ISSN 2066-6594 Ann. Acad. Rom. Sci.
Ser. Math. Appl.
Vol. 12, No. 1-2/2020

ROBUST STATIC OUTPUT FEEDBACK STACKELBERG STRATEGY FOR MARKOV JUMP DELAY STOCHASTIC SYSTEMS *

Hiroaki Mukaidani †, Ramasamy Saravanakumar‡, Hua Xu §, Weihua Zhuang ¶

DOI https://doi.org/10.56082/annalsarscimath.2020.1-2.476

Dedicated to Dr. Vasile Drăgan on the occasion of his 70th anniversary

Abstract

In this study, a robust static output feedback (SOF) Stackelberg strategy for a class of uncertain Markov Jump linear stochastic delay systems (UMJLSDSs) is investigated. After introducing certain preliminaries, a SOF Stackelberg strategy is derived. It is shown that the strategy set is established by solving two constraint optimization problems and cross-coupled stochastic matrix equations that consist of bilinear matrix inequalities (BMIs). In order to obtain the corresponding solutions of the constraint optimization problems and cross coupled stochastic matrix equations (CCSMEs), an algorithm based

*Accepted for publication in revised form on July 20, 2020
*This paper was presented in part in a paper accepted for presentation at 2020 IFAC World Congress [1].

†mukaida@hiroshima-u.ac.jp, Graduate School of Advanced Science and Engineering, Hiroshima University, 1-4-1 Kagamiyama, Higashi-Hiroshima, 739-8527 Japan.
‡saravana@hiroshima-u.ac.jp, Graduate School of Advanced Science and Engineering, Hiroshima University, 1-4-1 Kagamiyama, Higashi-Hiroshima, 739-8527 Japan.
§xu@mbaib.gsbs.tsukuba.ac.jp, Graduate School of Business Sciences, The University of Tsukuba, 3-29-1, Otsuka, Bunkyo-ku, Tokyo, 112-0012 Japan.
¶wzhuang@uwaterloo.ca, Department of Electrical and Computer Engineering, University of Waterloo, 200 University Avenue West, Waterloo, ON N2L3G1, Canada.

on the Krasnoselskii iterative algorithm is proposed instead of solving BMI. It is also shown that weak convergence can be achieved using this approach. A practical example is provided to demonstrate the effectiveness and convergence of the proposed algorithm.
MSC: 49N90, 70E60, 93B36, 93E20

keywords: Stackelberg strategy, uncertain Markov Jump linear stochastic delay systems (UMJLSDSs), BMIs, Krasnoselskii iterative algorithm.

1 Introduction

Over the past decade, various control problems, stabilization problems, and dynamic games for Markov jump linear stochastic systems (MJLSSs) have been studied. Researchers have been challenged to solve problems for MJLSSs with different characteristics such as time delays, deterministic uncertainties, external disturbances, and information availability in designing strategies. Control and stabilization problems for MJLSSs with delays have been studied [2, 3, 4, 5, 6]. A Pareto suboptimal control and a Nash equilibrium via state feedback strategy have been derived for Markov jump delay stochastic delay systems (MJDSSs) [7]. Moreover, static output feedback (SOF) control problems for MJLSSs have been investigated [8], and some SOF dynamic games for MJLSSs with external disturbances have also been studied [9].

Over the past twenty years, researchers have addressed challenges related to numerical computation methods to solve various cross coupled stochastic matrix equations (CCSMEs) and cross-coupled stochastic matrix inequalities (CCSMIs), which are well known NP-hard problems [10, 11]. There are often difficulties involved in solving such CCSMEs and CCSMIs when SOF strategies are considered in a problem. In [12, 13], an iterative computational algorithm has been presented to solve high-order cross coupled matrix equations involving SOF strategies. Several theoretical results have been established regarding the existence conditions of the solutions; however, further study is required to develop effective numerical algorithms with an appropriate convergence property. Furthermore, uncertain Markov jump linear stochastic systems (UMJLSSs) are often used to describe various practical systems with deterministic uncertainties, such as the modeling errors in system matrices and stochastic changes in operating points [14]. Although there have been recent advances in the robust SOF Nash and the state feedback Stackelberg strategies for uncertain Markov Jump linear stochastic delay systems (UMJLSDSs) [15, 16], studies on robust SOF Stackelberg games for the UMJLSDSs remain open. This is a challenging problem in the area of dynamic games because it involves several uncertain factors, as described above.

This paper investigates a robust Stackelberg game for UMJLSDSs. It is distinctly different from [16] in that SOF strategies are considered herein; it also differs from existing studies [9, 14] in which deterministic uncertainties are not considered. Furthermore, not stochastic delay systems but UMJLSDS is studied as an extension of recent results in [1]. In order to solve the present problem, an integrated application of Markov switching, dynamic game, stochastic control, and robust control is required. The contributions of this paper are threefold. First, owing to the deterministic uncertainties, two constraint optimization problems are formulated to determine the cost bounds of the leader and followers based on the guaranteed cost control technique [12]. It is shown that a robust SOF Stackelberg strategy set can be obtained by solving these optimization problems with the CCSMI constraints and the CCSMEs. The existence conditions for the solutions of the constraint optimization problems are derived using the Karush-Kuhn-Tucker (KKT) conditions. Second, a computational framework for validating a heuristic algorithm is proposed, in which the Krasnoselskii iterative algorithm [17] is used to obtain the solution set of the CCSMEs. A novel convergence condition combined with the Krasnoselskii iteration is introduced to achieve weak convergence of the algorithm. Finally, to demonstrate the effectiveness and reliability of the proposed computational framework and algorithm, a practical example based on a chemical refining process is presented.

Notation: The notations used in this paper are fairly standard: I_n denotes the $n \times n$ identity matrix, $|| \cdot ||$ the Euclidean norm of a matrix; $\mathbb{E}[\ \cdot \mid r_t = i]$ stands for the conditional expectation operator with respect to the event $\{r_t = i\}$; $\mathbb{M}^s_{n,m}$ denotes space of all $\boldsymbol{S} = (S(1), \ldots, S(s))$ with $S(i)$ being $n \times m$ matrix, $i \in \mathcal{D}$, $\mathcal{D} = \{1,\ 2, \ldots, s\}$. Moreover, the components of $\boldsymbol{S} + \boldsymbol{TU}$ are defined as $\boldsymbol{S} + \boldsymbol{TU} = (S(1) + T(1)U(1), \ldots, S(s) + T(s)U(s))$; $\mathscr{L}^2_F([0,\ \infty),\ \mathbb{R}^k)$ denotes the space of all measurable functions $u(t,\ \omega)\ :\ [0,\ \infty) \times \Omega \to \mathbb{R}^k$, which are F_t-measurable for every $t \geq 0$, and $\mathbb{E}[\int_0^\infty |u(t)|^2 dt \mid r_0 = i] < \infty$, $i \in \mathcal{D}$; $C([-h, 0]; \mathbb{R}^n)$, $h > 0$ denotes the family of continuous functions ϕ from $[-h, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\| = \sup_{-h \leq \theta \leq 0} \|\phi(\theta)\|$; $\lambda_{\max}[\cdot]$ and $\lambda_{\min}[\cdot]$ denote its largest and smallest eigenvalue, respectively.

2 Preliminary Results

Let $w(t)$, $t \geq 0$, be the one-dimensional Wiener process that is defined on a given filtered probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{t\geq 0},\ \mathcal{P})$, and r_t, $t \geq 0$, be a right continuous homogeneous Markov process taking values in a finite state space $\mathcal{D} = \{1,\ 2, \ldots, s\}$. Without loss of generality, it is assumed that $\{w(t)\}_{t\geq 0}$ and $\{r_t\}_{t\geq 0}$ are independent stochastic processes. Furthermore, the transition probabilities are given by

$$\mathbf{P}\{r_{t+\Delta t} = j | r_t = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & \text{if } i \neq j \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & \text{else} \end{cases} \tag{1}$$

where $\Delta t > 0$, $\pi_{k\ell} \geq 0$, $k \neq \ell$, $\pi_{kk} = -\sum_{\ell=1,\ \ell\neq k}^{s} \pi_{k\ell}$, $\lim_{\Delta t\to 0} \frac{o(\Delta t)}{\Delta t} = 0$.

Consider the following UMJLSDS

$$\begin{aligned} dx(t) &= [A(r_t,t)x(t) + A_h(r_t)x(t-h) + B_v(r_t)v(t)]dt \\ &\quad + A_p(r_t,t)x(t)dw(t), \end{aligned} \tag{2a}$$

$$x(t) = \phi(t),\ t \in [-h,\ 0], \tag{2b}$$

$$z(t) = H(r_t)x(t), \tag{2c}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector. $v(t) \in \mathbb{R}^{m_v}$ denotes the external disturbance. $z(t) \in \mathbb{R}^{n_z}$ denotes the controlled output. $w(t) \in \mathbb{R}$ denotes a one-dimensional standard Wiener process defined in the filtered probability space. h, $(h > 0)$ denotes the time-delay of the UMJLSDSs. $\phi(t)$ denotes a real-valued initial function. Without loss of generality, it is assumed that, for all $\psi \in [-h,\ 0]$, there exists a scalar $\sigma > 0$ such that $\|x(t+\psi)\| \leq \sigma\|x(t)\|$ [18, 19].

