Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 12, No. 1-2/2020

# TWO-PARAMETER SECOND-ORDER DIFFERENTIAL INCLUSIONS IN HILBERT SPACES*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70th anniversary

**Abstract**

In a real Hilbert space $H$, let us consider the boundary-value problem $-\varepsilon u''(t) + \mu u'(t) + Au(t) + Bu(t) \ni f(t),\ t \in [0, T];\ u(0) = u_0,\ u'(T) = 0$, where $T > 0$ is a given time instant, $\varepsilon, \mu$ are positive parameters, $A : D(A) \subset H \to H$ is a (possibly set-valued) maximal monotone operator, and $B : H \to H$ is a Lipschitz operator. In this paper, we investigate the behavior of the solutions to this problem in two cases: **(i)** $\mu > 0$ fixed, $0 < \varepsilon \to 0$, and **(ii)** $\varepsilon > 0$ fixed and $0 < \mu \to 0$. Notice that if $\mu = 1$ and $\varepsilon$ is a positive small parameter, the above problem is a Lions-type regularization of the Cauchy problem $u'(t) + Au(t) + Bu(t) \ni f(t),\ t \in [0, T];\ u(0) = u_0$, which was recently studied by L. Barbu and G. Moroşanu [Commun. Contemp. Math. 19 (2017)]. Our abstract results are illustrated with examples related to the heat equation and the telegraph differential system.
**MSC**: 34G25, 47J35, 47H05, 35K20, 35L50

**keywords:** Lions regularization, approximation, maximal monotone operator, Lipschitz operator, heat equation, telegraph differential system.

*Accepted for publication on May 11, 2020
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# 1 Introduction

Let $H$ be a real Hilbert space with scalar product $(\cdot,\cdot)$ and the induced norm $\|x\| = \sqrt{(x,x)},\ x \in H$.

Consider in $H$ the following boundary-value problem, denoted $(P_{\varepsilon\mu})$,

$$\begin{cases} -\varepsilon u''(t) + \mu u'(t) + Au(t) + Bu(t) \ni f(t),\ 0 < t < T, & (E_{\varepsilon\mu}) \\ u(0) = u_0,\ u'(T) = 0, & (BC) \end{cases}$$

where $T > 0$ is a given time instant, $\varepsilon, \mu > 0$ are parameters, and $A$, $B$ satisfy the following assumptions

$(H_A)$ $\quad A : D(A) \subset H \to H$ is a (possibly set-valued) maximal monotone operator;

$(H_B)$ $\quad B : D(B) = H \to H$ is a Lipschitz operator, i.e., there exists a constant $L > 0$ such that $\forall x, y \in H,\ \|Bx - By\| \le L\|x - y\|$.

Recall that an operator $A : D(A) \subset H \to H$ is said to be monotone if its graph $G(A) = \{[x,y] \in D(A) \times H;\ y \in Ax\}$ is a monotone subset of $H \times H$, i.e., $(x_1 - x_2, y_1 - y_2) \ge 0\ \ \forall [x_1, y_1], [x_2, y_2] \in G(A)$. If, in addition, $G(A)$ is not properly contained in the graph of any other monotone operator in $H$ then $A$ is called maximal monotone. It is well-known that a monotone operator $A$ is maximal monotone if and only if the range of $I + \lambda A$ is all of $H$ for all $\lambda > 0$ (equivalently for some $\lambda > 0$). In this case the so-called resolvent operator $J_\lambda = (I + \lambda A)^{-1}$ is everywhere defined, single-valued and nonexpansive (i.e., Lipschitz with constant $L = 1$). If $\phi : H \to (-\infty, +\infty]$ is a proper (i.e., not identically $+\infty$), convex, lower semicontinuous function then the subdifferential operator $\partial\phi$, defined by

$$\partial\phi(x) = \{y \in H;\ \phi(x) - \phi(v) \le (y, x - v)\ \forall v \in D(\phi)\},$$

is maximal monotone.

If $A$ is a linear, maximal monotone operator, then, equivalently, $-A$ is the infinitesimal generator of a $C_0$-semigroup of linear contractions $\{S(t) : H \to H;\ t \ge 0\}$, $\|S(t)\| \le 1$ for all $t \ge 0$ (cf. Lumer-Phillips Theorem in [16, p. 14]; see also [11, pp. 74-82, in particular Corollary 3.20]). Obviously, in this case we have equality in $(E_{\varepsilon\mu})$. If, in addition, $A$ is self-adjoint ($A = A^*$), then $A$ is the subdifferential of $\phi : H \to (-\infty, +\infty]$,

$$\phi(x) = \begin{cases} \frac{1}{2}\|A^{1/2}x\|^2, & x \in D(A^{1/2}), \\ +\infty, & x \in H \setminus D(A^{1/2}). \end{cases}$$

For details on semigroups of linear operators, see [10]-[12], [16]. For information on monotone operators, convex functions and first-order differential equations (inclusions) associated with monotone operators, we refer the reader to [4]-[7], [13], [15], [17].

In this paper, we are interested in the behavior of the solution of $(P_{\varepsilon\mu})$ in two distinct cases:

$(i)$ $\nu > 0$ fixed and $0 < \varepsilon \to 0$;

$(ii)$ $\varepsilon > 0$ fixed and $0 < \mu \to 0$.

In the next section of this paper (Section 2) we provide some preliminary results which will be useful in proving the main results. In Section 3 we state and prove our main results related to the cases $(i)$ and $(ii)$. Finally, Section 4 is devoted to some examples.

# 2 Preliminary Results

For $\mu > 0$ fixed, consider the following Cauchy problem, denoted $(P_0)$,

$$\begin{cases} \mu u'(t) + Au(t) + Bu(t) \ni f(t), \quad 0 < t < T, & (E) \\ u(0) = u_0. & (IC) \end{cases}$$

Note that Eq. $(E)$ is obtained by setting $\varepsilon = 0$ in Eq. $(E_{\varepsilon\mu})$. For $\mu = 1$, problem $(P_0)$ was considered in [3] as the reduced model for problem $(P_{\varepsilon 1})$ (i.e., problem $(P_{\varepsilon\mu})$ with $\mu = 1$). Recall that $(P_{\varepsilon 1})$ is an elliptic like regularization of $(P_0)$ with $\mu = 1$ in the sense of Lions (see [14] for similar linear problems).

Now, let us state some existence results for problem $(P_0)$.

