{\varepsilon_0\beta_1}{64}$, so we have $h(z(T)) \in \text{int}(R_F(T))$ and the proof is complete.

References

[1] T.M. Atanacković, S. Pilipović, D. Zorica. Properties of the Caputo-Fabrizio fractional derivative and its distributional settings. *Frac. Calc. App. Anal.* 21:29-44, 2018.

[2] J.P. Aubin, H. Frankowska. *Set-valued Analysis,* Birkhäuser, Basel, 1990

[3] D. Băleanu, K. Diethelm, E. Scalas, J. J. Trujillo. *Fractional Calculus Models and Numerical Methods,* World Scientific, Singapore, 2012.

[4] M. Caputo. *Elasticità e Dissipazione,* Zanichelli, Bologna, 1969.

[5] M. Caputo, M. Fabrizio. A new definition of fractional derivative without singular kernel. *Progress Fract. Diff. Appl.* 1:1-13, 2015.

[6] M. Caputo, M. Fabrizio. Applications of new time and spatial fractional derivatives with exponential kernels. *Progress Fract. Diff. Appl.* 2:1-11, 2016.

[7] A. Cernea. On a second-order evolution inclusion. *Results Nonlin. Anal.* 1:99-106, 2018.

[8] A. Cernea. On the solutions of a fractional differential inclusion of Caputo-Fabrizio type. *J. Nonlin. Evol. Equations Appl.* to appear.

[9] F. H. Clarke. *Optimization and Nonsmooth Analysis,* Wiley Interscience, New York, 1983.

[10] K. Diethelm. *The Analysis of Fractional Differential Equations,* Springer, Berlin, 2010.

[11] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. *Theory and Applications of Fractional Differential Equations,* Elsevier, Amsterdam, 2006.

[12] K. Miller, B. Ross. *An Introduction to the Fractional Calculus and Differential Equations,* John Wiley, New York, 1993.

[13] I. Podlubny. *Fractional Differential Equations,* Academic Press, San Diego, 1999.

[14] E.S. Polovinkin, G.V. Smirnov. An approach to differentiation of many-valued mapping and necessary condition for optimization of solution of differential inclusions. *Differential Equations* 22:660-668, 1986.

[15] M.A. Refai, K. Pal. New aspects of Caputo-Fabrizio fractional derivative. *Progress Fract. Diff. Appl.* 5:157-166, 2019.

[16] A. Shaikh, A. Tassaddiq, K.S. Nisar, D. Băleanu. Analysis of differential equations involving Caputo-Fabrizio fractional operator and its applications to reaction-diffusion equations. *Adv. Difference Equations* 2019(178):1-14, 2019.

[17] Ş. Toprakseven. The existence and uniqueness of initial-boundary value problems of the Caputo-Fabrizio differential equations. *Univ. J. Math. Appl.* 2:100-106, 2019.

[18] H. D. Tuan. On controllability and extremality in nonconvex differential inclusions. *J. Optim. Theory Appl.* 85:437-474, 1995.

[19] J. Warga. Controllability, extremality and abnormality in nonsmooth optimal control. *J. Optim. Theory Appl.* 41:239-260, 1983.

[20] S. Zhang, L. Hu, S. Sun. The uniqueness of solution for initial value problems for fractional differential equations involving the Caputo-Fabrizio derivative. *J. Nonlin. Sci. Appl.* 11:428-436, 2018.