Let $A(r_t,t)$ and $A_p(r_t,t) \in \mathbb{R}^{n\times n}$ be matrices with the following forms:

$$A(r_t,t) = A(r_t) + D(r_t)\Theta(r_t,t)E_a(r_t), \tag{3a}$$

$$A_p(r_t,t) = A_p(r_t) + D_p(r_t)\Theta(r_t,t)E_{pa}(r_t). \tag{3b}$$

In coefficients $\boldsymbol{A}$, $\boldsymbol{A}_h$, $\boldsymbol{A}_p \in \mathbb{M}_{n,n}^s$ and $\boldsymbol{B}_v \in \mathbb{M}_{n,m_v}^s$, $A(i)$, $A_h(i)$, $A_p(i)$, $D(i)$, $D_p(i)$, $E_a(i)$, $E_{pa}(i)$ and $B_v(i)$, $i \in \mathcal{D}$, are constant matrices; $\Theta(r_t,t) \in \mathbb{R}^{n_p\times n_q}$ is time-varying unknown real matrix satisfying $\Theta^T(r_t,t)\Theta(r_t,t) \leq I_{n_q}$ for every i [14, 15, 16].

First, the related definition and lemmas are introduced.

Definition 1 [18, 19] *The UMJLSDS is said to be stochastically stable if, when $v(t) \equiv 0$, for all finite $\phi(t) \in \mathbb{R}^n$ defined on $[-h,\ 0]$ and initial mode $r_0 = i \in \mathcal{D}$, there exists an $\tilde{M} > 0$ satisfying*

$$\mathbb{E}\left[\int_0^\infty x^T(t,\phi,r_0)x(t,\phi,r_0)dt \middle| \phi, r_0 = i\right] \leq x^T(0)\tilde{M}x(0). \tag{4}$$

Lemma 1 [15] *Let* $A \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{n \times n_p}$, $E \in \mathbb{R}^{n_q \times n}$ *and* $\Theta(r_t, t) \in \mathbb{R}^{n_p \times n_q}$ *satisfying* $\Theta^T(r_t, t)\Theta(r_t, t) \leq I_{n_q}$ *be given matrices. Then, for any matrix* $P = P^T > 0$*, there exist positive scalars* $\varepsilon > 0$ *and* $\lambda > 0$ *such that*

$$\begin{aligned}
&(A + D\Theta(r_t, t)E)^T P(A + D\Theta(r_t, t)E) \\
&\quad \leq A^T PA + \varepsilon^{-1} A^T PDD^T PA + (\varepsilon + \lambda)E^T E, \qquad (5a)\\
&D^T PD \leq \lambda I_{n_p}. \qquad (5b)
\end{aligned}$$

The following result has been proved [15].

Lemma 2 *Let* γ *denote the required disturbance attenuation level. Consider a set of symmetric positive semidefinite matrices* $\boldsymbol{W} \geq 0$, $U > 0$ *and positive scalars* $\lambda(i)$, $\varepsilon(i)$ *and* $\mu(i)$*, such that the following CCSMIs hold for every* $i \in \mathcal{D}$*:*

$$\Lambda(\boldsymbol{W}, U, \mu(i), \varepsilon(i), \lambda(i), i) < 0, \qquad (6a)$$

$$D_p^T(i)W(i)D_p(i) \leq \lambda(i)I_{n_b}, \qquad (6b)$$

where $i = 1, \ldots, s$*,*

$$\Lambda(\boldsymbol{W}, U, \mu(i), \varepsilon(i), \lambda(i), i) := \begin{bmatrix} \Phi_{11}(i) & W(i)A_h(i) \\ A_h^T(i)W(i) & -U \end{bmatrix},$$

$$\begin{aligned}
\Phi_{11}(i) := {} & W(i)A(i) + A^T(i)W(i) + \mu^{-1}(i)W(i)D(i)D^T(i)W(i) \\
& + \mu(i)E_a^T(i)E_a(i) + H^T(i)H(i) + U + \sum_{j=1}^{s} \pi_{ij}W(j) \\
& + A_p^T(i)W(i)A_p(i) + \varepsilon^{-1}(i)A_p^T(i)W(i)D_p(i)D_p^T(i)W(i)A_p(i) \\
& + (\varepsilon(i)+\lambda(i))E_{pa}^T(i)E_{pa}(i)+\gamma^{-2}W(i)B_v(i)B_v^T(i)W(i).
\end{aligned}$$

Then, we have the following results:

i) *The UMJLSDS* (2) *is stochastically stable internally with* $v(t) \equiv 0$, $v(t) \in \mathscr{L}_2[0, \infty)$.

ii) *The following inequality holds:*

$$\|z\|_{E_2}^2 < \gamma^2 \|v\|_2^2 + \mathcal{E}(W(i), U), \qquad (7)$$

where

$$\|z\|_{E_2}^2 := \mathbb{E}\left[\int_0^\infty \|z(t)\|^2 dt\right], \ \|v\|_2^2 := \mathbb{E}\left[\int_0^\infty \|v(t)\|^2 dt\right],$$

$$\mathcal{E}(W(i), U) := x^T(0)W(i)x(0) + \int_{-h}^{0} \phi^T(s)U\phi(s)ds.$$

iii) *The worst-case disturbance is given by*

$$v^*(t) = F^*_\gamma(r_t)x(t) = \gamma^{-2}B_v^T(r_t)W(r_t)x(t). \tag{8}$$

It is said to be robustly stochastically stable with disturbance attenuation level γ if the condition i) holds [20, 21].

It should be noted that the Lebesgue space $\mathscr{L}_2[0,\infty)$ consists of square-integrable functions on the interval $[0,\infty)$ equipped with the norm $\|\cdot\|_2$. On the other hand, $\|\cdot\|_{E_2}$ denotes the norm in $\mathscr{L}_2\big((\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{t\geq 0},\ \mathcal{P}), [0,\infty)\big)$ [20, 21].

The following corollary can be established by tracing the proof of Lemma 2 with some change.

Corollary 1 *Define the corresponding cost function for UMJLSDS* (2) *with* $v(t)\equiv 0$ *as follows:*

$$\tilde{J} := \mathbb{E}\left[\int_0^\infty x^T(t,\phi,r_0)Q(r_t)x(t,\phi,r_0)dt\middle|\phi, r_0 = i\ \right], \tag{9}$$

where $Q(r_t) = Q^T(r_t) > 0$.

Consider a set of symmetric positive semidefinite matrices $\boldsymbol{P} \geq 0$, $V > 0$ *and positive scalars* $\kappa(i)$, $\delta(i)$ *and* $\nu(i)$, *such that the following CCSMIs holds:*

$$\Gamma(\boldsymbol{P}, V, \nu(i), \delta(i), \kappa(i), i) \leq 0, \tag{10a}$$

$$D_p^T(i)P(i)D_p(i) \leq \kappa(i)I_{n_b}, \tag{10b}$$

where $i = 1, \ldots, s$,

$$\Gamma(\boldsymbol{P}, V, \nu(i), \delta(i), \kappa(i), i) := \begin{bmatrix} \Psi_{11}(i) & P(i)A_h(i) \\ A_h^T(i)P(i) & -V \end{bmatrix},$$

$$\begin{aligned}\Psi_{11}(i) := {} & P(i)A(i) + A^T(i)P(i) + \nu^{-1}(i)P(i)D(i)D^T(i)P(i) \\ & + \nu(i)E_a^T(i)E_a(i) + Q(i) + V + \sum_{j=1}^{s}\pi_{ij}P(j) \\ & + A_p^T(i)P(i)A_p(i) + \delta^{-1}(i)A_p^T(i)P(i)D_p(i)D_p^T(i)P(i)A_p(i) \\ & + (\delta(i) + \kappa(i))E_{pa}^T(i)E_{pa}(i).\end{aligned}$$

Then,we have the following inequality

$$\tilde{J} \leq x^T(0)P(i)x(0) + \int_{-h}^{0}\phi^T(s)V\phi(s)ds. \tag{11}$$

3 Problem Formulation

Consider the following UMJLSDS:

$$dx(t) = \Big[A(r_t,t)x(t) + A_h(r_t)x(t-h) + B_0(r_t)u_0(t) + \sum_{\ell=1}^{N} B_\ell(r_t)u_\ell(t) + B_v(r_t)v(t)\Big]dt + A_p(r_t,t)x(t)dw(t), \quad (12a)$$

$$x(t) = \phi(t),\ t \in [-h,\ 0], \quad (12b)$$

$$z(t) = \begin{bmatrix} H(r_t)x(t) \\ G_0(r_t)u_0(t) \\ G_1(r_t)u_1(t) \\ \vdots \\ G_N(r_t)u_N(t) \end{bmatrix}, \quad (12c)$$

$$y_k(t) = C_k(r_t)x(t), \quad (12d)$$

where $u_k(t) \in \mathbb{R}^{m_k}$, $k = 0,\ 1, \ldots, N$, denote the k-th control input, and $y_k(t) \in \mathbb{R}^{r_k}$, $k = 0,\ 1, \ldots, N$ denote the k-th output. It is assumed that $u_0(t)$ denotes the input controlled by the leader and $u_k(t)$ denote input controlled by the k-th follower, $k = 1, \ldots, N$. Without loss of generality, it is assumed that $G_k^T(r_t)G_k(r_t) = I_{m_k}$. Furthermore, in order to eliminate the dependence of the cost performance on $x(0)$, it is assumed that $\mathbb{E}[x(0)] = 0$, $\mathbb{E}[x(0)x^T(0)] = M_0 \geq 0$. Furthermore, $LL^T := \int_{-h}^{0} \phi(s)\phi^T(s)ds$ is defined.

The robust Stackelberg game with multiple players is investigated unlike the robust Nash game in [15]. Here, the problem under consideration is formulated as follows.

