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STATIC OUTPUT FEEDBACK DESIGN IN AN ANISOTROPIC NORM SETUP*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70th anniversary

Abstract

The design of static output feedback controllers in an anisotropic norm setup is considered. The aim is to determine a stabilizing static output feedback for a given four block system such that the resulting closed loop system has the a-anisotropic norm less than a given $\gamma > 0$. The solvability conditions are expressed in terms of the solution of a rank minimization problem with linear matrix inequalities constraints. Based on the specific form of these constraints it is shown that a solution of this problem may be obtained solving a semidefinite programming problem.
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1 Introduction

The problems of robust optimal control and filtering received much attention over the last seven decades. Early solutions for these problems were presented by Kwakernaak and Sivan [11] and robustness issues due to modelling errors were considered in e.g. [18]. When the exogenous signals are of white noise type, then H_2-norm minimization is applied, leading to the Kalman filter [7] and Linear Quadratic Gaussian (LQG) control. An alternative modelling of the exogenous inputs is based on deterministic bounded energy signals. Such formulations lead to the H_∞-norm based framework ([23]) and are applied in both filtering ([6], [17]) and control ([24]). Many practical applications, however, require a compromise between the H_2 and the H_∞-norm minimization since the latter may not be suitable when the considered signals are strongly coloured (e.g. periodic signals). On the other hand, H_∞-optimization may poorly perform when these signals are weakly coloured (e.g. white noise). For such cases mixed H_2/H_∞ norm minimization problems have been formulated and analysed (see, e.g. [2], [5], [14]). A promising alternative to accomplish such compromise is to use the so-called *a-anisotropic norm* ([8], [21]) since it offers and intermediate topology between the H_2 and H_∞ norms. More precisely, consider the m dimensional coloured signal $w(t), t = 0, 1, ...$ generated by the discrete-time stable filter G

$$\begin{aligned} x_f(t+1) &= A_f x_f(t) + B_f v(t) \\ w(t) &= C_f x_f(t) + D_f v(t), t = 0, 1, ... \end{aligned} \tag{1}$$

where $A_f \in \mathcal{R}^{n_f \times n_f}$, $B_f \in \mathcal{R}^{n_f \times m}$, $C_f \in \mathcal{R}^{m \times n_f}$, $D_f \in \mathcal{R}^{m \times m}$ and where $v \in \mathcal{R}$ are independent Gaussian white noises with $E[v(t)] = 0$ and $E[v(t)v^T(t)] = I_m$. Then, the a-anisotropic norm $|||F|||_a$ of a discrete-time stable system F with the state-space realisation

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), t = 0, 1, ... \end{aligned} \tag{2}$$

is defined as

$$|||F|||_a = \sup_{G \in \mathcal{G}_a} \frac{\|FG\|_2}{\|G\|_2}, \tag{3}$$

$\mathcal{G}_a$ denoting the set of all stochastic systems of form (1) with the *mean anisotropy* $\bar{A}(G) \leq a$. The mean anisotropy of stationary Gaussian sequences was introduced in [8] and it represents an entropy theoretic measure of the deviation of a probability distribution from Gaussian distributions with zero mean and scalar covariance matrices. In [9], it is proved based on the Szegö-Kolmogorov theorem ([15]) that the mean anisotropy of a signal generated by an m-dimensional Gaussian white noise $v(t)$ with zero mean and identity covariance applied to a stable linear system G with m outputs has the form

$$\bar{A}(G) = -\frac{1}{2} \ln \det \left(\frac{mE\left[\tilde{w}(0)\tilde{w}(0)^T\right]}{Tr\left(E\left[w(0)w(0)^T\right]\right)} \right), \tag{4}$$

where $E[\tilde{w}(0)\tilde{w}(0)^T]$ is the covariance of the prediction error $\tilde{w}(0) := w(0) - E[w(0)|(w(k), k < 0]$. In the case when the output w of the filter G is a zero mean Gaussian white noise (i.e. its optimal estimate is just zero), $w(0)$ cannot be estimated from its past values and $\tilde{w}(0) = w(0)$ which leads to $\bar{A}(G) = 0$.