**Lemma 1.** *Assume that $(H_A)$ and $(H_B)$ hold.*

*If $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;H)$ then there exists a unique $u \in W^{1,\infty}(0,T;H)$ satisfying $(P_0)$ (in the sense that $u$ satisfies Eq. $(E)$ for a.a. $t \in (0,T)$ and $u(0) = u_0$).*

*If, in particular, $A$ is linear, maximal monotone, then $u \in C^1([0,T];H)$, with $u(t) \in D(A)$ for all $t \in [0,T]$, satisfying Eq. $(E)$ for all $t \in [0,T]$.*

*Proof.* One can assume $\mu = 1$ without any loss of generality. The first statement of the lemma follows from [15, Theorem 2.1, p. 48 and Remark 2.1, p. 53]. In order to prove the second statement, denote by $\{S(t) : H \to$

$H;\, t \geq 0\}$ the $C_0$-semigroup of linear contractions generated by $-A$. One can write

$$\begin{aligned} u(t) &= S(t)u_0 + \int_0^t S(t-s)[f(s) - Bu(s)]\, ds \\ &= S(t)u_0 + \int_0^t S(s)[f(t-s) - Bu(t-s)]\, ds, \quad t \in [0,T]\,, \end{aligned}$$

and so, for all $t \in [0,T]$, we have

$$\begin{aligned} u'(t) &= S(t)(f(0) - Au_0 - Bu_0) + \int_0^t S(s)[f'(t-s) - (B \circ u)'(t-s)]\, ds \\ &= S(t)(f(0) - Au_0 - Bu_0) + \int_0^t S(t-s)[f'(s) - (B \circ u)'(s)]\, ds\,, \end{aligned}$$

which shows that in fact $u' \in C([0,T];\, H)$. We have used the fact that $B \circ u$ is a Lipschitz function, hence absolutely continuous on $[0,T]$, so its derivative belongs to $L^1(0,T;\, H)$ (see, e.g., [15, Theorem 4.3, p. 10]). Since $A$ is closed, it follows that, for all $t \in [0,T]$, $u(t) \in D(A)$ and $u'(t) + Au(t) + Bu(t) = f(t)$. $\square$

**Lemma 2.** *Assume that $A$ is the subdifferential of a proper, convex, lower semicontinuous function $\phi : H \to (-\infty, \infty]$ and $(H_B)$ holds. Then, for any $u_0 \in H$ with $\phi(u_0) < \infty)$ and $f \in L^2(0,T;H)$, there exists a unique function $u \in W^{1,2}([0,T];H)$ satisfying problem $(P_0)$.*

*Proof.* One can assume $\mu = 1$. If $B = 0$, the result belongs to H. Brezis (see, e.g., [15, p. 56]). So, we only need to extend the result to the case when $B$ satisfies $(H_B)$, $B \neq 0$. We shall follow the idea from the proof of Lemma 2.2 in [3]. Let $X = C([0,T];H)$ be equipped with the Bielecki norm

$$\|g\|_X = \sup_{t \in [0,T]} e^{-\gamma t} \|g(t)\|, \quad g \in X,$$

where $\gamma > 0$ is a constant which will be specified later. Obviously, $(X, \|\cdot\|_X)$ is a Banach space. Consider the operator $P : X \to X$ which associates with each $v \in X$ the solution $u \in W^{1,2}(0,T;H)$ of the Cauchy problem

$$u' + Au \ni f - Bv, \quad t \in [0,T]; \quad u(0) = u_0. \tag{1}$$

Let $v_1$, $v_2 \in X$ and $u_1 = Pv_1$, $u_2 = Pv_2$. Using the fact that $(u_1, v_1)$, $(u_2, v_2)$ satisfy (1) and monotonicity of $A = \partial\phi$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_1(t) - u_2(t)\|^2 \leq (Bv_1(t) - Bv_2(t), u_1(t) - u_2(t)),$$

for a.a. $t \in (0,T)$. Therefore,

$$\frac{1}{2}\|u_1(t)-u_2(t)\|^2 \leq L\int_0^t \|v_1(s)-v_2(s)\|\cdot\|u_1(s)-u_2(s)\|\,ds,$$

for all $t \in [0,T]$. According to a Gronwall type lemma (see, e.g., [15, p. 47]), we get from the above inequality

$$\begin{aligned}\|u_1(t)-u_2(t)\| &\leq L\int_0^t \|v_1(s)-v_2(s)\|\,ds \\ &= L\int_0^t e^{\gamma s}e^{-\gamma s}\|v_1(s)-v_2(s)\|\,ds,\end{aligned}$$

for all $t \in [0,T]$. This implies

$$\begin{aligned}\|u_1(t)-u_2(t)\| &\leq L\|v_1-v_2\|_X\int_0^t e^{\gamma s}ds, \\ &= \frac{L}{\gamma}\|v_1-v_2\|_X\big(e^{\gamma t}-1\big), \quad t \in [0,T].\end{aligned}$$

Therefore,

$$\|u_1-u_2\|_X \leq \frac{L}{\gamma}\|v_1-v_2\|_X.$$

Hence $P$ is a contraction if $\gamma > L$. By the Banach Contraction Principle, $P$ has a unique fixed point $u \in X$ which in fact belongs to $W^{1,2}(0,T;H)$ and is a solution to problem $(P_0)$ with $\mu = 1$. □

**Lemma 3.** *Assume $(H_A)$, $B=0$, $u_0 \in D(A)$, $f \in L^2(0,T;H)$. Then, for every $\varepsilon > 0$, $\mu \geq 0$, there exists a unique function $u \in W^{2,2}(0,T;H)$ satisfying problem $(P_{\varepsilon\mu})$ (i.e., $u$ satisfies Eq. $(E_{\varepsilon\mu})$ for a.a. $t \in (0,T)$ and $u(0) = u_0$).*

*Proof.* The proof is similar to the proof of Proposition 2.1 in [3]. Let $y_0 \in Au_0$. Denote $v(t) = u(t) - u_0$. With this change problem $(P_{\varepsilon\mu})$ (with $B=0$) becomes

$$\begin{cases}-\varepsilon v'' + \mu v' + \tilde{A}v \ni f(t) - y_0, \quad 0 < t < T\,, \\ v(0) = 0, \ v'(T) = 0\,,\end{cases}$$

where $\tilde{A}$ is the operator defined by $\tilde{A}v = A(v+u_0) - y_0$, $\forall v \in D(\tilde{A})$, where $D(\tilde{A}) = \{v \in H;\, v = z - u_0,\ z \in D(A)\}$. Obviously, $\tilde{A}$ is a maximal monotone operator and $[0,0] \in G(\tilde{A})$. So, we can assume from now on $u_0 = 0$ and $[0,0] \in G(A)$.

Denote $\mathcal{H} := L^2(0, T;\, H)$. This is a real Hilbert space with respect to its usual scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|_{\mathcal{H}}$.

For $D(Q) = \{u \in H^2(0, T;\, H);\, u(0) = u'(T) = 0\}$, let us define the operator $Q : D(Q) \subset \mathcal{H} \to \mathcal{H}$ by

$$Qu = -\varepsilon u'' + \mu u' \quad \forall u \in D(Q).$$

It is a simple exercise to show that $Q$ is maximal monotone in $\mathcal{H}$. Then, if $\bar{A}$ denotes the canonical extension of $A$ to $\mathcal{H}$, then for any $\lambda > 0$ the sum $Q + \bar{A}_\lambda$ is also maximal monotone, where $\bar{A}_\lambda$ denotes the Yosida approximation of $\bar{A}$: $\bar{A}_\lambda = \overline{(A_\lambda)}$. Therefore, for every $\lambda > 0$ and every $f \in \mathcal{H}$, there exists a unique $u_\lambda \in D(Q)$ which satisfies

$$\begin{cases} -\varepsilon u_\lambda'' + \mu u_\lambda' + A_\lambda u_\lambda + \lambda u_\lambda = f, \;\; 0 < t < T\,, \\ u_\lambda(0) = 0, \;\; u_\lambda'(T) = 0\,. \end{cases} \tag{2}$$