Problem : For a given $\gamma > 0$, find a robust SOF Stackelberg strategy set and a worst case disturbance,

$$u_k(t) = u_k^*(t) = F_k^*(r_t)y_k(t) = F_k^*(r_t)C_k(r_t)x(t),\ k = 1, \ldots, N, \quad (13a)$$

$$v(t) = v^*(t) = F_\gamma^*(r_t)x(t) \quad (13b)$$

such that

(i) *$u_k(t) = u_k^*(t)$, $k = 0, 1, \ldots, N$, make UMJLSDS* (12) *stochastically stable when $v(t) = 0$ and the following inequality holds:*

$$\|z\|_{E_2}^2 < \gamma^2\|v\|_2^2 + \mathcal{G}(\tilde{W}(i), \tilde{U}), \quad (14)$$

where $\mathcal{G}(\tilde{W}, \tilde{U}) := x^T(0)\tilde{W}(i)x(0) + \int_{-h}^{0} \phi^T(s)\tilde{U}\phi(s)ds$.

(ii) *When $v(t) = v^*(t) = F_\gamma^*(r_t)x(t)$ is applied, let us consider the following inequality for a leader's fixed strategy $u_0(t) = F_0^*(r_t)C_0(r_t)x(t)$, given below:*

$$\begin{aligned}\tilde{J}_k(u_0, u_1, \ldots, u_N, v^*, i) &= \mathbf{Tr}\big[M_0 P_k(i) + LL^T V_k\big] \\ &\geq J_k(u_0, u_1, \ldots, u_N, v^*, i), \end{aligned} \tag{15}$$

where

$$\Gamma_k(\boldsymbol{P}_k, V_k, F_1(i), \ldots, F_N(i), \nu_k(i), \delta_k(i), \kappa_k(i), i) \leq 0, \tag{16a}$$

$$D_p^T(i)P_k(i)D_p(i) \leq \kappa_k(i) I_{n_b}, \tag{16b}$$

and $k = 1, \ldots, N$, $i = 1, \ldots, s$,

$$\begin{aligned}
&\Gamma_k(\boldsymbol{P}_k, V_k, F_1(F_0(i), i), \ldots, F_N(F_0(i), i), \nu_k(i), \delta_k(i), \kappa_k(i), i) \\
&\quad := \begin{bmatrix} \tilde{\Psi}_{11}(i) & P_k(i)A_h(i) \\ A_h^T(i)P_k(i) & -V_k \end{bmatrix}, \\
&\tilde{\Psi}_{11}(i) := P_k(i)\tilde{A}_\gamma(i) + \tilde{A}_\gamma^T(i)P_k(i) + \nu_k^{-1}(i)P_k(i)D(i)D^T(i)P_k(i) \\
&\qquad + \nu_k(i)E_a^T(i)E_a(i) + Q_k(i) + C_k^T(i)F_k^T(F_0, i)R_k(i)F_k(F_0, i)C_k(i) \\
&\qquad + V_k + \sum_{j=1}^{s} \pi_{ij}P_k(j) + A_p^T(i)P_k(i)A_p(i) \\
&\qquad + \delta_k^{-1}(i)A_p^T(i)P_k(i)D_p(i)D_p^T(i)P_k(i)A_p(i) \\
&\qquad + (\delta_k(i) + \kappa_k(i))E_{pa}^T(i)E_{pa}(i), \\
&\tilde{A}_\gamma(i) := A_\gamma(i) + B_0(i)F_0(i)C_0(i) + \sum_{\ell=1}^{N} B_\ell(i)F_\ell(F_0(i), i)C_\ell(i), \\
&J_k(u_0, u_1, \ldots, u_N, v^*, i) = J_k(F_0(i)C_0(i)x, F_k(F_0(i), i)C_k(i)x, v^*, i) \\
&\quad = \mathbb{E}\bigg[\int_0^\infty x^T(t, \phi)\Big(Q_k(r_t) + C_k^T(r_t)F_k^T(F_0(r_t), r_t) \\
&\qquad \times R_k(r_t)F_k(F_0(r_t), r_t)C_k(r_t)\Big)x(t, \phi)dt\bigg|\phi, r_0 = i\bigg], \\
&Q_k(r_t) = Q_k^T(r_t) > 0, \ R_k(r_t) = R_k^T(r_t) > 0, \\
&dx(t) = \bigg[\Big(A(r_t, t) + B_0(r_t)F_0(r_t)C_0(r_t) \\
&\qquad + \sum_{\ell=1}^{N} B_\ell(r_t)F_\ell(F_0(r_t), r_t)C_\ell(r_t)\Big)x(t) + A_h(r_t)x(t-h)
\end{aligned}$$

$$+ B_v(r_t)v^*(t)\Big]dt + A_p(r_t,t)x(t)dw(t).$$

A follower's strategy set $(\bar{u}_1^0(u_0), \ldots, \bar{u}_N^0(u_0))$ *minimizes the following upper bound:*

$$\begin{aligned}
&\tilde{J}_k(u_0, \bar{u}_1^0(u_0), \ldots, \bar{u}_k^0(u_0), \ldots, \bar{u}_N^0(u_0), v^*, i) \\
&\quad = \min_{u_k^0(u_0)} \tilde{J}_k(u_0, \bar{u}_1^0(u_0), \ldots, u_k^0(u_0), \ldots, \bar{u}_N^0(u_0), v^*, i) \\
&\quad = \min \mathbf{Tr}\big[M_0 P_k(i) + LL^T V_k\big] \qquad (17)
\end{aligned}$$

with respect to $u_k^0(u_0)$ *for each* k. *That is, the following Nash equilibrium condition for the followers holds.*

$$\begin{aligned}
&\tilde{J}_k(u_0, \bar{u}_1^0(u_0), \ldots, \bar{u}_k^0(u_0), \ldots, \bar{u}_N^0(u_0), v^*, i) \\
&\quad \le \tilde{J}_k(u_0, \bar{u}_1^0(u_0), \ldots, u_k^0(u_0), \ldots, \bar{u}_N^0(u_0), v^*, i), \\
&k = 1, \ldots, N. \qquad (18)
\end{aligned}$$

(iii) *For any mapping* $\mathcal{H}_k$ *such that* $u_k^0 = \mathcal{H}_k u_0 = u_k(u_0) \in \mathbb{R}^{m_k}$, $k = 1, \ldots, N$ *with* $v(t) = v^*(t) = F_\gamma^* x(t)$, *the following inequality holds:*

$$\begin{aligned}
\tilde{J}_0(u_0, \bar{u}_1^0(u_0), \ldots, \bar{u}_N^0(u_0), v^*, i) &= \mathbf{Tr}\left[M_0 P_0(i) + LL^T V_0\right] \\
&\ge J_0(u_0, u_1, \ldots, u_N, v^*, i), \ \forall u_0, \quad (19)
\end{aligned}$$

where $P_0(i)$ *is the solution of the following CCSMIs*

$$\Gamma_0(\boldsymbol{P}_0, V_0, F_0(i), F_1(i), \ldots, F_N(i), \nu_0(i), \delta_0(i), \kappa_0(i), i) \le 0, \qquad (20a)$$

$$D_p^T(i) P_0(i) D_p(i) \le \kappa_0(i) I_{n_b}, \qquad (20b)$$

and $i = 1, \ldots, s$,

$$\begin{aligned}
&\Gamma_0(\boldsymbol{P}_0, V_0, F_0(i), F_1(i), \ldots, F_N(i), \nu_0(i), \delta_0(i), \kappa_0(i), i) \\
&\quad := \begin{bmatrix} \hat{\Psi}_{11}(i) & P_0(i) A_h(i) \\ A_h^T(i) P_0(i) & -V_0 \end{bmatrix}, \\
&\hat{\Psi}_{11}(i) := P_0(i)\hat{A}_\gamma(i) + \hat{A}_\gamma^T(i) P_0(i) \\
&\qquad + \nu_0^{-1}(i) P_0(i) D(i) D^T(i) P_0(i) + \nu_0(i) E_a^T(i) E_a(i) \\
&\qquad + Q_0(i) + C_0^T(i) F_0^T(i) R_0(i) F_0(i) C_0(i) + V_0 + \sum_{j=1}^{s} \pi_{ij} P_0(j) \\
&\qquad + A_p^T(i) P_0(i) A_p(i) + \delta_0^{-1}(i) A_p^T(i) P_0(i) D_p(i) D_p^T(i) P_0(i) A_p(i)
\end{aligned}$$

$$
\begin{aligned}
&\qquad + (\delta_0(i) + \kappa_0(i))E_{pa}^T(i)E_{pa}(i),\\
&\hat{A}_\gamma(i) := A_\gamma(i) + \sum_{\ell=0}^{N} B_\ell(i)F_\ell(i)C_\ell(i),\\
&J_0(u_0, u_1, \ldots, u_N, v^*, i)\\
&\quad = \mathbb{E}\Bigg[\int_0^\infty x^T(t,\phi)\Big(Q_0(r_t) + C_0^T(r_t)F_0^T(r_t)\\
&\qquad \times R_0(r_t)F_0(r_t)C_0(r_t)\Big)x(t,\phi)dt\Big|\phi, r_0 = i\Bigg],\\
&Q_0(r_t) = Q_0^T(r_t) > 0,\ R_0(r_t) = R_0^T(r_t) > 0.\\
&dx(t) = \Bigg[\Bigg(A(r_t,t) + B_0(r_t)F_0(r_t)C_0(r_t) + \sum_{\ell=1}^{N} B_\ell(r_t)F_\ell(r_t)C_\ell(r_t)\Bigg)x(t)\\
&\qquad + A_h(r_t)x(t-h) + B_v(r_t)v^*(t)\Bigg]dt + A_p(r_t,t)x(t)dw(t).
\end{aligned}
$$

A leader's strategy minimizes the following upper bound:

$$
\begin{aligned}
&\tilde{J}_0(\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_N, v^*, i)\\
&\quad = \min_{u_0} \tilde{J}_0(u_0, \bar{u}_1^0(u_0), \ldots, \bar{u}_N^0(u_0), v^*, i)\\
&\quad = \min \mathbf{Tr}\ \big[M_0P_0(i) + LL^TV_0\big]. \qquad (21)
\end{aligned}
$$

In this case, the following inequality holds.

$$
\tilde{J}_0(\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_N, v^*, i) \le \tilde{J}_0(u_0, \bar{u}_1^0(u_0), \ldots, \bar{u}_N^0(u_0), v^*, i), \qquad (22a)
$$

$$
\bar{u}_k = \bar{u}_k^0(\bar{u}_0),\ k = 1, \ldots, N. \qquad (22b)
$$

In the following subsections, the existence conditions of the robust SOF Stackelberg strategy set are established through the solution of two constraint optimization problems.

