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APPROXIMATING FIXED POINTS OF ALMOST CONVEX CONTRACTIONS IN METRIC SPACES*

Vasile Berinde†

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70^{th} anniversary

Abstract

In this paper we introduce a new class of mappings, obtained by merging the concepts of *almost contraction* and *convex contraction* and called *almost convex contractions*. This general class includes both the *almost contractions* and *convex contractions*. Existence fixed point theorems for almost convex contractions of order 2 and of order p are established.
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1 Introduction

In a series of papers published in the period 1981-1983, Istrăţescu [37]-[39] introduced and studied the concept of *convex contraction*. Except for a few echoes ([36], [50]) this concept did not generate very much interest in

*Accepted for publication on March 18, 2020
†vberinde@cunbm.utcluj.ro Department of Mathematics and Computer Science, North University Center at Baia Mare, Technical University of Cluj-Napoca, Victoriei 76, 430122 Baia Mare, Romania

the field but, recently, after more than 30 years from the date of its first publication, it suddenly attracted several authors, see [3]-[9], [34], [35], [42], [43], [44], [45], [60], to mention just a partial list.

On the other hand, the author introduced in 2004 [17] the class of *weak contractions*, later called *almost contractions*, a kind of contractive mappings that immediately attracted much interest amongst researchers in the filed (there are more than 411 papers that cite [17], according to Google Scholar, and more than 187 citing papers, according to Web of Science), see [1]- [2], [5], [10], [18]-[25], [29], [30], [32], [33], [59]-[64] etc. For a recent survey on this topic, we refer to the book chapter [27].

Having in view the fact that the two concepts mentioned above are essentially different, the main aim of this note is to show that we could merge them to obtain a new class of mappings, namely, the class of *almost convex contractions* that includes both *almost contractions* and *convex contractions* as particular cases.

Some fixed point theorems are presented for the class of almost convex contractions.

2 Almost contractions

One of the most useful results in nonlinear analysis, which, together with its local variants and generalisations, has many applications in solving nonlinear functional equations, optimization, variational inequalities etc., is the *Banach contraction mapping principle*, which can be briefly stated as follows.

Theorem B. *Let (X, d) be a complete metric space and $T : X \to X$ a strict contraction, i.e., a map satisfying*

$$d(Tx, Ty) \leq a \cdot d(x, y), \quad \forall\, x, y \in X, \tag{1}$$

where $0 < a < 1$ is a constant. Then T is a Picard operator (that is, T has a unique fixed point in X, say x^*, and Picard iteration $\{T^n x_0\}$ converges to x^* for all $x_0 \in X$).

It is easy to see that any contraction mapping satisfying (1) is continuous. Kannan [40] in 1968 has proved a fixed point theorem which extends Theorem B to mappings that need not be continuous on X (but are continuous *at* their fixed point, see [54]), by considering instead of (1) the next contractive condition: there exists a constant $b \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \quad \text{for all } \; x, y \in X\,. \tag{2}$$

Following the Kannan's theorem, a lot of papers were devoted to obtaining fixed point or common fixed point theorems for various classes of contractive type conditions that do not require the continuity of T, see, for example, [55], [56], [18] and references therein.

One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [28], is based on a condition similar to (2): there exists a constant $c \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big], \qquad \text{for all } \ x, y \in X. \tag{3}$$

On the other hand, in 1972, Zamfirescu [65] obtained a very interesting fixed point theorem which gather together all three contractive conditions mentioned above, i.e., condition (1) of Banach, condition (2) of Kannan and condition (3) of Chatterjea, in a rather unexpected way: if T is such that, for any pair $x, y \in X$, at least one of the conditions (1), (2) and (3) holds, then T is a Picard operator. Note that considering conditions (1), (2) and (3) all together is not trivial since, as shown later by Rhoades [53], the contractive conditions (1), (2) and (3), are independent to each other.