It is proved (see, for instance [21]) that the anisotropic norm has the property:

$$\frac{1}{\sqrt{m}}||F||_2 = |||F|||_0 \leq |||F|||_a \leq ||F||_\infty = \lim_{a \to \infty} |||F|||_a \tag{5}$$

In [9], conditions for the anisotropic norm boundedness are given in terms of a non convex optimization problem while in [10] a convex form of the Bounded Real Lemma (BRL) type result with respect to the anisotropic norm was obtained. One of the leading motivations to use the anisotropic norm is the fact $|||F|||_a \leq ||F||_\infty$ making it a relaxed version of the H_∞-norm for many practical cases in which the driving noise signals can be characterised not just by their finite energy, but as outputs of a colouring linear systems in a certain class, where the colouring filters are of a finite anisotropy. In a case study presented in [20] it is shown that for a TU-154 type aircraft landing system, the H_∞ controller is more efficient than the corresponding H_2 controller for a windshear profile (which is a coloured rather than a white noise process) but, as could be expected, is more conservative, in the sense of higher gains and subsequently larger control actions; moreover, the anisotropic-norm based controller (based on an appropriate anisotropic norm bound) is less conservative than the H_∞ controller and requires significantly smaller control actions.

The aim of the present paper is to derive a tractable characterisation of the static output feedback synthesis problem. Static Output Feedback (SOF) synthesis is very useful, when the design of fixed structured controllers such as PID (Proportional, Integral, Derivative) is required. Such controllers are very common both in the process control and aerospace control applications. However, it is well known that SOF synthesis is an NP-hard problem. Nevertheless, under the H_2 and H_∞ setups, the non-smooth optimization based software packages *hinfstruct* [1] and HIFOO [4] provide efficient solutions. Although the SOF problem within the anisotropic-norm setup has already been considered in [20], the solution there involves a couple of Linear Matrix Inequalities (LMI) with a non-convex coupling condition, limiting its application to some practical problems.

We next state in Section 2 the problem and provide some preliminaries. An approximate solution is suggested in Section 3, whereas a solution based on solving a rank minimization problem is offered in Section 4. It is shown that based on the particularities of the constraints arising in this rank minimization problem, it may solved using a semidefinite programming problem.

2 Problem Formulation and Some Useful Known Results

Consider the following four blocks plant

$$\begin{array}{rcl} x(t+1) & = & Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) & = & C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) & = & C_2 x(t), t = 0, 1, ... \end{array} \tag{6}$$

The state output feedback (SOF) problem considered in the present paper consists in determining a static control matrix K, such that the closed loop system obtained by taking $u(t) = Ky(t)$, namely

$$\begin{array}{rcl} x(t+1) & = & (A + B_2 K C_2)\, x(t) + B_1 w(t) \\ z(t) & = & (C_1 + D_{12} K C_2)\, x(t) + D_{11} w(t), t = 0, 1, ... \end{array} \tag{7}$$

is stable and its a-anisotropic norm is less than a given $\gamma > 0$. The solution of this problem will be detailed in Section 4. In the following, some known results concerning the used norms are briefly reminded.

Definition 1. The H_2-type norm of the discrete-time stable system (2) is defined as

$$\|F\|_2 = \left[\lim_{\ell \to \infty} \frac{1}{\ell} \sum_{t=0}^{\ell} E\left[y^T(t) y(t) \right] \right]^{\frac{1}{2}},$$

where $\{y(t)\}_{t\in\mathcal{Z}_+}$ with $\mathcal{Z}_+$ denoting the set of all positive integers, is the output of the system (1) with zero initial conditions generated by the sequence $\{w(t)\}_{t\in\mathcal{Z}_+}$ of independent random vectors with the property that $E\left[w(t)\right]=0$ and $E\left[w(t)w^T(t)\right]=I_m$.

The next result provides a method to compute the H_2 norm of the system of (2)(see e.g. [16]).

Lemma 1. The H_2 type norm of the stable system (2) is given by $\|F\|_2 = \left(Tr\left(B^TXB+D^TD\right)\right)^{\frac{1}{2}}$ where $X\geq 0$ is the solution of the generalised Lyapunov equation $X=A^TXA+C^TC$.