3.1 Disturbance Attenuation Condition

First, the disturbance attenuation condition is investigated. Consider the closed-loop UMJLSDS and the cost functions. For arbitrary $u_k(t)$, $k = 0, 1, \ldots, N$, the closed-loop UMJLSDS is established as

$$
dx(t) = \Big[\bar{A}_F(r_t,t)x(t) + A_h(r_t)x(t-h) + B_v(r_t)v(t)\Big]dt
$$

$$+ A_p(r_t,t)x(t)dw(t), \tag{23a}$$

$$z(t) = \begin{bmatrix} H(r_t)x(t) \\ G_0(r_t)F_0(r_t)C_0(r_t) \\ G_1(r_t)F_1(r_t)C_1(r_t) \\ \vdots \\ G_N(r_t)F_N(r_t)C_N(r_t) \end{bmatrix} x(t), \tag{23b}$$

where

$$\bar{A}_F(i,t) = \bar{A}_F(i) + D(i)\Theta(i,t)E_a(i),$$
$$\bar{A}_F(i) := A(i) + \sum_{\ell=0}^{N} B_\ell(i)F_\ell(i)C_\ell(i).$$

Thus, we have the following CCSMIs, using Lemma 2:

$$\tilde{\Lambda}(\tilde{\boldsymbol{W}}, \tilde{U}, \tilde{\mu}(i), \tilde{\varepsilon}(i), \tilde{\lambda}(i), i) < 0, \tag{24a}$$
$$D_p^T(i)\tilde{W}(i)D_p(i) \le \tilde{\lambda}(i)I_{n_b}, \tag{24b}$$

where $i = 1, \ldots, s$,

$$\tilde{\Lambda}(\tilde{\boldsymbol{W}}, \tilde{U}, \tilde{\mu}(i), \tilde{\varepsilon}(i), \tilde{\lambda}(i), i) := \begin{bmatrix} \tilde{\Phi}_{11}(i) & \tilde{W}(i)A_h(i) \\ A_h^T(i)\tilde{W}(i) & -\tilde{U} \end{bmatrix},$$
$$\tilde{\Phi}_{11}(i) := \tilde{W}(i)\bar{A}_F(i) + \bar{A}_F^T(i)\tilde{W}(i) + \tilde{\mu}^{-1}(i)\tilde{W}(i)D(i)D^T(i)\tilde{W}(i)$$
$$+ \tilde{\mu}(i)E_a^T(i)E_a(i) + H^T(i)H(i) + \sum_{\ell=0}^{N} C_\ell^T(i)F_\ell^T(i)F_\ell(i)C_\ell(i)$$
$$+ \tilde{U} + \sum_{j=1}^{s} \pi_{ij}\tilde{W}(j) + A_p^T(i)\tilde{W}(i)A_p(i)$$
$$+ \tilde{\varepsilon}^{-1}(i)A_p^T(i)\tilde{W}(i)D_p(i)D_p^T(i)\tilde{W}(i)A_p(i)$$
$$+ (\tilde{\varepsilon}(i)+\tilde{\lambda}(i))E_{pa}^T(i)E_{pa}(i)+\gamma^{-2}\tilde{W}(i)B_v(i)B_v^T(i)\tilde{W}(i).$$

Furthermore, the worst-case disturbance is given by

$$v^*(t) = F_\gamma^*(r_t)x(t) = \gamma^{-2}B_v^T(r_t)\tilde{W}(r_t)x(t). \tag{25}$$

3.2 Stackelberg Strategy

Second, the Stackelberg game for the UMJLSDS is considered. Let us consider the following optimization problem related to a follower's Nash strategy for UMJLSDS:

$$\begin{aligned}&\min_{u_k^0(u_0)} \tilde{J}_k(u_0, \bar{u}_1^0(u_0), \ldots, u_k^0(u_0), \ldots, \bar{u}_N^0(u_0), v^*, i)\\ &\quad = \min_{\boldsymbol{\Sigma}_k} \mathbf{Tr}\big[M_0 P_k(i) + LL^T V_k\big], \qquad (26)\\ &\text{s.t.}\boldsymbol{\Sigma}_k := (\boldsymbol{P}_k, V_k, F_k(i), \nu_k(i), \delta_k(i), \kappa_k(i)) \text{ satisfies (16)},\end{aligned}$$

where $u_0 = u_0(t)$ is the fixed leader's strategy.

In order to obtain the strategy set $(F_1(F_0, i), \ldots, F_N(F_0, i))$, the Karush-Kuhn-Tucker (KKT) conditions are derived. Define the following Lagrangian:

$$\mathscr{L}_k(i) = \mathbf{Tr}\ \big[M_0 P_k(i) + LL^T V_k\big] + \sum_{m=1}^{s} \mathbf{Tr}\ [S_k(m)\Delta_k(m)], \qquad (27)$$

where $S_k(m)$ is the symmetric matrix of the Lagrange multiplier, and we set $r_0 = i$. Furthermore, we have

$$\begin{aligned}\Delta_k(i) &:= \Delta_k(\boldsymbol{P}_k, V_k, F_1(i), \ldots, F_N(i), \nu_k(i), \delta_k(i), \kappa_k(i), i)\\ &= \tilde{\Psi}_{11}(i) + P_k(i)A_h(i)V_k^{-1}A_h^T(i)P_k(i). \qquad (28)\end{aligned}$$

In this case, we have the following cross coupled stochastic matrix equations (CCSMEs):

$$\begin{aligned}&\frac{\partial \mathscr{L}_k(i)}{\partial P_k(i)} = \Delta_k^1(i)\\ &\quad = \Delta_k^1(\boldsymbol{S}_k, P_k(i), V_k, F_1(i), \ldots, F_N(i), \nu_k(i), \delta_k(i), i) = 0, \qquad (29a)\\ &\frac{\partial \mathscr{L}_k(i)}{\partial S_k(i)} = \Delta_k(i) = 0, \qquad (29b)\\ &\frac{1}{2} \cdot \frac{\partial \mathscr{L}_k(i)}{\partial F_k(F_0, i)} = \Delta_k^2(i) = \Delta_k^2(F_k(i), P_k(i), S_k(i), i) = 0, \qquad (29c)\end{aligned}$$

where $k = 1, \ldots, N$, $i = 1, \ldots, s$,

$$\begin{aligned}\Delta_k^1(i) &= M_0 + S_k(i)\tilde{A}_\gamma^T(i) + \tilde{A}_\gamma(i)S_k(i)\\ &\quad + \nu_k^{-1}(i)\big[S_k(i)P_k(i)D(i)D^T(i) + D(i)D^T(i)P_k(i)S_k(i)\big]\\ &\quad + \sum_{j=1}^{s} \pi_{ji}S_k(j) + A_p(i)S_k(i)A_p^T(i)\\ &\quad + \delta_k^{-1}(i)\big[A_p(i)S_k(i)A_p^T(i)P_k(i)D_p(i)D_p^T(i)\\ &\quad + D_p(i)D_p^T(i)P_k(i)A_p(i)S_k(i)A_p^T(i)\big]\end{aligned}$$

$$
\begin{aligned}
&\quad + S_k(i)P_k(i)A_h(i)V_k^{-1}A_h^T(i) + A_h(i)V_k^{-1}A_h^T(i)P_k(i)S_k(i), \\
\Delta_k^2(i) &= \left[R_k(i)F_k(F_0,i)C_k(i) + B_k^T(i)P_k(i)\right]S_k(i)C_k^T(i).
\end{aligned}
$$

It should be noted that the derivative with respect to V_k, $\nu_k(i)$, $\delta_k(i)$ and $\kappa_k(i)$, $k = 1, \dots, N$ is not needed because this optimization part can be performed by means of the LMI instead of the KKT condition.