Multiplication of the above equation by $u_\lambda$ and integration over $[0, T]$ lead us to

$$\underbrace{-\varepsilon (u_\lambda', u_\lambda)|_0^T}_{=0} + \varepsilon \|u_\lambda'\|_{\mathcal{H}}^2 + \frac{\mu}{2} \|u_\lambda(T)\|^2 + \lambda \|u_\lambda\|_{\mathcal{H}}^2 \leq \|f\|_{\mathcal{H}} \|u_\lambda\|_{\mathcal{H}}\,. \tag{3}$$

On the other hand, since $u_\lambda(t) = \int_0^t u_\lambda'(s)\, ds$, we have

$$\|u_\lambda(t)\| \leq \int_0^t \|u_\lambda'(s)\|\, ds \leq \int_0^T \|u_\lambda'(s)\|\, ds \leq T^{1/2} \|u_\lambda'\|_{\mathcal{H}}.$$

This inequality along with (3) gives

$$\varepsilon \|u_\lambda'\|_{\mathcal{H}}^2 \leq T^{1/2} \|f\|_{\mathcal{H}} \|u_\lambda'\|_{\mathcal{H}}\,.$$

Therefore $u_\lambda'$ is bounded in $\mathcal{H}$ and so $u_\lambda$ is bounded in $C([0, T];\, H)$. We also have for a.a. $t \in (0, T)$

$$\frac{d}{dt}(u_\lambda', A_\lambda u_\lambda) = (u_\lambda'', A_\lambda u_\lambda) + \underbrace{((A_\lambda u_\lambda)', u_\lambda')}_{\geq 0},$$

which implies

$$\begin{aligned} 0 \;&\geq\; \varepsilon \int_0^T (u_\lambda'', A_\lambda u_\lambda)\, dt \\ &=\; \int_0^T (\mu u_\lambda' + A_\lambda u_\lambda + \lambda u_\lambda - f, A_\lambda u_\lambda)\, ds \\ &=\; \|A_\lambda u_\lambda\|_{\mathcal{H}}^2 + \mu \langle u_\lambda', A_\lambda u_\lambda \rangle + \lambda \langle u_\lambda, A_\lambda u_\lambda \rangle - \langle f, A_\lambda u_\lambda \rangle\,. \end{aligned}$$

It follows that $A_\lambda u_\lambda$ is bounded in $\mathcal{H}$ for $0 < \lambda \le \lambda_0$, where $\lambda_0 > 0$ is a fixed number. From Eq. (2) we obtain that $u''_\lambda$ is also bounded in $\mathcal{H}$ for $0 < \lambda \le \lambda_0$.

Now, by using the information obtained so far, we will show that $u_\lambda$ converges in $C([0,T];\, H)$ as $\lambda \to 0^+$. To this purpose, we calculate for $\lambda,\, \nu \in (0, \lambda_0]$

$$\begin{aligned} &-\varepsilon \int_0^T (u''_\lambda - u''_\nu, u_\lambda - u_\nu)\, ds + \mu \int_0^T (u'_\lambda - u'_\nu, u_\lambda - u_\nu)\, ds \\ &+ \int_0^T (A_\lambda u_\lambda - A_\nu u_\nu, u_\lambda - u_\nu)\, ds = - \int_0^T (\lambda u_\lambda - \nu u_\nu, u_\lambda - u_\nu)\, ds\,. \end{aligned}$$

Thus, denoting $J_\lambda = (I + \lambda A)^{-1}$, we have

$$\begin{aligned} & \underbrace{-\varepsilon (u'_\lambda - u'_\nu, u_\lambda - u_\nu)|_0^T}_{=0} + \varepsilon \|u'_\lambda - u'_\nu\|_{\mathcal{H}} + \frac{\mu}{2} \|u_\lambda(T) - u_\nu(T)\|^2 \\ + \quad & \underbrace{\langle A J_\lambda u_\lambda - A J_\nu u_\nu, J_\lambda u_\lambda - J_\nu u_\nu \rangle}_{\ge 0} \\ = \quad & -\langle A_\lambda u_\lambda - A_\nu u_\nu, \lambda A_\lambda u_\lambda - \nu A_\nu u_\nu \rangle - \langle \lambda u_\lambda - \nu u_\nu, u_\lambda - u_\nu \rangle\,. \end{aligned}$$

This leads to

$$\|u'_\lambda - u'_\nu\|_{\mathcal{H}}^2 \le Const.\, (\lambda + \nu),\ \ \lambda,\, \nu \in (0, \lambda_0]\,,$$

i.e., $u'_\lambda$ is Cauchy, hence convergent, in $\mathcal{H}$. It follows that $u_\lambda$ is convergent in $C([0,T];\, H)$ since

$$\begin{aligned} \|u_\lambda(t) - u_\nu(t)\| &= \| \int_0^t (u'_\lambda - u'_\nu)\, ds \| \\ &\le \int_0^t \|u'_\lambda - u'_\nu\|\, ds \\ &\le \int_0^T \|u'_\lambda - u'_\nu\|\, ds \\ &\le T^{1/2} \|u'_\lambda - u'_\nu\|_{\mathcal{H}}\,, \end{aligned}$$

for all $t \in [0,T]$, $\lambda,\, \nu \in (0, \lambda_0]$.

Summarizing, there exists $u \in W^{2,2}(0,T;\, H)$ such that

$$u_\lambda \to u \ \text{ in } C([0,T];\, H)\,, \tag{4}$$

$$u'_\lambda \to u' \ \text{ in } \mathcal{H}\,, \tag{5}$$

$$u''_\lambda \to u'' \quad \text{weakly in } \mathcal{H}\,, \tag{6}$$

as $\lambda \to 0^+$. Now, denoting

$$h_\lambda(t) := \|u'_\lambda(t) - u'(t)\|, \quad \lambda > 0,\ t \in [0,T],$$

one can apply the Arzelà-Ascoli Criterion, to derive that $h_\lambda$ converges in $C[0,T]$ to zero (see (5)). Indeed, for $t \in [0,T]$,

$$\begin{aligned} h_\lambda(t) &\leq \|u'_\lambda(t)\| + \|u'(t)\| \\ &= \|\int_t^T u''_\lambda\| + \|u'(t)\| \\ &\leq \int_0^T \|u''_\lambda\| + \|u'(t)\| \\ &\leq T^{1/2}\|u''_\lambda\|_{\mathcal{H}} + \|u'\|_{C([0,T];H)} < \infty. \end{aligned}$$

Moreover, $h_\lambda$ is equi-continuous since

$$\|u'_\lambda(t) - u'_\lambda(s)\| = \|\int_s^t u''_\lambda\| \leq T^{1/2}\|u''_\lambda\|_{\mathcal{H}}|t-s|^{1/2}, \ \ \lambda > 0,\ t,s \in [0,T].$$

Therefore, $u'_\lambda \to u'$ in $C([0,T];H)$ as $\lambda \to 0^+$. From this information and (4), we derive $u(0) = 0$ and $u'(T) = 0$. In what follows we prove that $u$ is the solution we are looking for by letting $\lambda \to 0^+$ in the equation

$$-\varepsilon u''_\lambda + \mu u'_\lambda + \bar{A}_\lambda u_\lambda + \lambda u_\lambda = f \quad \text{in } \mathcal{H}\,. \tag{7}$$

To this purpose let us observe that

$$\|J_\lambda u_\lambda - u\|_{\mathcal{H}} \leq \lambda\|A_\lambda u_\lambda\|_{\mathcal{H}} + \|u_\lambda - u\|_{\mathcal{H}}\,,$$

hence

$$J_\lambda u_\lambda \to u \quad \text{in } \mathcal{H}\,, \tag{8}$$

as $\lambda \to 0^+$.

