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ITERATIVE COMPUTING THE MINIMAL SOLUTION OF THE COUPLED NONLINEAR MATRIX EQUATIONS IN TERMS OF NONNEGATIVE MATRICES*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70^{th} anniversary

Abstract

We investigate a set of nonlinear matrix equations with nonnegative matrix coefficients which has arisen in applied sciences. There are papers where the minimal nonnegative solution of the set of nonlinear matrix equations is computed applying the different procedures. The alternate linear implicit method and its modifications have intensively investigated because they have simple computational scheme. We construct a new decoupled modification of the alternate linear implicit procedure to compute the minimal nonnegative solution of the considered set of equations. The convergence properties of the proposed iteration are derived and a sufficient condition for convergence is derived. The performance of the proposed algorithm is illustrated on several numerical examples. On the basis of the experiments we derive conclusions for applicability of the computational schemes.
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1 Introduction

In the field of control theory and application, the research on a positive system and its application is a hot topic. We need to solve a nonlinear matrix equation with an M-matrix very often. The general nonsymmetric matrix Riccati equation associated with M-matrices has many applications - in the Markov chains [8], in the transport theory [14] and many others. Nonsymmetric Riccati equation $XCX - XD - AX + B = 0$ arises from the game theory and more specially from the investigation of the open-loop Nash linear quadratic differential game [6, 1, 16].

A more general problem on connected to the properties of the stabilising solution of the game theoretic algebraic Riccati equation is investigated in [4, 5, 10].

Research on numerical methods to compute the minimal nonnegative solution of the set of nonsymmetric coupled Riccati equations (SNCRE) associated with M-matrices is important topic in recent years. Zhang and Tan [17] have investigated two numerical iteration methods for solving a SNCRE, i.e. the inexact Newton method and the alternate linear implicit method (ALI). They used these methods to compute the minimal nonnegative solution of the SNCRE. They have proved the convergence properties of these iterations.

In this paper we propose two new effective modifications of the ALI method. Convergence properties are discussed. Moreover, we show that $A_i - \tilde{X}_i C_i$ and $D_i - C_i \tilde{X}_i\,, i = 1, \ldots, s$ are M-matrices, where $(\tilde{X}_1, \ldots, \tilde{X}_s)$ is the minimal solution to the SNCRE. Numerical examples show the effectiveness of the new modifications.

The notation $\mathbf{R}^{s\times q}$ stands for $s \times q$ real matrices. We exploit the properties of nonnegative matrices. A matrix $A = (a_{ij}) \in \mathbf{R}^{m\times n}$ is a nonnegative matrix if the inequalities $a_{ij} \geq 0$ are satisfied for all $1 \leq i \leq m$ and $1 \leq j \leq n$. We use an elementwise order relation. The inequality $P \geq Q (P > Q)$ for $P = (p_{ij}), Q = (q_{ij})$ means that $p_{ij} \geq q_{ij} (p_{ij} > q_{ij})$ for all indexes i and j. A matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is said to be a Z-matrix if it has nonpositive off-diagonal entries. Any Z-matrix A can be written in the form $A = \alpha I - N$ with N being a nonnegative matrix. Each M-matrix is a Z-matrix with if $\alpha \geq \rho(N)$, where $\rho(N)$ is the spectral radius of N. It is called a nonsingular M-matrix if $\alpha > \rho(N)$ and a singular M-matrix if $\alpha = \rho(N)$. A square matrix is said to be a c-stable matrix if every eigenvalue has negative real part.

We exploit the following properties of M-matrices.

Lemma 1 *[3, 11] The following statements are equivalent for a Z-matrix (-W):*

(a) $-W$ is a nonsingular M-matrix;

(b) $(\theta I_n - W)$ is a nonsingular M-matrix, where $\theta < 0$ and I_n is the $n \times n$ unit matrix;

(c) $W^{-1} \leq 0$ (in elementwise order);

(d) All eigenvalues of W have negative real parts, i.e. W is stable.

Lemma 2 *[9] Let $A = (a_{ij})$ be an $n \times n$ M-matrix. If the elements of $B = (b_{ij})$ satisfy the relations:*

$$a_{ii} \geq b_{ii}\,, \qquad (a_{ij}) \leq (b_{ij}) \leq 0\,, i \neq j,\ i,j = 1,\ldots,n\,,$$

then B is also an M-matrix.