From $\Delta_k^1(i) = 0$, we have $S_k(i) > 0$. Therefore, from $\Delta_k^2(i) = 0$, each follower has the following strategy:

$$
u_k(t) = F_k^*(F_0, r_t)C_k(r_t)x(t),\ k = 1, \dots, N. \tag{30}
$$

Next, the leader's strategy is established. The following optimization problem for UMJLSDS related to the leader's strategy can be defined:

$$
\begin{aligned}
&\min_{u_0} \tilde{J}_0(u_0, \bar{u}_1^0(u_0), \dots, \bar{u}_N^0(u_0), v^*, i) = \min_{\mathbf{\Sigma}_0} \mathbf{Tr}\big[M_0P_0(i) + LL^TV_0\big], \qquad (31)\\
&\text{s.t.}\mathbf{\Sigma}_0 := (\boldsymbol{P}_0, \boldsymbol{P}_1, \dots, \boldsymbol{P}_N, \boldsymbol{S}_1, \dots, \boldsymbol{S}_N, V_0, V_1, \dots, V_N, F_0(i), F_1(i), \dots, F_N(i),\\
&\qquad\qquad \boldsymbol{\nu}_0, \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_N, \boldsymbol{\delta}_0, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_N, \boldsymbol{\kappa}_0, \boldsymbol{\kappa}_1, \dots, \boldsymbol{\kappa}_N)
\end{aligned}
$$

satisfies (20) and (29).

In order to solve the above-mentioned optimization problem, let us consider the following Lagrangian:

$$
\begin{aligned}
\mathscr{L}_0(i) &= \mathbf{Tr}\ \big[M_0P_0(i) + LL^TV_0\big] + \sum_{m=1}^{s} \mathbf{Tr}\ [S_0(m)\Delta_0(m)] \\
&\quad + \sum_{m=1}^{s}\sum_{k=1}^{N} \mathbf{Tr}\ \Big[T_k(m)\Delta_k(m) + X_k(m)\Delta_k^1(m) + Z_k(m)\Delta_k^2(m)\Big],
\end{aligned} \tag{32}
$$

where $S_0(m)$ and $T_k(m)$, $X_k(m)$, $Z_k(m)$, $k = 1, \dots, N$ are the symmetric matrix of the Lagrange multipliers, and

$$
\begin{aligned}
\Delta_0(i) &:= \Delta_0(\boldsymbol{P}_0, V_0, F_0(i), F_1(i), \dots, F_N(i), \nu_0(i), \delta_0(i), \kappa_0(i), i) \\
&= \hat{\Psi}_{11}(i) + P_0(i)A_h(i)V_0^{-1}A_h^T(i)P_0(i).
\end{aligned} \tag{33}
$$

As a necessary condition, the following equations can be derived by using the KKT condition:

$$
\begin{aligned}
&\frac{\partial \mathscr{L}_0(i)}{\partial P_0(i)} = \Gamma_0^1(i) \\
&\quad = \Gamma_0^1(\boldsymbol{S}_0, P_0(i), V_0, F_0(i), F_1(i), \dots, F_N(i), \nu_0(i), \delta_0(i), i) = 0,
\end{aligned} \tag{34a}
$$

$$\frac{\partial \mathscr{L}_0(i)}{\partial P_k(i)} = \Gamma_k^1(i) = \Gamma_k^1(\boldsymbol{T}_k, P_k(i), S_k(i), V_k, X_k(i), Z_k(i), F_0(i), F_1(i), \dots, F_N(i), \nu_k(i), \delta_k(i), i) = 0, \tag{34b}$$

$$\frac{\partial \mathscr{L}_0(i)}{\partial S_k(i)} = \Gamma_k^3(i) = \Gamma_k^3(\boldsymbol{X}_k, P_k(i), V_k, Z_k(i), F_0(i), F_1(i), \dots, F_N(i), \nu_k(i), \delta_k(i), i) = 0, \tag{34c}$$

$$\frac{1}{2} \cdot \frac{\partial \mathscr{L}_0(i)}{\partial F_k(i)} = \Gamma_k^2(i) = \Gamma_k^2(Z_k(i), P_0(i), S_0(i), P_k(i), S_k(i), T_k(i), X_k(i), F_k(i), i) = 0, \tag{34d}$$

$$\frac{1}{2} \cdot \frac{\partial \mathscr{L}_0(i)}{\partial F_0(i)} = \Gamma_0^2(i) = \Gamma_0^2(F_0(i), P_0(i), P_1(i), \dots, P_N(i), S_0(i), S_1(i), \dots, S_N(i), T_1(i), \dots, T_N(i), X_1(i), \dots, X_N(i), i) = 0, \tag{34e}$$

where $k = 1, \dots, N$, $i = 1, \dots, s$,

$$\frac{\partial \mathscr{L}_0(i)}{\partial S_0(i)} = \Delta_0(i) = 0, \ \frac{\partial \mathscr{L}_0(i)}{\partial T_k(i)} = \Delta_k(i) = 0,$$

$$\frac{\partial \mathscr{L}_0(i)}{\partial X_k(i)} = \Delta_k^1(i) = 0, \ \frac{\partial \mathscr{L}_0(i)}{\partial Z_k(i)} = \Delta_k^2(i) = 0,$$

$$\begin{aligned}
\Gamma_0^1(i) &= M_0 + S_0(i)\hat{A}_\gamma^T(i) + \hat{A}_\gamma(i)S_0(i) \\
&\quad + \nu_0^{-1}(i)\big[S_0(i)P_0(i)D(i)D^T(i) + D(i)D^T(i)P_0(i)S_0(i)\big] \\
&\quad + \sum_{j=1}^{s} \pi_{ji} S_0(j) + A_p(i)S_0(i)A_p^T(i) \\
&\quad + \delta_0^{-1}(i)\big[A_p(i)S_0(i)A_p^T(i)P_0(i)D_p(i)D_p^T(i) \\
&\quad + D_p(i)D_p^T(i)P_0(i)A_p(i)S_0(i)A_p^T(i)\big] \\
&\quad + S_0(i)P_0(i)A_h(i)V_0^{-1}A_h^T(i) + A_h(i)V_0^{-1}A_h^T(i)P_0(i)S_0(i),
\end{aligned}$$

$$\begin{aligned}
\Gamma_k^1(i) &= T_k(i)\hat{A}_\gamma^T(i) + \hat{A}_\gamma(i)T_k(i) \\
&\quad + \nu_k^{-1}(i)\big[T_k(i)P_k(i)D(i)D^T(i) + D(i)D^T(i)P_k(i)T_k(i) \\
&\quad + S_k(i)X_k(i)D(i)D^T(i) + D(i)D^T(i)X_k(i)S_k(i)\big] \\
&\quad + \sum_{j=1}^{s} \pi_{ji} T_k(j) + A_p(i)T_k(i)A_p^T(i) \\
&\quad + \delta_k^{-1}(i)\big[A_p(i)T_k(i)A_p^T(i)P_k(i)D_p(i)D_p^T(i) \\
&\quad + D_p(i)D_p^T(i)P_k(i)A_p(i)T_k(i)A_p^T(i)
\end{aligned}$$

$$
\begin{aligned}
&+ A_p(i)S_k(i)A_p^T(i)X_k(i)D_p(i)D_p^T(i)\\
&+ D_p(i)D_p^T(i)X_k(i)A_p(i)S_k(i)A_p^T(i)\big]\\
&+ T_k(i)P_k(i)A_h(i)V_k^{-1}A_h^T(i) + A_h(i)V_k^{-1}A_h^T(i)P_k(i)T_k(i)\\
&+ S_k(i)X_k(i)A_h(i)V_k^{-1}A_h^T(i) + A_h(i)V_k^{-1}A_h^T(i)X_k(i)S_k(i)\\
&+ \frac{1}{2}\Big(B_k(i)[Z_k(i)]^T C_k(i)S_k(i) + S_k(i)C_k^T(i)Z_k(i)B_k^T(i)\Big),\\
\Gamma_k^3(i) &= X_k(i)\hat{A}_\gamma(i) + \hat{A}_\gamma^T(i)X_k(i)\\
&+ \nu_k^{-1}(i)\big[P_k(i)D(i)D^T(i)X_k(i) + X_k(i)D(i)D^T(i)P_k(i)\big]\\
&+ \sum_{j=1}^{s}\pi_{ij}X_k(j) + A_p^T(i)X_k(i)A_p(i)\\
&+ \delta_k^{-1}(i)\big[A_p^T(i)P_k(i)D_p(i)D_p^T(i)X_k(i)A_p(i)\\
&+ A_p^T(i)X_k(i)D_p(i)D_p^T(i)P_k(i)A_p(i)\big]\\
&+ P_k(i)A_h(i)V_k^{-1}A_h^T(i)X_k(i) + X_k(i)A_h(i)V_k^{-1}A_h^T(i)P_k(i)\\
&+ \frac{1}{2}\Big(C_k^T(i)F_k^T(i)R_k(i) + P_k(i)B_k(i)\Big)[Z_k(i)]^T C_k(i)\\
&+ \frac{1}{2}C_k^T(i)[Z_k(i)]\Big(R_k(i)F_k(i)C_k(i) + B_k^T(i)P_k(i)\Big),\\
\Gamma_k^2(i) &= B_k^T(i)\big(P_0(i)S_0(i) + P_k(i)T_k(i) + X_k(i)S_k(i)\big)C_k^T(i)\\
&+ R_k(i)\Big(F_k(i)C_k(i)T_k(i) + \frac{1}{2}[Z_k(i)]^T C_k(i)S_k(i)\Big)C_k^T(i),\\
\Gamma_0^2(i) &= R_0(i)F_0(i)C_0(i)S_0(i)C_0^T(i) + B_0^T(i)\Big(P_0(i)S_0(i)\\
&+ \sum_{\ell=1}^{N}\big[P_\ell(i)T_\ell(i) + X_\ell(i)S_\ell(i)\big]\Big)C_0^T(i),\\
F_k(i) &= F_k(F_0, i),\ k = 1, \ldots, N.
\end{aligned}
$$

Therefore, if $C_0(i)S_0(i)C_0^T(i)$ is nonsingular, the gain of the leader's strategy $F_0(i)$ can be computed as follows:

$$u_0(t) = F_0^*(r_t)C_0(r_t)x(t). \tag{35}$$

It should also be noted that the derivative with respect to V_k, $\nu_k(i)$, $\delta_k(i)$ and $\kappa_k(i)$, $k = 0, 1, \ldots, N$ is not needed because these variables can be determined by solving the LMI optimization problem that will be explained later.