A more general result has been obtained by Ćirić [31], who considered the quasi-contraction condition:

$$d(Tx, Ty) \leq \lambda \max\left\{d(x, y),\, d(x, Tx),\, d(y, Ty),\, d(x, Ty),\, d(y, Tx)\right\}, \tag{4}$$

for all $x, y \in K$, where $0 < \lambda < 1$.

Zamfirescu's fixed point theorem [65] is a particular case of the next fixed point theorem [17] established for almost contractions, see also the papers [15], [16] and [18].

Theorem 1 ([17], Theorem 2.1) *Let (X, d) be a complete metric space and $T : X \to X$ an almost contraction, that is, a mapping for which there exist a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx)\,, \qquad \text{for all } \ x, y \in X. \tag{5}$$

Then

1) $Fix\,(T) = \{x \in X : Tx = x\} \neq \emptyset$*;*

2) *For any $x_0 \in X$, Picard iteration $\{x_n\}_{n=0}^{\infty}$, $x_n = T^n x_0$, converges to some $x^* \in Fix\,(T)$;*

3) *The following estimate holds*

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots;\, i = 1, 2, \ldots \tag{6}$$

Let us recall, see [56], that a mapping T possessing properties 1) and 2) above is called a *weakly Picard operator*.

Notice also that while any quasi-contraction is a Picard operator (that is, it has a *unique* fixed point), an almost contraction is a weakly Picard operator, i.e., it does not have a unique fixed point, in general, as shown by the next Example.

Example 1 Let $X = [0,1]$ be the unit interval with the usual norm and let $T : [0,1] \to [0,1]$ be given by $Tx = \frac{1}{2}$ for $x \in [0,2/3)$ and $Tx = 1$, for $x \in [2/3,1]$.

As T has two fixed points, that is, $Fix\,(T) = \left\{\frac{1}{2}\,, 1\right\}$, it does not satisfy neither Ćirić's condition (4), nor Banach, Kannan, Chatterjea, Zamfirescu or Ćirić [31] contractive conditions, but T satisfies the contraction condition (5).

Indeed, for $x, y \in [0,2/3)$ or $x, y \in [2/3,1]$, (5) is obvious. For $x \in [0,2/3)$ and $y \in [2/3,1]$ or $y \in [0,2/3)$ and $x \in [2/3,1]$ we have $d(Tx,Ty) = 1/2$ and $d(y,Tx) = |y - 1/2| \in [1/6,1/2]$, in the first case, and $d(y,Tx) = |y - 1| \in [1/3,1]$, in the second case, which show that it suffices to take $L = 3$ in order to ensure that (5) holds for $0 < \delta < 1$ arbitrary and all $x, y \in X$.

There exist many recent developments on almost contractions (also called weak contractions or Berinde operators), see [1]- [2], [18]-[27], [32], [33], [59]-[64], to mention just a few papers devoted to this topic.

3 Convex contractions

Definition 1 ([37]) *Let (X,d) be a metric space. A self map $T : X \to X$ is called a* convex contraction *if*

$$d(T^2x, T^2y) \le a \cdot d(Tx,Ty) + b \cdot d(x,y), \forall x, y \in X, \tag{7}$$

where a, b are constants satisfying $0 < a, b < 1$ and $a + b < 1$.

Example 2 *If $b = 0$, then by the convex contraction condition (8) we obtain the Banach contraction condition 1:*

$$d(Tx,Ty) \le a \cdot d(x,y), \forall x, y \in X,$$

subject to a change of notation.

If $a = 0$, then by the convex contraction condition (8), we obtain the well known "asymptotic" contraction condition:

$$d(T^2x, T^2y) \leq b \cdot d(x, y),$$

that ensures the existence of a fixed point (even in the case when 2 is replaced by a given integer n).

Example 3 ([37])

Let $X = [0, 1]$ with the usual metric and let $T : [0, 1] \to [0, 1]$ be defined by

$$Tx = \frac{x^2 + 1/2}{2}, x \in [0, 1].$$

Then T is not a Banach contraction, although $Fix\,(T) = \{0\}$.