Note that $\bar{A}_\lambda u_\lambda \in \bar{A}J_\lambda u$, $\lambda > 0$. Taking into account the demiclosedness of $\bar{A}$ and (4)-(6), (8), we can now pass to the (weak) limit in (7) to obtain that $u$ is a solution of equation $(E_{\varepsilon,\mu})$ with $B = 0$.

The uniqueness of $u$ follows easily by using the monotonicity of $A$. Indeed, if $\hat{u}$ would be another solution, then the monotonicity of $A$ leads to

$$-\varepsilon(u'' - \hat{u}'', u - \hat{u}) + \mu(u' - \hat{u}', u - \hat{u}) \leq 0 \quad \text{for a.a. } t \in (0,T)\,.$$

Integration over $[0,T]$ gives

$$\varepsilon\|u' - \hat{u}'\|^2_{\mathcal{H}} \leq 0 \ \Rightarrow\ u' = \hat{u}' \ \Rightarrow\ u = \hat{u}\,,$$

since $u(0) - \hat{u}(0) = 0$. □

Now we are ready to state and prove an existence result for the full $(P_{\varepsilon\mu})$.

**Lemma 4.** *Assume* $(H_A)$, $(H_B)$, $u_0 \in D(A)$, $f \in L^2(0,T;\, H)$, *and* $0 < \varepsilon < \mu^2/(4L)$. *Then problem* $(P_{\varepsilon\mu})$ *has a unique solution* $u = u_{\varepsilon\mu} \in W^{2,2}(0,T;\, H)$.

*Proof.* As in the proof of the previous lemma, we can assume without any loss of generality that $u_0 = 0$, $0 \in D(A)$, and $0 \in A0$.

Denote again $\mathcal{H} = L^2(0,T;\, H)$ and keep the symbols $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_{\mathcal{H}}$ for its scalar product and the induced norm.

Define the operator $G_\varepsilon : D(G_\varepsilon) \subset \mathcal{H} \to \mathcal{H}$ by

$$\begin{aligned} D(G_\varepsilon) \quad &= \quad \{v \in W^{2,2}(0,T;\, H);\, v(0) = 0,\, Mv(T) + v'(T) = 0,\, \exists \zeta \in \mathcal{H} \\ &\qquad \text{such that } \zeta(t) \in A(e^{Mt}v(t)) \text{ for a.a. } t \in (0,T)\}\,, \end{aligned}$$

$$G_\varepsilon v = -\varepsilon e^{-Mt}(e^{Mt}v)'' + \mu\Big(\frac{M}{2}v + v'\Big) + e^{-Mt}A(e^{Mt}v) \ \forall v \in D(G_\varepsilon)\,,$$

where $M$ is a constant satisfying $2L/\mu < M < \mu/2\varepsilon$. Obviously, $0 \in D(G_\varepsilon)$ and $0 \in G_\varepsilon 0$. Let $v_1$, $v_2 \in D(G_\varepsilon)$. Denoting $w = v_1 - v_2$, we have

$$\begin{aligned} \langle G_\varepsilon v_1 \quad &- \quad G_\varepsilon v_2, w\rangle \\ &\geq \quad -\varepsilon \int_0^T (e^{-Mt}(e^{Mt}w)'', w)\, ds + \frac{\mu M}{2}\|w\|_{\mathcal{H}}^2 + \mu \int_0^T (w', w)\, ds \\ &= \quad \underbrace{-\varepsilon((e^{Mt}w)', e^{-Mt}w)|_0^T}_{=0} + \varepsilon \int_0^T ((e^{Mt}w)', (e^{-Mt}w)')\, ds \\ &\qquad + \frac{\mu M}{2}\|w\|_{\mathcal{H}}^2 + \frac{\mu}{2}\|w(T)\|^2 \\ &= \quad \frac{M}{2}(\mu - 2M\varepsilon)\|w\|_{\mathcal{H}}^2 + \varepsilon\|w'\|_{\mathcal{H}}^2 + \frac{1}{2}\|w(T)\|^2 \\ &\geq \quad \frac{M}{2}(\mu - 2M\varepsilon)\|w\|_{\mathcal{H}}^2\,, \end{aligned}$$

which shows that $G_\varepsilon$ is strongly monotone. Now, let us prove that $G_\varepsilon$ is maximal monotone. To this purpose consider the equation

$$\frac{\mu M}{2}v + G_\varepsilon v \ni g\,,$$

for an arbitrary but fixed $g \in \mathcal{H}$. Equivalently, $v$ satisfies

$$\begin{cases} -\varepsilon e^{-Mt}(e^{Mt}v)'' + \mu M v + \mu v' + e^{-Mt}A(e^{Mt}v) \ni g & \text{for a.a. } t \in (0,T)\,, \\ v(0) = 0,\ Mv(T) + v'(T) = 0\,. \end{cases}$$

Using the change $u(t) = e^{Mt}v(t)$ we see that this problem can be rewritten as

$$\begin{cases} -\varepsilon u'' + \mu u' + Au \ni e^{Mt}g(t), \ 0 < t < T\,, \\ u(0) = 0, \ u'(T) = 0\,, \end{cases}$$

which has a unique solution $u \in W^{2,2}(0,T;\,H)$ (cf. Lemma 3). Thus $G_\varepsilon$ is indeed maximal monotone in $\mathcal{H}$.

Now, the operator $R : D(R) = \mathcal{H} \to \mathcal{H}$, defined by

$$Rv = \frac{\mu M}{2}v + e^{-Mt}B(e^{Mt}v), \ \ v \in \mathcal{H}\,,$$

is strongly monotone (since $\mu M > 2L$) and maximal monotone (since it is Lipschitz continuous on $\mathcal{H}$). It follows that the sum $G_\varepsilon + R$ is maximal monotone and strongly monotone, hence its range is all of $\mathcal{H}$. So, the function $t \mapsto e^{-Mt}f(t)$ belongs to the range of $G_\varepsilon + R$, i.e., there exists (a unique) $v \in D(G_\varepsilon)$ satisfying the problem

$$\begin{cases} -\varepsilon e^{-Mt}(e^{Mt}v)'' + \mu M v + \mu v' + e^{-Mt}A(e^{Mt}v) + e^{-Mt}B(e^{Mt}v) \ni e^{-Mt}f(t), \\ v(0) = 0, \ Mv(T) + v'(T) = 0\,. \end{cases}$$

Therefore, $u(t) = e^{Mt}v(t)$ is the unique solution of

$$\begin{cases} -\varepsilon u'' + \mu u' + Au + Bu \ni f, \quad \text{a.e. in } (0,T)\,, \\ u(0) = 0, \ u'(T) = 0\,. \end{cases}$$