We are now in a position to state the main contribution of this work.

Theorem 1 *Consider the UMJLSDS* (12) *with multiple players controlling* $u_k(t) = F_k(r_t)C_k(r_t)x(t)$, $k = 0, 1, \ldots, N$, *and a deterministic disturbance* $v(t) = F_\gamma(r_t)x(t)$. *For a given attenuation performance level* $\gamma > 0$, *suppose that,*

(i) *there exists a feasible solution set* $\tilde{W}(i)$, $i = 1, \ldots, s$, *for the CCSMIs* (24),

(ii) *there exist feasible solution sets of the optimization problems* (26) *and* (31) *under constraints CCSMIs* (16) *and* (20), *respectively,*

(iii) *there exist the solution sets for the CCSMEs* (29) *and* (34) *respectively.*

Then the robust SOF Stackelberg strategy set can be obtained in the forms of (30) *and* (35) *such that the conditions* (14), (18), *and* (22) *in the problem formulation are satisfied.*

4 Numerical Algorithm

In order to obtain the robust SOF Stackelberg strategy set, we must solve the optimization problems (26) and (31), the CCSMEs (29) and (34) that consist of the bilinear matrix inequalities (BMIs). However, despite the fact that Newton's method or other reliable numerical algorithms [12] can be applied, it is difficult to obtain the solution set even though the Newton's method or other reliable numerical algorithms [12] can be applied. Hence, the following algorithm based on iterative LMI optimization is proposed. It should be noted that the proposed algorithm comes originates from the Krasnoselskii iteration sequence [17] and can be established easily, because the calculation of a Jacobi matrix is not needed required when using the Newton's method. Furthermore, the weaker convergence is attained as compared with to the previous result in [12].

Step 1. *Set the initial values: choose* $F_k^{(0)}(i)$, $k = 0, 1, \ldots, N$, $Z_k^{(0)}(i)$, $k = 1, \ldots, N$, *and* $F_\gamma^{(0)}(i)$, $i = 1, \ldots, s$, *such that closed-loop* UMJLSDS (12a) *is stochastically stable; choose an appropriate* κ *value for* $\tilde{W}^{(0)}(i) = \kappa I_n$ *and compute* $F_\gamma^{(0)}(i) = \gamma^{-2} B_v^T(i)\tilde{W}^{(0)}(i)$;

Step 2-1. *Solve the following optimization problem* (36) *for variable* Ξ_k, $k = 1, \ldots, N$, $i = 1, \ldots, s$:

$$\min_{\Xi_k} \mathbf{Tr} \left[\sum_{j=1}^{s} M_0 P_k^{(n+1)}(j) + LL^T V_k^{(n+1)} \right], \tag{36a}$$

$$\begin{aligned}
&\boldsymbol{\Xi}_k := (\boldsymbol{P}_k^{(n+1)}, V_k^{(n+1)}, \tilde{\boldsymbol{\nu}}_k^{(n+1)}, \tilde{\boldsymbol{\delta}}_k^{(n+1)}, \tilde{\boldsymbol{\kappa}}_k^{(n+1)}),\\
&\text{s.t. } \boldsymbol{\Xi}_k \text{ satisfies (36b) and (36c)},\\
&\tilde{\Gamma}_k(\boldsymbol{P}_k, V_k, \nu_k(i), \delta_k(i), \kappa_k(i), i)\\
&\quad := \begin{bmatrix} \tilde{\boldsymbol{\Psi}}_{11}(i) & P_k(i)A_h(i) & P_k(i)D(i) & \tilde{\Psi}^{14}(i) \\ A_h^T(i)P_k(i) & -V_k & 0 & 0 \\ D^T(i)P_k(i) & 0 & -\nu_k(i)I_{n_p} & 0 \\ \tilde{\Psi}^{14T}(i) & 0 & 0 & -\delta_k(i)I_{n_p} \end{bmatrix} \le 0, \qquad (36b)\\
&D_p^T(i)P_k(i)D_p(i) \le \kappa_k(i)I_{n_b}, \qquad (36c)
\end{aligned}$$

where $i = 1, \ldots, s$,

$$\begin{aligned}
\tilde{\boldsymbol{\Psi}}_{11}(i) := {}& P_k(i)\hat{A}_\gamma^{(n)}(i) + \hat{A}_\gamma^{(n)T}(i)P_k(i) + \nu_k(i)E_a^T(i)E_a(i)\\
&+ Q_k(i) + F_k^{(n)T}(i)R_k(i)F_k^{(n)}(i) + V_k + \sum_{j=1}^{s} \pi_{ij}P_k(j)\\
&+ A_p^T(i)P_k(i)A_p(i) + (\delta_k(i) + \kappa_k(i))E_{pa}^T(i)E_{pa}(i),\\
\tilde{\Psi}^{14}(i) := {}& A_p^T(i)P_k(i)D_p(i);
\end{aligned}$$

Step 2-2. *Solve the following* CCSMEs *for* $S_k^{(n+1)}(i)$, $k = 1, \ldots, N$, $i = 1, \ldots, s$:

$$\begin{aligned}
&\Delta_k^1(\boldsymbol{S}_k^{(n+1)}, P_k^{(n+1)}(i), V_k^{(n+1)}(i), F_1^{(n)}(i), \ldots, F_N^{(n)}(i)\\
&\quad, \nu_k^{(n+1)}(i), \delta_k^{(n+1)}(i), i) = 0; \qquad (37)
\end{aligned}$$

Step 2-3. *Compute* $F_k^{(n+1)}(i)$, $k = 1, \ldots, N$, $i = 1, \ldots, s$:

$$\begin{aligned}
F_k^{(n+1)}(i) = {}& -\big[R_k(i)\big]^{-1} B_k^T(i)P_k^{(n+1)}(i)S_k^{(n+1)}(i)C_k^T(i)\\
&\times \big[C_k(i)S_k^{(n+1)}(i)C_k^T(i)\big]^{-1}; \qquad (38)
\end{aligned}$$

Step 3-1. *Solve the following optimization problem* (39) *for variable* $\Xi_0(i)$, $i = 1, \ldots, s$:

$$\begin{aligned}
&\min_{\boldsymbol{\Xi}_0} \mathbf{Tr}\left[\sum_{j=1}^{s} M_0 P_0^{(n+1)}(j) + LL^T V_0^{(n+1)}\right], \qquad (39a)\\
&\boldsymbol{\Xi}_0 := (\boldsymbol{P}_0^{(n+1)}, V_0^{(n+1)}, \boldsymbol{\nu}_0^{(n+1)}, \boldsymbol{\delta}_0^{(n+1)}, \boldsymbol{\kappa}_0^{(n+1)}),\\
&\text{s.t. } \boldsymbol{\Xi}_0 \text{ satisfies (39b) and (39c)},
\end{aligned}$$

$$\hat{\hat{\Gamma}}_0(\boldsymbol{P}_0, V_0, \nu_0(i), \delta_0(i), \kappa_0(i), i)$$
$$:= \begin{bmatrix} \hat{\boldsymbol{\Psi}}_{11}(i) & P_0(i)A_h(i) & P_0(i)D(i) & \hat{\Psi}^{14}(i) \\ A_h^T(i)P_0(i) & -V_0 & 0 & 0 \\ D^T(i)P_0(i) & 0 & -\nu_0(i)I_{n_p} & 0 \\ \hat{\Psi}^{14T}(i) & 0 & 0 & -\delta_0(i)I_{n_p} \end{bmatrix} \leq 0, \quad (39b)$$
$$D_p^T(i)P_0(i)D_p(i) \leq \kappa_0(i)I_{n_b}, \quad (39c)$$

where

$$\hat{\boldsymbol{\Psi}}_{11}(i) := P_0(i)\hat{A}_\gamma^{(n)}(i) + \hat{A}_\gamma^{(n)T}(i)P_0(i) + \nu_0(i)E_a^T(i)E_a(i) + Q_0(i) + F_0^{(n)T}(i)R_0(i)F_0^{(n)}(i) + V_0 + \sum_{j=1}^{s} \pi_{ij}P_0(j) + A_p^T(i)P_0(i)A_p(i) + (\delta_0(i) + \kappa_0(i))E_{pa}^T(i)E_{pa}(i),$$
$$\hat{\Psi}^{14}(i) := A_p^T(i)P_0(i)D_p(i),$$
$$\hat{A}_\gamma^{(n)}(i) := A(i) + \sum_{\ell=0}^{N} B_\ell(i)F_\ell^{(n)}(i)C_\ell(i) + B_v(i)F_\gamma^{(n)}(i);$$

Step 3-2. *Solve the following* CCSMEs *for* $\boldsymbol{S}_0^{(n+1)}$, $i = 1, \ldots, s$:

$$\Gamma_0^1(\boldsymbol{S}_0^{(n+1)}, P_0^{(n+1)}(i), V_0^{(n+1)}, F_0^{(n)}(i), F_1^{(n+1)}(i), \ldots, F_N^{(n+1)}(i), \nu_0^{(n+1)}(i), \delta_0^{(n+1)}(i), i) = 0; \quad (40)$$