Definition 2. The H_∞ norm of the stable discrete-time system (2) is defined as

$$\|F\|_\infty = \sup_{\theta\in[0,2\pi)} \lambda_{\max}^{\frac{1}{2}}\left(F^T\left(e^{-j\theta}\right)F\left(e^{j\theta}\right)\right),$$

where $\lambda_{\max}$ denotes the maximal eigenvalue and $F(\cdot)$ is the transfer function of the system.

The H_∞ norm is characterised by the following result, well-known as the Bounded Real Lemma (BRL).

Lemma 2. The stable system (2) has the norm $\|F\|_\infty<\gamma$ for a certain $\gamma>0$ if and only if the Riccati equation

$$\begin{aligned} P = {} & A^TPA+\left(A^TPB+C^TD\right)\left(\gamma^2I-B^TPB-D^TD\right)^{-1} \\ & \times\left(A^TPB+C^TD\right)^T+C^TC \end{aligned}$$

has a stabilizing solution $P\geq 0$ such that $\gamma^2I-B^TPB-D^TD>0$.

It is recalled ([5]) that a symmetric solution P of the above Riccati equation is called a *stabilising solution* if the system $x(t+1)=\left(A+B\tilde{K}\right)x(t)$ is stable, where by definition

$$\tilde{K} := \left(\gamma^2I-B^TPB-D^TD\right)^{-1}\left(A^TPB+C^TD\right)^T.$$

To conclude this section, we state the Bounded Real LemmaL-like result to characterise the anisotropic norm [9]. Note that for a that tends to infinity, that the result of Lemma 2 is recovered.

Theorem 3. The stable system of (2) satisfies $|||F|||_a\leq\gamma$ for a given $\gamma>0$ if and only if there exists $q\in\left(0,\min\left(\gamma^{-2},\quad\|F\|_\infty^{-2}\right)\right)$ such that the Riccati equation

$$\begin{aligned} X = {} & A^TXA+\left(A^TXB+C^TD\right)\left(\tfrac{1}{q}I-B^TXB-D^TD\right)^{-1} \\ & \times\left(A^TXB+C^TD\right)^T+C^TC \end{aligned} \tag{8}$$

has a stabilising solution $X \geq 0$ satisfying the following conditions

$$\Psi_q := \frac{1}{q} I - B^T X B - D^T D > 0 \tag{9}$$

and

$$\det\left(\frac{1}{q} - \gamma^2\right) \Psi_q^{-1} \leq e^{-2a}. \tag{10}$$

Remark 1. An equivalent formulation of the above result can be obtained replacing the Riccati equation (8) with the linear matrix inequality (LMI)

$$\left[\begin{array}{cc} -X + A^T X A + C^T C & A^T X B + C^T D \\ B^T X A + D^T C & -\frac{1}{q} I + B^T X B + D^T D \end{array}\right] < 0 \tag{11}$$

with $X > 0$.

3 Approximate Characterisation of the Anisotropic Norm

Denoting $\eta = \sqrt{1/q}$ we first restate the result of Theorem 3 above, as

$$||F||_\infty < \eta$$

so that

$$det(\eta^2 - \gamma^2)\Psi_q^{-1} \leq e^{-2a}$$

or equivalently

$$\eta^2 - \gamma^2 \leq (det\Psi_q)^{1/m} e^{-2a/m}$$

However, it is well known that for positive semidefinite matrices the following inequality holds (see e.g. [3])

$$(det\Psi_q)^{1/m} \leq \frac{Tr\Psi_q}{m}$$

Combining those two inequalities, we readily obtain

$$\eta^2 - \gamma^2 \leq \frac{Tr\Psi_q}{m} e^{-2a/m}$$

Noting now that $\Psi_q = \eta^2 I - B^T X B - D^T D$ the latter inequality becomes

$$\gamma^2 \geq \eta^2 - \left(\eta^2 - \frac{Tr(B^T X B + D^T D)}{m}\right) e^{-2a/m}.$$

After arranging terms, we obtain

$$\gamma^2 \geq \eta^2(1 - e^{-2a/m}) + \frac{Tr(B^T XB + D^T D)}{m} e^{-2a/m}$$

Noting that X of Theorem 3 provides and upper bound on the corresponding X of Lemma 1, and also noting that η is an upper bound on the H_∞-norm of the system of (1), we readily obtain the following result.