□

# 3 Main Results

Let us first consider the case **(i)**: $\mu > 0$ fixed and $0 < \varepsilon \to 0$. We have the following approximation result, which is a refinement of Theorem 2.1 in [3]:

**Theorem 1.** *Assume that* $(H_A)$ *and* $(H_B)$ *are satisfied and, in addition,* $\mu$ *is a fixed positive number,* $u_0 \in D(A)$*, and* $f \in W^{1,1}(0,T;H)$*. Then, for every* $0 < \varepsilon < \mu^2/(8L)$*, the problems* $(P_{\varepsilon\mu})$ *and* $(P_0)$ *have unique solutions,* $u_\varepsilon \in W^{2,2}(0,T;H)$ *and respectively* $u \in W^{1,\infty}(0,T;H)$*, and the following estimates hold*

$$\|u_\varepsilon - u\|_{C([0,T];H)} = \mathcal{O}(\varepsilon^{1/4}) \quad \textit{and } \|u_\varepsilon - u\|_{L^2(0,T;H)} = \mathcal{O}(\varepsilon^{1/2}). \qquad (9)$$

*Proof.* The existence and uniqueness of solutions $u \in W^{1,\infty}(0,T;H)$ and $u_\varepsilon \in W^{2,2}(0,T;H)$ to the problems $(P_0)$ and $(P_{\varepsilon\mu})$, with $\mu > 0$ fixed, are guaranteed by Lemma 1 and Lemma 4, respectively. Denote $r_\varepsilon = u_\varepsilon - u$. Note that

$$r_\varepsilon + u = u_\varepsilon \in W^{2,2}(0,T;H), \quad r_\varepsilon \in W^{1,\infty}(0,T;H),$$

and

$$\begin{cases} -\varepsilon(r_\varepsilon + u)'' + \mu r_\varepsilon' + Au_\varepsilon - Au + Bu_\varepsilon - Bu \ni 0, \quad \text{a.e. in } (0,T), \\ r_\varepsilon(0) = 0, \quad (r_\varepsilon + u)'(T) = 0. \end{cases} \tag{10}$$

In what follows we show that

$$\|r_\varepsilon\|_{L^2(0,T;H)} = \mathcal{O}(\varepsilon^{1/2}), \quad \|r_\varepsilon'\|_{L^2(0,T;H)} = \mathcal{O}(1). \tag{11}$$

For $\varepsilon \in (0, \mu^2/(8L))$, denote $s_\varepsilon(t) = e^{-Mt} r_\varepsilon(t)$, where $M = M_\varepsilon$ satisfies the inequalities $2L/\mu < M < \mu/(2\varepsilon)$. So, (10) can be written as

$$\begin{cases} -\varepsilon e^{-Mt}(e^{Mt}s_\varepsilon + u)'' + \mu s_\varepsilon' + \mu M s_\varepsilon + e^{-Mt}(Au_\varepsilon - Au) \\ +e^{-Mt}(Bu_\varepsilon - Bu) \ni 0, \quad \text{a.e. in } (0,T), \\ s_\varepsilon(0) = 0, \quad (e^{Mt}s_\varepsilon + u)'(T) = 0. \end{cases} \tag{12}$$

Now, take the scalar product in $H$ of Eq. $(12)_1$ and $s_\varepsilon$, use the monotonicity of $A$, and integrate over $[0,T]$ to obtain

$$\begin{aligned} - \quad \varepsilon \quad & \int_0^T \langle (e^{Mt}s_\varepsilon + u)'', e^{-Mt}s_\varepsilon \rangle dt + \frac{\mu}{2}\|s_\varepsilon(T)\|^2 + \mu M \|s_\varepsilon\|_{\mathcal{H}}^2 \\ + \quad & \int_0^T e^{-Mt}\langle Bu_\varepsilon - Bu, s_\varepsilon \rangle dt \le 0, \end{aligned}$$

where $\mathcal{H} = L^2(0,T;H)$. Therefore,

$$\begin{aligned} \varepsilon \quad & \int_0^T \langle (e^{Mt}s_\varepsilon + u)', -Me^{-Mt}s_\varepsilon + e^{-Mt}s_\varepsilon' \rangle dt + \mu M \|s_\varepsilon\|_{\mathcal{H}}^2 \\ + \quad & \int_0^T e^{-Mt}\langle Bu_\varepsilon - Bu, s_\varepsilon \rangle dt \le 0, \end{aligned}$$

which implies (by using the Cauchy-Schwarz and Hölder inequalities)

$$\varepsilon \|s_\varepsilon'\|_{\mathcal{H}}^2 + M(\mu - \varepsilon M)\|s_\varepsilon\|_{\mathcal{H}}^2 + \int_0^T e^{-Mt}\langle Bu_\varepsilon - Bu, s_\varepsilon \rangle dt$$

$$\leq M\varepsilon\|u'\|_{\mathcal{H}}\|s_\varepsilon\|_{\mathcal{H}} + \varepsilon\|u'\|_{\mathcal{H}}\|s'_\varepsilon\|_{\mathcal{H}}. \tag{13}$$

On the other hand, it follows by using $(H_B)$

$$-\int_0^T e^{-Mt}\langle Bu_\varepsilon - Bu, s_\varepsilon\rangle dt \leq L\int_0^T e^{-Mt}\|r_\varepsilon\| \cdot \|s_\varepsilon\|\, dt = L\|s_\varepsilon\|^2_{\mathcal{H}}. \tag{14}$$

Now, take $M = 4L/\mu$. We can infer from (13) and (14)

$$L\|s_\varepsilon\|^2_{\mathcal{H}} + \varepsilon\|s'_\varepsilon\|^2_{\mathcal{H}} \leq c\varepsilon + \frac{L}{2}\|s_\varepsilon\|^2_{\mathcal{H}} + \frac{\varepsilon}{2}\|s'_\varepsilon\|^2_{\mathcal{H}},$$

where $c$ is a positive constant (we have used the elementary inequality $ab \leq (a^2+b^2)/2$). It follows that

$$\|s_\varepsilon\|_{\mathcal{H}} = \mathcal{O}(\varepsilon^{1/2}), \quad \|s'_\varepsilon\|_{\mathcal{H}} = \mathcal{O}(1).$$

Since $r_\varepsilon(t) = e^{Mt}s_\varepsilon(t)$, we have $(9)_2$. Using the obvious equality

$$\|r_\varepsilon\|^2 = 2\int_0^t (r_\varepsilon(s), r'_\varepsilon(s))\, ds, \quad t \in [0,T],$$

it follows

$$\|r_\varepsilon\|_{C([0,T];H)} = \|u_\varepsilon - u\|_{C([0,T];H)} = \mathcal{O}(\varepsilon^{1/4}).$$

□

If $A$ is a subdifferential operator, we have the following better result:

**Theorem 2.** *Assume that $(H_B)$ is satisfied, and $A$ is the subdifferential of a proper, convex and lower semicontinuous function $\phi : H \to (-\infty, +\infty]$. Assume, in addition, that $\mu$ is a fixed positive number, $u_0 \in D(A)$, and $f \in L^2(0,T;H)$. Then, for every $\varepsilon \in (0, \mu^2/(8L))$, the problems $(P_{\varepsilon\mu})$ and $(P_0)$ have unique solutions, $u_\varepsilon \in W^{2,2}(0,T;H)$ and respectively $u \in W^{1,2}(0,T;H)$, and the following estimates hold*

$$\|u_\varepsilon - u\|_{C([0,T];H)} = \mathcal{O}(\varepsilon^{1/4}) \quad \textit{and } \|u_\varepsilon - u\|_{L^2(0,T;H)} = \mathcal{O}(\varepsilon^{1/2}). \tag{15}$$

*Proof.* One can use Lemma 2 and Lemma 4, and repeat the proof of Theorem 1 above. Notice that the regularity of $u$ in this case (i.e., $u \in W^{1,2}(0,T;H)$) is sufficient to follow the same proof. □

**Remark 1.** *Assume that $B = 0$ in both Theorem 1 and Theorem 2, $\varepsilon > 0$ is arbitrary, and the other assumptions are maintained. Then the conclusions remain valid. In order to prove this assertion, we can use the reasonings from the proof of Theorem 1, taking into account Lemma 3 (where $\varepsilon > 0$ was arbitrary) instead of Lemma 4.*

**Remark 2.** *According to Theorem 1 and Theorem 2, the problem $(P_{\varepsilon\mu})$ with $\mu > 0$ fixed is a genuine regularization of the reduced problem $(P_0)$, since $u_\varepsilon$ is more regular than $u$.*

*It is also worth pointing out that the condition $u'_\varepsilon(T) = 0$ is more convenient than the condition $u_\varepsilon(T) = u_1$ used in [2], which generates a boundary layer phenomenon near $t = T$ since $u$ does not satisfy in general the condition $u(T) = u_1$.*

Now, let us investigate the case **(ii)**: $\varepsilon > 0$ fixed, and $0 < \mu \to 0$. We shall assume $B = 0$, otherwise the problem $(P_{\varepsilon\mu})$ may not have solutions if the condition $0 < \varepsilon < \mu^2/(4L)$ (used in Lemma 4) is not satisfied. Indeed, let us consider in $H = \mathbb{R}$ the following simple problem, denoted $(P)$,

$$\begin{cases} -u''(t) + \mu u'(t) - u(t) = 0, \ 0 < t < T, \\ u(0) = u_0, \ u'(T) = 0, \end{cases}$$

i.e., $\varepsilon = 1$, $A = 0$, $L = 1$, and $f = 0$. Assume that $0 < \mu < 2$, so the condition $\varepsilon < \mu^2/(4L)$ is not fulfilled. Denote $a = \sqrt{4 - \mu^2}/2$. The general solution of the above equation is given by

$$u(t) = e^{\mu t/2}[c_1 \cos(at) + c_2 \sin(at)],$$

with $c_1$, $c_2 \in \mathbb{R}$. Using the boundary conditions $u(0) = u_0$, $u'(T) = 0$, we find $c_1 = u_0$ and

$$[(\mu/2)\sin(aT) + a\cos(aT)]c_2 = u_0[-(\mu/2)\cos(aT) + a\sin(aT)]\,. \tag{16}$$

Let us try to identify a $T > 0$ such that the coefficient of $c_2$ in (16) be null, i.e.,

$$\sqrt{4 - \mu^2} = -\mu \tan(aT)\,. \tag{17}$$

For a given $\mu \in (0, 2)$, choose $T = T_\delta = (\pi + \delta)/\sqrt{4 - \mu^2}$, where $\delta \in (0, \pi)$. Then, Eq. (17) becomes

$$\frac{\sqrt{4 - \mu^2}}{\mu} = \underbrace{-\tan\left(\frac{\pi + \delta}{2}\right)}_{>0}\,.$$

which is equivalent to

$$\mu = \frac{2}{\sqrt{1+\tan^2\left(\frac{\pi+\delta}{2}\right)}}. \tag{18}$$

Obviously, the right hand side of Eq. (18) is an increasing function of $\delta \in (0,\pi)$ whose range is the interval $(0,2)$. Therefore, for each $\mu \in (0,2)$ there exists a unique $\delta \in (0,\pi)$ satisfying (18). Thus, for $T = T_\delta$ chosen above, the coefficient of $c_2$ in Eq. (16) equals zero. On the other hand, if $u_0 \in \mathbb{R} \setminus \{0\}$, then the right hand side of Eq. (16) is not equal to zero. Indeed, assuming the contrary would give

$$\begin{aligned} \frac{\mu}{2} &= a\tan(aT) \\ &= \frac{\sqrt{4-\mu^2}}{2}\underbrace{\tan\left(\frac{\pi+\delta}{2}\right)}_{<0}, \end{aligned}$$

which is impossible. Hence, there is no $c_2$ satisfying Eq. (16), i.e., the problem $(P)$ has no solution.

**Theorem 3.** *Assume that* $(H_A)$ *is satisfied,* $B = 0$*,* $\varepsilon$ *is a fixed positive number,* $u_0 \in D(A)$*, and* $f \in L^2(0,T;H)$*. Then, for every* $\mu > 0$*, the problems* $(P_\mu) \equiv (P_{\varepsilon\mu})$ *and* $(P_0) \equiv (P_{\varepsilon 0})$*, have unique solutions,* $u_\mu \in W^{2,2}(0,T;H)$ *and respectively* $u \in W^{2,2}(0,T;H)$*, and the following estimates hold*

$$\|u_\mu - u\|_{C([0,T];H)} = \mathcal{O}(\mu^{1/2}) \quad \text{and } \|u_\mu - u\|_{L^2(0,T;H)} = \mathcal{O}(\mu). \tag{19}$$

*Proof.* We know from Lemma 3 that, for each $\mu > 0$, there exists a unique $u_\mu \in W^{2,2}(0,T;H)$ satisfying the problem $(P_\mu)$, i.e.,

$$\begin{cases} -\varepsilon u_\mu''(t) + \mu u_\mu'(t) + Au_\mu(t) \ni f(t),\ 0<t<T, & (E_\mu) \\ u_\mu(0) = u_0,\ u_\mu'(T) = 0. & (BC) \end{cases}$$

By the same lemma (Lemma 3), there exists a unique $u \in W^{2,2}(0,T;H)$ which satisfies the problem $(P_0)$, i.e.,

$$\begin{cases} -\varepsilon u''(t) + Au(t) \ni f(t),\ 0<t<T, & (E_0) \\ u(0) = u_0,\ u'(T) = 0. & (BC) \end{cases}$$

Now, subtract $(E_0)$ from $(E_\mu)$, multiply the resulting equation by $u_\mu - u$, and integrate over $[0,T]$ to get

$$-\varepsilon\int_0^T (u_\mu''-u'', u_\mu-u)\,dt + \mu\int_0^T (u_\mu', u_\mu-u)\,dt + \int_0^T (Au_\mu - Au, u_\mu-u)\,dt = 0.$$