Step 3-3. *Solve the following* CCSMEs *for* $\boldsymbol{X}_k^{(n+1)}$, $k = 1, \ldots, N$, $i = 1, \ldots, s$:

$$\Gamma_k^3(\boldsymbol{X}_k^{(n+1)}, P_k^{(n+1)}(i), V_k^{(n+1)}, Z_k^{(n)}(i), F_0^{(n)}(i), F_1^{(n+1)}(i), \ldots, F_N^{(n+1)}(i), \nu_k^{(n+1)}(i), \delta_k^{(n+1)}(i), i) = 0; \quad (41)$$

Step 3-4. *Solve the following* CCSMEs *for* $\boldsymbol{T}_k^{(n+1)}$, $k = 1, \ldots, N$, $i = 1, \ldots, s$:

$$\Gamma_k^1(\boldsymbol{T}_k^{(n+1)}, P_k^{(n+1)}(i), S_k^{(n+1)}(i), V_k^{(n+1)}, X_k^{(n+1)}(i), Z_k^{(n)}(i), F_0^{(n)}(i), F_1^{(n+1)}(i), \ldots, F_N^{(n+1)}(i), \nu_k^{(n+1)}(i), \delta_k^{(n+1)}(i), i) = 0; \quad (42)$$

Step 3-5. *Solve the following* CCSMEs *for* $\boldsymbol{Z}_k^{(n+1)}$, $k = 1, \ldots, N$, $i = 1, \ldots, s$:

$$[Z_k^{(n+1)}(i)]^T = -2\big[R_k(i)\big]^{-1}\Big(B_k^T(i)\big(P_0^{(n+1)}(i)S_0^{(n+1)}(i)$$

$$
\begin{aligned}
&+ P_k^{(n+1)}(i)T_k^{(n+1)}(i) + X_k^{(n+1)}(i)S_k^{(n+1)}(i)\big) \\
&+ R_k(i)F_k^{(n+1)}(i)C_k(i)T_k^{(n+1)}(i)\Big)C_k^T(i) \\
&\quad \times \big[C_k(i)S_k^{(n+1)}(i)C_k^T(i)\big]^{-1};
\end{aligned} \tag{43}
$$

Step 3-6. *Compute* $F_0^{(n+1)}(i)$, $i = 1, \dots, s$*:*

$$
\begin{aligned}
F_0^{(n+1)}(i) = -\big[R_0(i)\big]^{-1}B_0^T(i)\Bigg(&P_0^{(n+1)}(i)S_0^{(n+1)}(i) \\
&+ \sum_{\ell=1}^{N}\big[P_\ell^{(n+1)}(i)T_\ell^{(n+1)}(i) + X_\ell^{(n+1)}(i)S_\ell^{(n+1)}(i)\big]\Bigg)C_0^T(i) \\
&\times \big[C_0(i)S_0^{(n+1)}(i)C_0^T(i)\big]^{-1};
\end{aligned} \tag{44}
$$

Step 4. *Solve the following optimization problem for* $\tilde{W}^{(n+1)}(i)$ *for variables* $\mathbf{\Sigma}_2$*:*

$$
\min_{\mathbf{\Sigma}_2} \sum_{j=1}^{s} \mathbf{Tr}[\tilde{W}^{(n+1)}(j) + LL^T\tilde{U}^{(n+1)}], \tag{45a}
$$

$\mathbf{\Sigma}_2 := (\tilde{\boldsymbol{W}}^{(n+1)}, \tilde{U}^{(n+1)}, \tilde{\boldsymbol{\mu}}^{(n+1)}, \tilde{\boldsymbol{\varepsilon}}^{(n+1)}, \tilde{\boldsymbol{\lambda}}^{(n+1)})$,

s.t. $\mathbf{\Sigma}_2$ satisfies (45b) and (45c),

$$
\begin{aligned}
&\hat{\Lambda}(\tilde{\boldsymbol{W}}, \tilde{U}, \tilde{\mu}(i), \tilde{\varepsilon}(i), \tilde{\lambda}(i), i) \\
&:= \begin{bmatrix}
\mathbf{\Phi}_{11}(i) & \tilde{W}(i)A_h(i) & \tilde{W}(i)D(i) & \tilde{\Phi}^{14}(i) & \tilde{W}(i)B_v(i) \\
A_h^T(i)\tilde{W}(i) & -\tilde{U} & 0 & 0 & 0 \\
D^T(i)\tilde{W}(i) & 0 & -\tilde{\mu}(i)I_{n_p} & 0 & 0 \\
\tilde{\Phi}^{14T}(i) & 0 & 0 & -\tilde{\varepsilon}(i)I_{n_p} & 0 \\
B_v^T(i)\tilde{W}(i) & 0 & 0 & 0 & -\gamma^2 I_{m_v}
\end{bmatrix} < 0,
\end{aligned} \tag{45b}
$$

$$
D_p^T(i)\tilde{W}(i)D_p(i) \leq \tilde{\lambda}(i)I_{n_b}, \tag{45c}
$$

where $i = 1, \dots, s$*,*

$$
\begin{aligned}
\mathbf{\Phi}_{11}(i) := &\tilde{W}(i)\bar{A}^{(n)}(i) + \bar{A}^{(n)T}(i)\tilde{W}(i) + \tilde{\mu}(i)E_a^T(i)E_a(i) \\
&+ H^T(i)H(i) + \sum_{\ell=0}^{N} C_\ell^T F_\ell^T(i)F_\ell(i)C_\ell + \tilde{U} \\
&+ \sum_{j=1}^{s} \pi_{ij}\tilde{W}(j) + A_p^T(i)\tilde{W}(i)A_p(i) + (\tilde{\varepsilon}(i) + \tilde{\lambda}(i))E_{pa}^T(i)E_{pa}(i),
\end{aligned}
$$

$$\tilde{\Phi}^{14}(i) := A_p^T(i)\tilde{W}(i)D_p(i),$$
$$\bar{A}^{(n)}(i) := A(i) + \sum_{\ell=0}^{N} B_\ell(i)F_\ell^{(n+1)}(i)C_\ell(i);$$

Step 5. *For any appropriate value of* $\eta \in (0, 1)$*, set*

$$\boldsymbol{z}^{(n+1)} = (1-\eta)\boldsymbol{z}^{(n)} + \eta\mathscr{T}(\boldsymbol{z}^{(n)}), \tag{46}$$

where

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{P}_0 & \boldsymbol{P}_1 & \cdots & \boldsymbol{P}_N & \boldsymbol{S}_0 & \boldsymbol{S}_1 & \cdots & \boldsymbol{S}_N & \boldsymbol{X}_1 & \cdots & \boldsymbol{X}_N \\ \boldsymbol{T}_1 & \cdots & \boldsymbol{T}_N & \boldsymbol{Z}_1 & \cdots & \boldsymbol{Z}_N & \boldsymbol{W} & \boldsymbol{F}_0 & \boldsymbol{F}_1 & \cdots & \boldsymbol{F}_N \end{bmatrix}.$$

Furthermore, $\mathscr{T}$ *denotes the mapping from Step 2.1 to Step 4 related to the renewable.*

Step 6. *If the iterative algorithm consisting of Steps 2 to 5 converges, we have obtained the iterative solutions; otherwise, if the number of iterations reaches a preset threshold, declare that there is no strategy set. Stop.*

Theorem 2 *If* $\mathscr{T}$ *is a monotone nonexpansive mapping with a fixed point, then* $\{\boldsymbol{z}^{(n)}\}$ *converges weakly to a fixed point.*

Proof: The proof is based on the existing result in [17]. Since this result can be proved by suing the similar discussion on [16], it is omitted.

5 Numerical Example

In order to demonstrate the effectiveness of the proposed method for time-delay stochastic systems, numerical results are presented for the linearized model of a chemical refining process with transport lag [22]. It should be noted that although the state feedback strategy has been solved in [16], the SOF strategy is considered in this paper. It should also be noted that the considered model has been slightly modified. The catalyst feed $u_2(t) = \delta F_B/3V_R$ is split into $\alpha u_2(t)$, $0 < \alpha < 1$ entering the vessel directly and a remaining $(1-\alpha)u_2(t)$ entering the vessel owing to prior mixing with the reactant feed. The system matrices are given as follows.

$$s = 2,\ \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -0.3 & 0.3 \\ 0.7 & -0.7 \end{bmatrix},$$

$$
\begin{aligned}
&A(1) = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ 6.40 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix}, \\
&A_h(1) = \textbf{block diag}\left(\begin{matrix} 1.92 & 1.92 & 1.87 & 1.724 \end{matrix} \right), \\
&B_v(1) = \begin{bmatrix} 0 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix},\ D(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.1 \end{bmatrix}, \\
&A(2) = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ 6.40 & 0.347 & -16.5 & -1.04 \\ 0 & 0.833 & 5.0 & -3.96 \end{bmatrix}, \\
&A_h(2) = \textbf{block diag}\left(\begin{matrix} 1.20 & 1.20 & 1.70 & 1.240 \end{matrix} \right), \\
&B_v(2) = \begin{bmatrix} 0 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix},\ D(2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.2 \end{bmatrix}, \\
&B_0(i) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ B_1(i) = \begin{bmatrix} 0 \\ 0.4 \\ 0 \\ 0 \end{bmatrix},\ B_2(i) = \begin{bmatrix} 0 \\ 1.6 \\ 0 \\ 0 \end{bmatrix}, \\
&A_p(i) = 0.1A(i),\ D_p(i) = 0.1D(i), \\
&E_a(i) = \left[\begin{matrix} 0 & 0 & 0.5 & 0 \end{matrix} \right],\ E_{pa}(i) = 0.1E_a(i), \\
&H(i) = \left[\begin{matrix} 0 & 0 & 0 & 1 \end{matrix} \right], \\
&C_0(i) = C_1(i) = C_2(i) = \left[\begin{matrix} 0 & 1 & 0 & 0 \end{matrix} \right],\ i = 1,\ 2, \\
&h = 1,\ k = 1,\ 2, \\
&\phi(t) = \left[\begin{matrix} 1 & 0.5 & -1 & 2 \end{matrix} \right]^T,\ -1 \le t \le 0, \\
&Q_0(i) = 0.5I_4,\ Q_1(i) = 1.3I_4,\ Q_2(i) = 0.8I_4, \\
&R_0(i) = 1.5,\ R_1(i) = 2.5,\ R_2(i) = 1.8,\ i = 1,\ 2.
\end{aligned}
$$