Lemma 4. Consider the system F given by (2). Let η and σ respectively satisfy

$$||F||_\infty < \eta \text{ and } ||F||_2 < \sigma$$

The a-anisotropic norm of the system of (2) is then upper bounded by the following linear interpolation between its H_∞ and H_2 norms. Namely,

$$\gamma^2 \geq \eta^2(1 - e^{-2a/m}) + \frac{\sigma^2}{m} e^{-2a/m}$$

In view of (5) one may interpret the a-anisotropic norm, the following approximate relation

$$|||F|||_a^2 \approx |||F|||_\infty^2(1 - e^{-2a/m}) + |||F|||_0^2 e^{-2a/m}$$

Thus Lemma 4 provides a useful insight to the a-anisotropic norm, which can lead to using mixed H_2/H_∞ optimisation, in the exact proportions dictated by the Lemma.

4 Static Output Feedback

We next provide a solution to the problem of synthesis of SOF control synthesis under the anisotropic norm, using iterative solution for LMIs. To this end, we define the cost function to be

$$J(K) = |||F_{c\ell}(K)|||_a \tag{12}$$

where $F_{c\ell}(K)$ is given by (7). Using Theorem 3 and Remark 1, if follows that the closed loop system $F_{c\ell}$ is stable and it has the a-anisotropic norm less than a given $\gamma > 0$ if and only if there exist a $q \in \left(0, \min(\gamma^{-2}, \|F_{c\ell}\|_\infty^{-2})\right)$ and a symmetric matrix $X > 0$ such that

$$\begin{bmatrix} \mathcal{E}_1(X,K) & \mathcal{E}_2(X,K) \\ (1,2)^T & -\frac{1}{q}I + B_1^T XB_1 + D_{11}^T D_{11} \end{bmatrix} < 0 \tag{13}$$

where

$$\mathcal{E}_1(X,K) := -X + (A+B_2KC_2)^T X (A+B_2KC_2) + (C_1+D_{12}KC_2)^T (C_1+D_{12}KC_2)$$

$$\mathcal{E}_2(X,K) := (A+B_2KC_2)^T XB_1 + (C_1+D_{12}KC_2)^T D_{11}$$

and

$$\frac{1}{q} - \gamma^2 < e^{-\frac{2a}{m}} \left(\det\left(\frac{1}{q}I - B_1^T XB_1 - D_{11}^T D_{11}\right)\right)^{1/m} \tag{14}$$

Based on Schur complements arguments, one can see that the inequality (13) is equivalent with

$$\begin{bmatrix} -X & 0 & (A+B_2KC_2)^T & (C_1+D_{12}KC_2)^T \\ 0 & -\frac{1}{q}I & B_1^T & D_{11}^T \\ A+B_2KC_2 & B_1 & -X^{-1} & 0 \\ C_1+D_{12}KC_2 & D_{11} & 0 & -I \end{bmatrix} < 0.$$

Multiplying the above inequality to the left and to the right by $diag(I,I,X,I)$ one obtains that is is equivalent with

$$\begin{bmatrix} -X & 0 & (A+B_2KC_2)^T X & (C_1+D_{12}KC_2)^T \\ 0 & -\frac{1}{q}I & B_1^T X & D_{11}^T \\ X(A+B_2KC_2) & XB_1 & -X & 0 \\ C_1+D_{12}KC_2 & D_{11} & 0 & -I \end{bmatrix} < 0 \tag{15}$$

which may be re-written as

$$\mathcal{Z} + \mathcal{P}^T K\mathcal{Q} + \mathcal{Q}^T K^T \mathcal{P} < 0, \tag{16}$$

where one denoted

$$\mathcal{Z} := \begin{bmatrix} -X & 0 & A^T X & C_1^T \\ 0 & -\frac{1}{q}I & B_1^T X & D_{11}^T \\ XA & XB_1 & -X & 0 \\ C_1 & D_{11} & 0 & -I \end{bmatrix}, \ \mathcal{P}^T := \begin{bmatrix} 0 \\ 0 \\ XB_2 \\ D_{12} \end{bmatrix},$$

$$\mathcal{Q} := \begin{bmatrix} C_2 & 0 & 0 & 0 \end{bmatrix}. \tag{17}$$

According with the so-called Projection lemma (see e.g. [16]), the inequality (16) is feasible with respect to K if and only if the following conditions are accomplished