2 A set of Riccati equations

Consider a SNCRE:

$$\mathcal{R}_i(X_1,\ldots,X_s) := X_i C_i X_i - X_i D_i - A_i X_i + B_i + \sum_{j\neq i} e_{ij} X_j = 0\,, \qquad (1)$$

$i = 1,\ldots,s$ introduced in [17]. The matrix coefficients are $A_i = (a^i_{kp}) \in R^{m\times m}, B_i \in R^{m\times n}, C_i \in R^{n\times m}, D_i = (d^i_{kp}) \in R^{n\times n}$ and $(X_1,\ldots,X_s)$ is a solution of the set of equations (1) with $X_i \in R^{m\times n}, i = 1,\ldots,s$. Entries of $E = (e_{ij})$ are nonnegative constants.

The couple of matrices $(\tilde{X}_1,\ldots,\tilde{X}_s)$ is the minimal nonnegative solution to (1) if $\tilde{X}_i \leq X_i, i = 1,\ldots,s$ (elementwise order) for any nonnegative solution $(X_1,\ldots,X_s)$ to (1).

Define the ALI iterative method introduced by Zhang and Tan [17] with initial matrices $X_i^{(0)} = 0 \in R^{n\times n} (m = n)$. The method uses positive constants $\gamma_i, i = 1,\ldots,s$ which are computed via ((31),[17]):

$$\gamma_i = \max\{max_j a^i_{jj},\ max_j d^i_{jj}\}\,. \qquad (2)$$

$k = 0,1,2,\ldots$:

$$\begin{aligned} &Y_i^{(k)}(\gamma_i I_n + D_i - C_i X_i^{(k)}) = (\gamma_i I_n - A_i) X_i^{(k)} + B_i + \textstyle\sum_{j\neq i} e_{ij} X_j^{(k)} \qquad (3)\\ &(\gamma_i I_n + A_i - Y_i^{(k)} C_i) X_i^{k+1} = Y_i^{(k)}(\gamma_i I_n - D_i) + B_i + \textstyle\sum_{j\neq i} e_{ij} Y_j^{(k)}\,. \end{aligned}$$

We propose two modifications of iteration (3).

The first one is:

$$
\begin{aligned}
& X_i^{(0)} = 0, i = 1, 2, \ldots s\,, \\
& \text{For } k = 0, 1, 2, \ldots \text{ one computes :} \\
& Y_i^{(k)}(\gamma_i I_n + D_i) = (\gamma_i I_n - A_i + X_i^{(k)} C_i) X_i^{(k)} + B_i + \textstyle\sum_{j\neq i} e_{ij} X_j^{(k)} \\
& (\gamma_i I_n + A_i) X_i^{(k+1)} = Y_i^{(k)}(\gamma_i I_n - D_i + C_i Y_i^{(k)}) + B_i + \textstyle\sum_{j\neq i} e_{ij} Y_j^{(k)}\,.
\end{aligned}
\tag{4}
$$

Motivation for iteration (4) is that the inversion of the matrices $(\gamma_i I_n + D_i)$ and $(\gamma_i I_n + A_i)$ are executed in the beginning of the iterative process, i.e. it is only one time.

The second iteration is based on the transformation $\gamma_i I_n + D_i - C_i X_i^{(k)} = L_i^{(k)} - U_i^{(k)}, i = 1, 2, \ldots s,\ k = 0, 1, 2, \ldots$, where $L_i^{(k)}$ is the lower triangular part of the given matrix and $U_i^{(k)}$ is the strictly upper triangular part. Then

$$
\begin{aligned}
& X_i^{(0)} = 0, i = 1, 2, \ldots s\,, \\
& \text{For } k = 0, 1, 2, \ldots \text{ one computes :} \\
& Y_i^{(k)} L_i^{(k)} = (\gamma_i I_n - A_i) X_i^{(k)} + X_i^{(k)} U_i^{(k)} + B_i + \textstyle\sum_{j\neq i} e_{ij} X_j^{(k)} \\
& (\gamma_i I_n + A_i) X_i^{(k+1)} = Y_i^{(k)}(\gamma_i I_n - D_i + C_i Y_i^{(k)}) + B_i + \textstyle\sum_{j\neq i} e_{ij} Y_j^{(k)}\,.
\end{aligned}
\tag{5}
$$