Next, we select $\gamma = 2.2$. Using the proposed iterative algorithms, the leader's strategy set $u_c(t) = F_c(r_t)C_c(r_t)x(t)$, and the worst case disturbance is computed as follows:

$$
F_0(1) = \left[2.1386 \times 10^{-1}\right],\ F_0(2) = \left[1.7092 \times 10^{-1}\right],
$$

$$
\begin{aligned}
F_1(1) &= \begin{bmatrix} -5.9503 \times 10^{-2} \end{bmatrix}, \ F_1(2) = \begin{bmatrix} -5.5634 \times 10^{-2} \end{bmatrix}, \\
F_2(1) &= \begin{bmatrix} -2.2368 \times 10^{-1} \end{bmatrix}, \ F_2(2) = \begin{bmatrix} -2.0705 \times 10^{-1} \end{bmatrix}, \\
F_\gamma(1) &= \begin{bmatrix} -3.3510 \times 10^{-3} & 2.9249 \times 10^{-3} \\ -8.9851 \times 10^{-4} & 3.0307 \times 10^{-4} \end{bmatrix}, \\
F_\gamma(2) &= \begin{bmatrix} -3.9648 \times 10^{-3} & 3.6591 \times 10^{-3} \\ -1.6243 \times 10^{-3} & 4.2115 \times 10^{-4} \end{bmatrix}.
\end{aligned}
$$

We employ the proposed Krasnoselskii iterative algorithm to obtain the converged solutions and the strategies. In particular, Step 2 is only demonstrated. The initial gains are set as $F_0^{(0)}(i) = F_1^{(0)}(i) = F_2^{(0)}(i) = Z_1^{(0)}(i) = Z_2^{(0)}(i) = \begin{bmatrix} 0.5 \end{bmatrix}$, for $i = 1,\ 2$. The initial condition is selected by the trial and error method such that the closed loop system remains stable. The algorithms converge after six iterations, with an accuracy of 10^{-9}.

In Step 5 of the heuristic algorithms, the value of η is set to 1.0×10^{-4}. The values $\psi^{(n)} := \|\boldsymbol{z}^{(n+2)} - \boldsymbol{z}^{(n+1)}\| / \|\boldsymbol{z}^{(n+1)} - \boldsymbol{z}^{(n)}\|$ in the iterations of the Krasnoselskii iterative algorithm are listed in Table 1, which shows that the proposed algorithm generates a non-increasing sequence for the cost. Therefore, the weak convergence property is satisfied.

Table 1. The sequence value in the iterations.

n	$\psi^{(n)}$
1	1.000
2	9.951×10^{-1}
3	9.980×10^{-1}
4	9.979×10^{-1}
5	9.980×10^{-1}
6	9.978×10^{-1}

6 Conclusion

In this work, the robust multi-player SOF Stackelberg game for the UMJLS-DSs has been studied. The primary challenge is that the considered Stackelberg game involves several uncertain factors, such as time delays, deterministic uncertainties, external disturbances, and information availability in designing strategies. In order to address these difficulties, an integrated application of Markov switching, dynamic game, stochastic control, and robust control is created. As a result, a robust SOF Stackelberg strategy set is obtained by solving the optimization problems with the CCSMIs constraints and the high-order CCSMEs. Moreover, to validate the heuristic algorithm, a computational framework that uses the Krasnoselskii iterative algorithm [17] to compute the solution set of the CCSMEs is proposed. The novel convergence condition combined with the Krasnoselskii iteration is introduced to achieve weak convergence of the algorithm. Finally, a modified numerical example from [22] was solved to demonstrate the effectiveness of the proposed algorithm.

References

[1] H. Mukaidani, S. Ramasamy, H. Xu, and W. Zhuang, Robust Stackelberg games via static output feedback strategy for uncertain stochastic systems with state delay, in *Proc. 21rst IFAC World Congress*, Berlin, Germany, July 2020.

[2] X. Mao, A. Matasov, and A. B. Piunovskiy, Stochastic differential delay equations with Markovian switching, *Bernoulli*, vol. 6, no. 1, pp. 73–90, 2000.

[3] S. Sathananthana, O. Adetonab, C. Beanea, and L. H. Keela, Feedback stabilization of Markov jump linear systems with time-varying delay, *Stochastic Analysis and Applications*, vol.26, no.3, pp. 577–594, 2008.

[4] H. Chen, C. Zhu, P. Hu, and Y. Zhang, Delayed-state-feedback exponential stabilization for uncertain Markovian jump systems with mode-dependent time-varying state delays, *Nonlinear Dynamics*, vol. 69, no. 3, pp. 1023–1039, 2012.

[5] R. Saravanakumar, M. S. Ali, and H. R. Karimi, Robust H_∞ control of uncertain stochastic Markovian jump systems with mixed time-varying delays, *Int. J. Systems Science*, vol. 48, no. 4, pp. 862–872, 2017.

[6] L. Qiu L, Y. Shi, B. Xu, and F. Yao, Robust stochastic stability and delayed-state-feedback stabilisation of uncertain Markovian jump linear systems with random time delays, *IET Control Theory & Applications*, vol. 9, no. 12, pp. 1878–1886, 2015.

[7] H. Mukaidani, M. Unno, T. Yamamoto, and H. Xu, Nash Strategy for Markov jump stochastic delay systems, in *Proc. 52nd IEEE Conf. Decision and Control*, Florence, Italy, pp. 1198–1203, Dec. 2013.

[8] M. Dolgov and U. D. Hanebeck, Static output-feedback control of Markov jump linear systems without mode observation, *IEEE Trans. Automatic Control*, vol. 62, no. 10, pp. 5401–5406, 2017.

[9] H. Mukaidani, H. Xu, and V. Dragan, Static output feedback Stackelberg strategy of infinite horizon Markov jump linear stochastic systems with H_∞ constraint. in *Proc. 57th IEEE Conf. Decision and Control*, Miami Beach, FL, USA, pp. 1935–1940, Dec. 2018.

[10] V. D. Blondel and J. N. Tsitsiklis, A survey of computational complexityresults in systems and control, *Automatica*, vol. 36, no.9, pp. 1249-1274, 2000.

[11] M. Fu and Z. Q. Luo, Computational complexity of a problem arising from the fixed order output feedback design, *Systems and Control Letters*, vol. 30, no. 5, pp. 209–215, 1997.

[12] S. O. R.Moheimani and I. R. Petersen, Optimal guaranteed cost control of uncertain systems via static and dynamic output feedback, *Automatica*, vol. 32, no. 4, pp. 575–579, 1996.

[13] J. Gadewadikar, F. L. Lewis, and M. Abu-Khalaf, Necessary and sufficient conditions for H_∞ static output-feedback control, *J. Guidance, Control, and Dynamics*, vol. 29, no. 4, pp. 915–920, 2006.

[14] E. -K.Boukas, *Stochastic Switching Systems: Analysis and Design.* Boston, MA, USA: Birkhäuser, 2006.

[15] H. Mukaidani, S. Ramasamy, H. Xu, and W. Zhuang, Robust Nash static output feedback strategy for uncertain Markov jump delay stochastic systems, in *Proc. 58th IEEE Conf. Decision and Control*, Nice, France, pp. 5826–5831, Dec. 2019.

[16] H. Mukaidani, S. Ramasamy, H. Xu and W. Zhuang, Stackelberg strategy for uncertain Markov jump delay stochastic systems, *IEEE Control Systems Letters*, vol. 4, no. 4, pp. 1006–1011, 2020.

[17] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, *J. Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.

[18] Z. Wang, H. Qiao, and K. J. Burnham, On stabilization of bilinear uncertain time-delay stochastic systems with Markovian jumping parameters, *IEEE Trans. Automatic Control*, vol. 47, no. 4, pp. 640–646, 2002.

[19] Y. -Y. Cao and J. Lam, Robust H_∞ control of uncertain Markovian jump systems with time-delay, *IEEE Trans. Automatic Control*, vol. 45, no. 1, pp. 77–83, 2000.

[20] M. S. Mahmoud and P. Shi, Robust stability, stabilization and H_∞ control of time-delay systems with Markovian jump parameters, *Int. J. Robust and Nonlinear Control*, vol. 13, no. 8, pp. 755–784, 2003.

[21] T. Senthilkumar and P. Balasubramaniam, Delay-dependent robust stabilization and H_∞ control for nonlinear stochastic systems with Markovian jump parameters and interval time-varying delays, *J. Optimization Theory and Applications*, vol. 151, no. 1, pp. 100–120, 2011.

[22] D. W. Ross, Controller design for time lag systems via a quadratic criterion, *IEEE Trans. Automatic Control*, vol. 16, no. 6, pp. 664-672, 1971.