But T is a convex contraction, as we have

$$\left|f^2(x) - f^2(y)\right| \leq \frac{1}{2}\left|f(x) - f(y)\right| + \frac{1}{4}\left|x - y\right|, x, y \in [0, 1],$$

with $a = \frac{1}{2}$ and $b = \frac{1}{4}$.

The first main result in [37] is the following fixed point theorem.

Theorem 2 ([37]) *Let (X, d) be a complete metric space and $T : X \to X$ a continuous (a, b)-convex contraction, i.e., a mapping satisfying*

$$d(T^2x, T^2y) \leq a \cdot d(Tx, Ty) + b \cdot d(x, y), \forall x, y \in X,$$

where $0 < a, b < 1$ and $a + b < 1$. Then

1) *$Fix\,(T) = \{x \in X : Tx = x\} = \{x^*\}$;*

2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to x^*.*

The corresponding version for p-convex contractions is stated as follows.

Theorem 3 ([37]) *Let (X, d) be a complete metric space and $T : X \to X$ be a continuous convex contraction of order p, i.e., a mapping satisfying*

$$d(T^px, T^py) \leq \sum_{i=0}^{p-1} a_i d(T^ix, T^iy), x, y \in X.$$

where $\sum_{i=0}^{p-1} a_i < 1$, and $a_0, a_1, , \cdots \geq 0$. Then

1) *$Fix\,(T) = \{x \in X : Tx = x\} = \{x^*\}$;*

2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to x^*.*

For many other results related to convex contractions, we refer to [37], [38], [39], [3]-[9], [43], [45], [60].

4 Almost convex contractions

In this section we aim to unify the concepts of convex contraction and almost contraction.

Definition 2 *Let (X, d) be a metric space. A self map $T : X \to X$ is called a* almost convex contraction *if*

$$d(T^2x, T^2y) \leq a_0 d(x, y) + a_1 d(Tx, Ty) + b_0 d(y, Tx) + b_1 d(Ty, T^2x), x, y \in X, \tag{8}$$

where a_0, b_0, a_1, b_1 are constants satisfying $a_0 + a_1 < 1$ and $a_0, b_0, a_1, b_1 \geq 0$.

Remark 1

1) In the particular case $b_0 = b_1 = 0$, from Definition 2 we get the convex contractions of order 2 introduced by Istrăţescu [37].

2) In the particular case $a_0 = b_0 = 0$, from Definition 2 we get the concept of almost contraction, introduced in [17].

Similarly, we can define the almost convex contractions of order p.

Definition 3 *Let (X, d) be a metric space. A self map $T : X \to X$ is called a* almost convex contraction of order p *if*

$$d(T^p x, T^p y) \leq \sum_{i=0}^{p-1} a_i d(T^i x, T^i y) + \sum_{i=0}^{p-1} b_i d(T^{i+1} x, T^i y), x, y \in X. \tag{9}$$

where $\sum_{i=0}^{p-1} a_i < 1$, and $a_0, b_0, a_1, b_1, \dots \geq 0$.

Remark 2 *1) In the particular case $b_0 = b_1 = \cdots = b_{p-1} = 0$, from Definition 3 we get the convex contractions of order p introduced by Istrăţescu [37].*

In order to prove our main result, we need the next Lemma.

Lemma 1 [47] *If $\{\Delta_n\}_{n \geq 0}$ is a sequence of non negative real numbers satisfying*

$$\Delta_{n+1} \leq \alpha_1 \Delta_n + \alpha_2 \Delta_{n-1}, n \geq 1, \tag{10}$$

where α_1, $\alpha_2 \in (0, 1)$ are such that $\alpha_1 + \alpha_2 \leq 1$, then

a) There exist $L > 0$ and $\theta \in (0, 1]$ such that

$$\Delta_n \leq L \cdot \theta^n, \text{ for all } n \geq 1. \tag{11}$$

b) If $\alpha_1 + \alpha_2 < 1$ then $\theta \in (0, 1)$.