3 Convergence properties

Lemma 3 *We construct the matrix sequences $\{X_i{}^{(k)}, Y_i{}^{(k)}\}_{k=0}^{\infty}$ using (4) with initial values $X_i^{(0)} = 0$. Then for any positive k , the following equalities hold: :*

(i) $\mathcal{R}_i(X_1{}^{(k)}, \ldots, X_s{}^{(k)}) \;=\; (Y_i^{(k)} - X_i^{(k)})(\gamma_i I_n + D_i)\,, \quad i = 1, \ldots, , s$

(ii) $\mathcal{R}_i(Y_1{}^{(k)}, \ldots, Y_s{}^{(k)}) = \; (\gamma_i I_n - A_i + X_i^{(k)} C_i)(Y_i^{(k)} - X_i^{(k)})$
$+(Y_i^{(k)} - X_i^{(k)})\, C_i Y_i^{(k)} + \sum_{j\neq i} e_{ij}(Y_j^{(k)} - X_j^{(k)})\,,$
$i = 1, \ldots, , s\,,$

(iii) $\mathcal{R}_i(Y_1{}^{(k)}, \ldots, Y_s{}^{(k)}) = (\gamma_i I_n + A_i)(X_i^{(k+1)} - Y_i^{(k)})\,, \qquad i = 1, \ldots, s\,,$

(iv) $\mathcal{R}_i(X_1{}^{(k+1)}, \ldots, X_s{}^{(k+1)}) = (X_i^{(k+1)} - Y_i^{(k)})(\gamma_i I_n - D + C_i Y_i^{(k)})$
$+X_i^{(k+1)} C_i\, (X_i^{(k+1)} - Y_i^{(k)}) + \sum_{j\neq i} e_{ij}(X_j^{(k+1)} - X_j^{(k)})\,,$
$i = 1, \ldots, s\,.$

In addition, the following equalities are true for any nonnegative matrices $\hat{X}_1, \ldots, \hat{X}_s$*:*

$$\text{(v)} \quad \begin{aligned} \mathcal{R}_i(\hat{X}_1, \ldots, \hat{X}_s) \quad &= (Y_i^{(k)} - \hat{X}_i)(\gamma_i I_n + D_i) + (\hat{X}_i - X_i^{(k)}) C_i X_i^{(k)} \\ &+ (\gamma_i I_n - A_i + \hat{X}_i C_i)(\hat{X}_i - X_i^{(k)}) + \textstyle\sum_{j \neq i} e_{ij} (\hat{X}_j - X_j^{(k)}) , \\ i = 1, \ldots, s \, , \end{aligned}$$

$$\text{(vi)} \quad \begin{aligned} \mathcal{R}_i(\hat{X}_1, \ldots, \hat{X}_s) &= (\gamma_i I_n + A_i)(X_i^{(k+1)} - \hat{X}_i) + \hat{X}_i C_i (\hat{X}_i - Y_i^{(k)}) \\ &+ (\hat{X}_i - Y_i^{(k)})(\gamma_i I_n - D_i + C_i Y_i^{(k)}) \\ &+ \textstyle\sum_{j \neq i} e_{ij} (\hat{X}_j - Y_j^{(k)}) , \quad i = 1, \ldots, s \, . \end{aligned}$$

Proof. The proof is completed by a direct calculation.

Theorem 1 *Assume* $A_i, D_i, i = 1, \ldots, s$ *are Z matrices. There are positive numbers* γ_i*, such that* $(\gamma_i \, I_n + A_i)$ *and* $(\gamma_i \, I_n + D_i)$ *are nonsingular M-matrices* $i = 1, \ldots, s$*. Matrices* $B_i, C_i, i = 1, \ldots, s$ *are nonnegative. Assume there exist nonnegative matrices* $\hat{X}_1, \ldots, \hat{X}_s$*, such that*
$\mathcal{R}_i(\hat{X}_1, \ldots, \hat{X}_s) \leq 0 \, , i = 1, \ldots, s \, .$

The matrix sequences $\{X_1^{(k)}, \ldots, X_s^{(k)}\}_{k=0}^{\infty}$ *defined by (4) satisfy the following properties:*