$$W_{\mathcal{P}}^T Z W_{\mathcal{P}} < 0 \tag{18}$$

and

$$W_{\mathcal{Q}}^T Z W_{\mathcal{Q}} < 0 \tag{19}$$

where $W_{\mathcal{P}}$ and $W_{\mathcal{Q}}$ are any bases of the null spaces of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Since a base of the null space of $\mathcal{P}$ is

$$W_{\mathcal{P}} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X^{-1}\left(B_2^T\right)^{\perp} & 0 \\ 0 & 0 & 0 & \left(D_{12}^T\right)^{\perp} \end{bmatrix},$$

where $(\cdot)^{\perp}$ denotes the null space of the matrix $(\cdot)$, one directly obtains from condition (18)

$$\begin{bmatrix} -X & 0 & A^T\left(B_2^{\perp}\right)^T & C_1^T\left(D_{12}^{\perp}\right)^T \\ 0 & -\frac{1}{q}I & B_1^T\left(B_2^{\perp}\right)^T & D_{11}^T\left(D_{12}^{\perp}\right)^T \\ B_2^{\perp}A & B_2^{\perp}B_1 & -B_2^{\perp}X^{-1}\left(B_2^{\perp}\right)^T & 0 \\ D_{12}^{\perp}C_1 & D_{12}^{\perp}D_{11} & 0 & -D_{12}^{\perp}\left(D_{12}^{\perp}\right)^T \end{bmatrix} < 0. \tag{20}$$

Further, taking into account that $W_{\mathcal{Q}} = diag\left(C_2^{\perp}, I, I, I\right)$, the condition (19) becomes

$$\begin{bmatrix} -\left(C_2^{\perp}\right)^T X C_2^{\perp} & 0 & \left(C_2^{\perp}\right)^T A^T X & \left(C_2^{\perp}\right)^T C_1^T \\ 0 & -\frac{1}{q}I & B_1^T X & D_{11}^T \\ XAC_2^{\perp} & XB_1 & -X & 0 \\ C_1C_2^{\perp} & D_{11} & 0 & -I \end{bmatrix} < 0. \tag{21}$$

Since the inequalities (20) and (21) depends both on $X > 0$ and its inverse X^{-1}, one obtains the rank minimization problem

$$\min_{X>0,\,Y>0} \operatorname{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \tag{22}$$

with the constraints (20) and (21), where in (20), X^{-1} is denoted by Y.

The next result proved in [13] is useful to show that the above rank minimization problem may be solved using a semidefinite programming problem (SDP).

Proposition 1. Let $D \in \mathcal{R}^{m\times n}(m \leq n)$ be a full row rank, $Q \leq 0$ and C invertible. Then the RMP

$$\min_{X \geq 0} \operatorname{rank} X$$

with the constraint

$$D\left(Q+CXC^T-\sum_i M_iXM_i^T\right)D^T\geq 0 \tag{23}$$

can be solved as the semidefinite programming problem

$$\min_{X\geq 0}\operatorname{trace} X$$

with the same constraint (23).

After swapping the second and the third columns and rows of the matrix from the left hand side of (20), using Schur complements arguments, it results that (20) is equivalent with

$$\begin{array}{l} -\begin{bmatrix} I & 0 \\ 0 & B_2^{\perp} \end{bmatrix}\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & B_2^{\perp} \end{bmatrix}^T + P \\ +\begin{bmatrix} 0 & \left(B_2^{\perp}\right)^T\left(I+A^T\right) \\ (I+A)B_2^{\perp} & 0 \end{bmatrix} < 0 \end{array}, \tag{24}$$

where one denoted

$$\begin{array}{l} P := \begin{bmatrix} 0 & C_1^T\left(D_{12}^{\perp}\right)^T \\ B_2^{\perp}B_1 & D_{11}\left(D_{12}^{\perp}\right)^T \end{bmatrix}\begin{bmatrix} \frac{1}{q}I & 0 \\ 0 & D_{12}^{\perp}\left(D_{12}^{\perp}\right)^T \end{bmatrix}^{-1} \\ \cdot\begin{bmatrix} 0 & B_1^T\left(B_2^{\perp}\right)^T \\ D_{12}^{\perp}C_1 & D_{12}^{\perp}D_{11} \end{bmatrix} \geq 0. \end{array}$$