The next theorem is the main result of this paper.

Theorem 4 *Let (X, d) be a complete metric space and $T : X \to X$ a continuous almost convex contraction. Then*

1) $Fix\,(T) \neq \emptyset$;

2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to $x^* \in Fix\,(T)$.*

Proof

We shall prove that T has at least a fixed point in X. To this end, let $x_0 \in X$ be arbitrary and $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration corresponding to T and the starting point x_0.

Take $x := x_{n-1}$, $y := x_n$ in 8 to obtain

$$d(T^2 x_{n-1}, T^2 x_n) \leq a_0 d(x_{n-1}, x_n) + a_1 d(Tx_{n-1}, Tx_n)\,,$$

which shows that

$$d(x_{n+1}, x_{n+2}) \leq a_0 d(x_{n-1}, x_n) + a_1 d(x_n, x_{n+1})\,. \tag{12}$$

Now, by denoting $\Delta_n = d(x_{n-1}, x_n)$ and applying Lemma 1, we obtain in a standard way that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence, hence convergent. Denote

$$x^* = \lim_{n\to\infty} x_n.$$

By the continuity of T we then deduce that x^* is a fixed point of T.

Remark 3 *1) In the particular case $b_0 = b_1 = 0$, by Theorem 4 we get Theorem 2 for convex contractions [37];*

2) In the particular case $a_0 = b_0 = 0$, by Theorem 4 we get Theorem 1 for almost contractions [17], when the assumption that T is continuous is superfluous;

3) For other fixed point theorems that consider hybrid contractive conditions involving the convex contraction condition, we refer to [7]-[9], where various combinations of classical contraction conditions (Kannan, Maia, Reich, Ćirić etc.) and convex contraction condition are considered.

4) Note that a sequence satisfying 12 is usually called subconvex*, see [12]-[14] and [6]-[9], which shows the the term "convex contraction" coined by Istrăţescu ([37], [38], [39]), is quite well motivated.*

A similar result, corresponding to the case of almost convex contractions of order p can be obtained by means of a more general auxiliary result, see Lemma 2 in [26].

Lemma 2 *Let k be a positive integer and $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq \alpha_1 a_n + \alpha_2 a_{n-1} + \cdots + \alpha_k a_{n-k+1} + b_n,\ n \geq k-1, \tag{13}$$

where $\alpha_1, \ldots, \alpha_k \in [0,1)$ and $\alpha_1 + \cdots + \alpha_k < 1$. If $\lim\limits_{n\to\infty} b_n = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Theorem 5 *Let (X,d) be a complete metric space and $T : X \to X$ a continuous almost convex contractions of order p. Then*

1) $Fix\,(T) \neq \emptyset$;

2) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n$, $n = 0,1,2,...$, converges to $x^* \in Fix\,(T)$.

5 Conclusions

We introduced a new class of mappings, obtained by merging the concepts of *almost contraction* and *convex contraction* and called them *almost convex contractions*. As it has been shown by the particular cases indicated above, this general class of mappings includes, amongst many other contractive type mappings, both the *almost contractions* and *convex contractions*.

We established two fixed point theorems, the first one for almost convex contractions of order 2 and then for almost convex contractions of order p.

The proof of Theorem 4 is based on the assumption that T is continuous, which has been used to prove that the limit x^* of the Picard sequence $\{x_n\}_{n=0}^{\infty}$ is a fixed point of T.

However, almost contractions are essentially discontinuous mappings, see Example 1 (but are continuous at their fixed points, see [25]).

On the other hand, in the case of Theorem 2, that is, for convex contractions of order 2, the continuity of T is not necessary to establish that x^* is a fixed point of T, see [9].

So, the problem is to study if the continuity of T is also not necessary in the more general case of almost convex contractions, too.

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