Integration by parts of the first term of the above equation, combined with the monotonicity of $A$, gives

$$\varepsilon\|u'_\mu - u'\|^2_{\mathcal{H}} + \underbrace{\mu\int_0^T (u'_\mu - u', u_\mu - u)\,dt}_{=(\mu/2)\|u_\mu - u\|^2|_0^T} + \mu\int_0^T (u', u_\mu - u)\,dt \leq 0,$$

where $\mathcal{H} = L^2(0,T;H)$. Therefore,

$$\varepsilon\|u'_\mu - u'\|^2_{\mathcal{H}} \leq \mu\|u'\|_{\mathcal{H}}\|u_\mu - u\|_{\mathcal{H}}. \tag{20}$$

On the other hand, we have

$$\begin{aligned} \|u_\mu(t) - u(t)\|^2 &= \|\int_0^t [u'_\mu(s) - u'(s)]\,ds\|^2 \\ &\leq \Big(\int_0^T \|u'_\mu(s) - u'(s)\|\,ds\Big)^2 \\ &\leq T\|u'_\mu - u'\|^2_{\mathcal{H}}, \quad t \in [0,T]. \end{aligned} \tag{21}$$

By combining (20) and (21), we get

$$\|u_\mu(t) - u(t)\|^2 \leq \frac{T\mu}{\varepsilon}\|u'\|_{\mathcal{H}}\|u_\mu - u\|_{\mathcal{H}},\ t \in [0,T]. \tag{22}$$

Hence,

$$\|u_\mu - u\|_{\mathcal{H}} \leq \frac{T^2}{\varepsilon}\|u'\|_{\mathcal{H}} \cdot \mu, \tag{23}$$

so $\|u_\mu - u\|_{\mathcal{H}} = \mathcal{O}(\mu)$, which is one of the desired estimates (see (19)). Finally, we can derive from (22) and (23) the other desired estimate: $\|u_\mu - u\|_{C([0,T];H)} = \mathcal{O}(\mu^{1/2})$ □

# 4 Applications

In this section we illustrate our results with two applications.

## 4.1 Lions regularization of the nonlinear heat equation

Consider the following problem, denoted $(\Pi_0)$,

$$\begin{cases} u_t - cu_{xx} + B(u) = f(t,x), \quad 0 < t < T,\ 0 < x < 1, \\ u(t,0) = 0 = u(t,1), \quad 0 < t < T, \\ u(0,x) = u_0(x), \quad 0 < x < 1, \end{cases}$$

which describes, in particular, the heat flow in a homogeneous metal rod, assuming that the temperature at the ends of the rod is zero. Here, $c > 0$ denotes the thermal diffusivity, $u_0$ is the initial temperature distribution, $f$ represents an external heat source, and $B$ is a function defined on $\mathbb{R}$ which is Lipschitz continuous (with a Lipschitz constant $L > 0$) and bounded. Note that $B$ might represent a distributed thermostatic control, in this case being a piecewise linear function with a finite number of knots. For an example of such a function $B$, see, e.g., [9, Fig. 2, p. 20].

Let us associate with the problem $(\Pi_0)$ the following Lions regularization, denoted $(\Pi_\varepsilon)$,

$$\begin{cases} -\varepsilon u_{tt} + u_t - cu_{xx} + B(u) = f(t,x), \quad 0 < t < T,\ 0 < x < 1, \\ u(t,0) = 0 = u(t,1), \quad 0 < t < T, \\ u(0,x) = u_0(x),\ u_t(T,x) = 0, \quad 0 < x < 1, \end{cases}$$

where $\varepsilon$ is a small positive parameter. Here, the parameter $\mu$ (the coefficient of $u_t$) is equal to 1.

We can state the following specific result:

**Theorem 4.** *Let the above conditions be satisfied. If $u_0 \in H_0^1(0,1) \cap H^2(0,1)$ and $f \in L^2(0,T;L^2(0,1)) = L^2((0,T) \times (0,1))$, then, for every $\varepsilon \in (0, 1/(8L))$, the problems $(\Pi_\varepsilon)$ and $(\Pi_0)$ have unique solutions, $u_\varepsilon \in W^{2,2}(0,T;L^2(0,1))$ and respectively $u \in W^{1,2}(0,T;L^2(0,1))$, and the following estimates hold true*

$$\|u_\varepsilon - u\|_{C([0,T];L^2(0,1))} = \mathcal{O}(\varepsilon^{1/4}) \quad \textit{and} \quad \|u_\varepsilon - u\|_{L^2(0,T;L^2(0,1))} = \mathcal{O}(\varepsilon^{1/2}). \quad (24)$$

*If, in addition, $B$ is a nondecreasing function, then the above assertions, including the estimates (24), hold true for every $\varepsilon > 0$.*

*If, in particular, $B = 0$ and $f \in W^{1,1}(0,T;L^2(0,1))$, then $u \in C^1([0,T]; L^2(0,1))$.*

*Proof.* Let $H = L^2(0,1)$ equipped with the usual scalar product and norm. Define $A : D(A) \subset H \to H$ by

$$D(A) = H_0^1(0,1) \cap H^2(0,1), \quad Av = -cv'' \ \ \forall v \in D(A).$$

It is well known that $A$ is self-adjoint, maximal monotone in $H$. More precisely, $A$ is the subdifferential of $\phi : H \to (-\infty, +\infty]$,

$$\phi(v) = \begin{cases} \frac{c}{2} \int_0^1 |v'(x)|^2 dx, & v \in H_0^1(0,1), \\ +\infty, & v \in H \setminus H_0^1(0,1). \end{cases}$$

So, for the first part of the theorem (Theorem 4) one can apply Theorem 2, since $(\Pi_\varepsilon)$ and $(\Pi_0)$ can be written as the following problems in $H = L^2(0,1)$

$$\begin{cases} -\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t), \ 0 < t < T, \\ u(0) = u_0, \ u'(T) = 0, \end{cases}$$

where $u(t) := u(t, \cdot) \in H$, $f(t) := f(t, \cdot) \in H$, and

$$\begin{cases} u'(t) + Au(t) + Bu(t) = f(t), \ 0 < t < T, \\ u(0) = u_0. \end{cases}$$

If, in addition, $B$ is a nondecreasing function, then it is the derivative (subdifferential) of a $C^1$ convex function (see, e.g., [15, p. 42]), and so is the canonical extension of $B$ to $H$, again denoted $B$. This extension is also Lipschitz, hence $A + B$ is maximal monotone, being a subdifferential operator. So, in this case, one can apply Remark 1.

The last assertion of the theorem follows from Lemma 1. □

## 4.2 Lions regularization of the telegraph differential system

Consider in $D_T = \{(t, x) : 0 \leq t \leq T, \, 0 \leq x \leq 1\}$ the following problem

$$\begin{cases} -\varepsilon u_{tt} + \mu u_t + v_x + ru = f_1(t, x), \\ -\varepsilon v_{tt} + cv_t + u_x + gv = f_2(t, x), \end{cases} \tag{25}$$

subject to

$$\begin{cases} u(0, x) = u_0(x), \ v(0, x) = v_0(x), \\ u_t(T, x) = 0, \ v_t(T, x) = 0, \ 0 < x < 1, \end{cases} \tag{26}$$

$$-v(t, 0) \in \alpha(u(t, 0)), \ v(t, 1) \in \beta(u(t, 1)), \ 0 < t < T, \tag{27}$$

where $\varepsilon$, $\mu$, $c$, $r$, $g$ are positive numbers; $\alpha$, $\beta$ are maximal monotone mappings $: \mathbb{R} \to \mathbb{R}$; and $f_1, \, f_2 \in L^2((0.T) \times (0, 1))$.