Since

$$\begin{bmatrix} \mu I & \left(B_2^{\perp}\right)^T\left(I+A^T\right) \\ (I+A)B_2^{\perp} & \mu I \end{bmatrix} > \begin{bmatrix} 0 & \left(B_2^{\perp}\right)^T\left(I+A^T\right) \\ (I+A)B_2^{\perp} & 0 \end{bmatrix},$$

$\forall\mu>0$, one may replace the last term from the left hand side of (24) by

$$\begin{bmatrix} \mu I & \left(B_2^{\perp}\right)^T\left(I+A^T\right) \\ (I+A)B_2^{\perp} & \mu I \end{bmatrix} > 0$$

for a certain $\mu>0$, obtaining thus

$$-\begin{bmatrix} I & 0 \\ 0 & B_2^{\perp} \end{bmatrix}\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & B_2^{\perp} \end{bmatrix}^T + \hat{P} < 0, \tag{25}$$

with $\hat{P} > 0$, where

$$\hat{P} := P + \begin{bmatrix} \mu I & \left(B_2^{\perp}\right)^T \left(I + A^T\right) \\ (I + A)B_2^{\perp} & \mu I \end{bmatrix} > 0.$$

Denote $\tilde{B}_2 \in \mathcal{R}^{\ell \times n}$ a full rank matrix such that $\begin{bmatrix} B_2^{\perp} \\ \tilde{B}_2 \end{bmatrix}$ is square and invertible. Then (25) may be written as

$$\begin{bmatrix} I_n & 0_{n\times\ell} \end{bmatrix} \left\{ - \begin{bmatrix} I_n & 0_n \\ 0_n & \begin{bmatrix} B_2^{\perp} \\ B_2^{*} \end{bmatrix} \end{bmatrix} \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \begin{bmatrix} I_n & 0_n \\ 0_n & \begin{bmatrix} B_2^{\perp} \\ B_2^{*} \end{bmatrix} \end{bmatrix}^T + \begin{bmatrix} \hat{P} & 0 \\ 0 & P^{*} \end{bmatrix} \right\} \begin{bmatrix} I_n \\ 0_{\ell\times n} \end{bmatrix} < 0 \qquad (26)$$

where $P^{*} \in \mathcal{R}^{\ell\times\ell}$ is an arbitrary positive definite matrix. The above condition coincides with the form (23) for $D := \begin{bmatrix} I_n & 0_{n\times\ell} \end{bmatrix}$, $X \leftarrow \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix}$, $Q = -\begin{bmatrix} P & 0 \\ 0 & P^{*} \end{bmatrix}$, $C = \begin{bmatrix} I_n & 0_n \\ 0_n & \begin{bmatrix} B_2^{\perp} \\ B_2^{*} \end{bmatrix} \end{bmatrix}$ and $M_i = 0$.

A similar reasoning may be easily used to show that (21) may be represented under (23) form after writing (21) in the equivalent form

$$\begin{bmatrix} -\left(C_2^{\perp}\right)^T X C_2^{\perp} & 0 & \left(C_2^{\perp}\right)^T A^T & \left(C_2^{\perp}\right)^T C_1^T \\ 0 & -\frac{1}{q} I & B_1^T & D_{11}^T \\ A C_2^{\perp} & B_1 & -X^{-1} & 0 \\ C_1 C_2^{\perp} & D_{11} & 0 & -I \end{bmatrix} < 0, \qquad (27)$$

obtained multiplying (21) to the left and to the right by $diag\left(I,\, I,\, X^{-1}, I\right)$.

Based on the above developments, the following result provides a solution of the static output feedback problem with bounded a-anisotropic norm.

Theorem 5. Assume that for a certain $q > 0$, $X > 0, Y > 0$ satisfy (20) and (27) in which X^{-1} is replaced by Y, so that rank $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} = n$. Then K obtained as a solution of the linear matrix inequality (16), provides a solution to the static output feedback problem for which the resulting closed loop system $F_{c\ell}$ defined by (7) has an a-anisotropic norm less than γ >0 for any γ satisfying the conditions (14) and $q \in (0, \min\{\gamma^{-2}, \|F_{c\ell}\|_{\infty}^{-2}\})$.

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