Note that the above problem is a Lions type regularization of the following problem, denoted $(P_{\mu c})$, associated with the telegraph differential system,

$$\begin{cases} \mu u_t + v_x + ru = f_1(t, x), \\ cv_t + u_x + gv = f_2(t, x), \ 0 < t < T, \ 0 < x < 1, \end{cases} \tag{28}$$

$$u(0, x) = u_0(x), \ v(0, x) = v_0(x), \ 0 < x < 1, \tag{29}$$

$$-v(t,0) \in \alpha(u(t,0)),\ v(t,1) \in \beta(u(t,1)),\ 0 < t < T, \tag{30}$$

where $\mu$, $c$, $r$, $g$ represent the inductance, capacitance, resistance, and capacitance of an electrical circuit (long line) (see, e.g., [1, Chapter 3] and [8]). If $\alpha$ and $\beta$ are linear functions, then (30) become equalities representing the classic Ohm's law at the ends $x = 0$ and $x = 1$. The problems (25), (26), (27) and (28), (29), (30) can be written as problems in $H = L^2(0,1) \times L^2(0,1)$, equipped with the scalar product

$$\big((z_1,z_2),(w_1,w_2)\big) = \int_0^1 z_1 w_1\,dx + \int_0^1 z_2 w_2\,dx \quad \forall (z_1,z_2),(w_1,w_2) \in H,$$

and the corresponding Hilbertian norm. These problems are associated with the operator $A : D(A) \subset H \to H$ defined by

$$D(A) = \{(z_1,z_2) \in H^1(0,1) \times H^1(0,1);\ -z_2(0) \in \alpha(z_1(0)),\ z_2(1) \in \beta(z_1(1))\},$$

$$A(z_1,z_2) = (z_2' + rz_1, z_1' + gz_2) \quad \forall (z_1,z_2) \in D(A),$$

and look as follows

$$\begin{cases} -\varepsilon(u''(t),v''(t)) + (\mu u'(t), cv'(t)) + A(u(t),v(t)) = f(t), \quad 0 < t < T, \\ (u,v)(0) = (u_0,v_0),\ (u',v')(T) = 0, \end{cases} \tag{31}$$

and

$$\begin{cases} (\mu u'(t), cv'(t)) + A(u(t),v(t)) = f(t), \quad 0 < t < T, \\ (u,v)(0) = (u_0,v_0), \end{cases} \tag{32}$$

where $u(t) := u(t,\cdot)$, $v(t) := v(t,\cdot)$, $f(t) := (f_1(t,\cdot), f_2(t,\cdot))$.

It is well-known (see [15, Lemma 4.1, p. 250]) that $A : D(A) \subset H \to H$ is maximal monotone (but it is not a subdifferential). So, if $c = \mu$, we can use our results in Section 3 to obtain specific results for the present problems (25), (26), (27) and (28), (29), (30). We leave to the reader this task. Here we consider a case which is a bit different from the cases **(i)**, **(ii)**, namely

**(iii)** $\varepsilon > 0$, $c > 0$ are fixed and $0 < \mu \to 0$.

If $\varepsilon$ is small then the case **(iii)** corresponds to the practical situation of a telegraph system where the inductance $\mu$ tends to zero. We have

**Theorem 5.** *Assume that $\varepsilon$ and $c$ are fixed positive numbers, $r \geq 0$, $g \geq 0$, and $\alpha$, $\beta$ are maximal monotone mappings. Then, for every $\mu \geq 0$, $u_0$, $v_0 \in H^1(0,1)$ such that $-v_0(0) \in \alpha(u_0(0))$, $v_0(1) \in \beta(u_0(1))$, and*

*$f_1$, $f_2 \in L^2((0,T)\times(0,1))$, the problem (31) has a unique solution $(u_\mu, v_\mu) \in W^{2,2}(0,T;\, L^2(0,1))^2$, and the following estimates hold true*

$$\|u_\mu - u\|_{C([0,T];L^2(0,1))} = \mathcal{O}(\mu^{1/2}), \quad \|v_\mu - v\|_{C([0,T];L^2(0,1))} = \mathcal{O}(\mu^{1/2}), \tag{33}$$

$$\|u_\mu - u\|_{L^2(0,T;L^2(0,1))} = \mathcal{O}(\mu), \quad \|v_\mu - v\|_{L^2(0,T;L^2(0,1))} = \mathcal{O}(\mu), \tag{34}$$

*where $(u,v)$ is the solution of (31) corresponding to $\mu = 0$ (this notation is used to avoid confusion with $(u_0, v_0)$).*

*Proof.* The proof is similar to that of Theorem 3, with some differences due to the fact that here we have a system of differential equations instead of a single differential equation. Existence and uniqueness of $(u_\mu, v_\mu) \in W^{2,2}(0,T;\, L^2(0,1))^2$ is derived in a standard manner (see Lemma 3). Now, subtracting (28) from (25), multiplying each resulting equation by $u_\mu - u$ and $v_\mu - v$, integrating over $(0,T)\times(0,1)$, and using the monotonicity of $A$, we get for $\mu > 0$

$$\begin{aligned}
&\varepsilon \int_0^T \left(\|(u_\mu)_t - u_t\|^2_{L^2(0,1)} + \|(v_\mu)_t - v_t\|^2_{L^2(0,1)}\right) dt \\
&+\mu \int_0^T ((u_\mu)_t - u_t + u_t, u_\mu - u)_{L^2(0,1)} dt \\
&+c \int_0^T ((v_\mu)_t - v_t, v_\mu - v)_{L^2(0,1)}\, dt \le 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varepsilon \int_0^T &\left(\|(u_\mu)_t - u_t\|^2_{L^2(0,1)} + \|(v_\mu)_t - v_t\|^2_{L^2(0,1)}\right) dt \\
&\le -\mu \int_0^T (u_t, u_\mu - u)_{L^2(0,1)} dt,
\end{aligned}$$

which implies

$$\varepsilon\|(u_\mu)_t - u_t\|^2_{\mathcal{H}} + \varepsilon\|(v_\mu)_t - v_t\|^2_{\mathcal{H}} \le \mu\|u_t\|_{\mathcal{H}}\|u_\mu - u\|_{\mathcal{H}}, \tag{35}$$

where $\mathcal{H} = L^2(0,T;L^2(0,1))$. On the other hand, we have for all $t \in [0,T]$

$$\|u_\mu(t,\cdot) - u(t,\cdot)\|^2_{L^2(0,1)} \le T\|(u_\mu)_t - u_t\|^2_{\mathcal{H}}. \tag{36}$$

Now, from (35) and (36) we deduce

$$\|u_\mu - u\|^2_{\mathcal{H}} \le \frac{T^2\mu}{\varepsilon}\|u_t\|_{\mathcal{H}}\|u_\mu - u\|_{\mathcal{H}},$$

and so

$$\|u_\mu - u\|_{\mathcal{H}} = \mathcal{O}(\mu), \tag{37}$$

which is the first estimate in (34). Then, using (35), (36) and (37), we deduce the first estimate in (33). The other desired estimates in (33) and (34) follow easily. □

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