(i) $\hat{X}_i \geq X_i^{(k+1)} \geq Y_i^{(k)} \geq X_i^{(k)}$ *for* $i = 1, \ldots, s \, , \quad k = 0, 1, \ldots;$

$$\text{(ii)} \quad \begin{aligned} &\mathcal{R}_i(X_1^{(k)}, \ldots, X_s^{(k)}) \geq 0 \, , \qquad \mathcal{R}_i(Y_1^{(k)}, \ldots, Y_s^{(k)}) \geq 0 \, , \\ &\mathcal{R}_i(X_1^{(k+1)}, \ldots, X_s^{(k+1)}) \geq 0 \, , \quad i = 1, \ldots, s \, , \quad k = 0, 1, \ldots \, . \end{aligned}$$

(iii) *The matrix sequences* $\{X_1^{(k)}, \ldots, X_s^{(k)}\}_{k=0}^{\infty}$ *converge to the nonnegative minimal solution* $\tilde{X}_1, \ldots, \tilde{X}_s$ *to the set of Riccati equations*
$\mathcal{R}_1(X_1, \ldots X_s)) = 0 \, , \ldots, \mathcal{R}_s(X_1, \ldots X_s) = 0$ *with* $\tilde{X}_i \leq \hat{X}_i \, , i = 1, \ldots, s$*.*

(iv) *Moreover, if* $A_i - \hat{X}_i C_i$ *and* $D_i - C_i \hat{X}_i \, , i = 1, \ldots, s$ *are nonsingular M-matrices, then* $A_i - \tilde{X}_i C_i$ *and* $D_i - C_i \tilde{X}_i \, , i = 1, \ldots, s$ *are nonsingular M-matrices, i.e. matrices* $-A_i + \tilde{X}_i C_i$ *and* $-D_i + C_i \tilde{X}_i \, , i = 1, \ldots, s$ *are c-stable.*

Proof. We begin with the facts that $(\gamma_i I_n + A_i)^{-1} \geq 0$, and $(\gamma_i I_n + D_i)^{-1} \geq 0, i = 1, \ldots, s$. We construct the matrix sequences
$\{{X_1}^{(k)}, \ldots, {X_s}^{(k)}, {Y_1}^{(k)}, \ldots, {Y_s}^{(k)}\}_{k=0}^{\infty}$ applying recursive equations (4) with $X_1^{(0)} = \ldots = X_s^{(0)} = 0$ and γ_i, computed by (2). We confirm the facts $\gamma_i I_n - D_i \geq 0$ and $\gamma_i I_n - A_i \geq 0 \, , i = 1, \ldots, s$.

For $k = 0$ we obtain $Y_i^{(0)}(\gamma_i I_n + D_i) = B_i \geq 0$ and thus $Y_i^{(0)} = B_i(\gamma_i I_n + D_i)^{-1} \geq 0$. And $Y_i^{(0)} \geq X_i^{(0)} = 0, i = 1, \ldots, s$. In addition, $\mathcal{R}_i({X_1}^{(0)}, \ldots, {X_s}^{(0)}) = B_i \geq 0, i = 1, \ldots, s$.

Applying Lemma 3 (ii), we get

$$\mathcal{R}_i({Y_1}^{(0)}, \ldots, {Y_s}^{(0)}) = (\gamma_i I_n - A_i)Y_i^{(0)} + Y_i^{(0)} C_i Y_i^{(0)} + \textstyle\sum_{j\neq i} e_{ij} Y_j^{(0)} \geq 0\,.$$

To compute $X_i^{(1)}$ we have

$$(\gamma_i I_n + A_i)X_i^{(1)} = W_i^{(0)} \geq 0\,,$$

where

$$W_i^{(0)} := Y_i^{(0)}(\gamma_i I_n - D_i + C_i Y_i^{(0)}) + B_i + \sum_{j\neq i} e_{ij} Y_j^{(0)}\,.$$

We obtain $X_i^{(1)}$ is nonnegative $i = 1, \ldots, s$.

Applying Lemma 3 (iii), we get

$$(X_i^{(1)} - Y_i^{(0)}) = (\gamma_i I_n + A_i)^{-1}\, \mathcal{R}_i({Y_1}^{(k)}, \ldots, {Y_s}^{(k)}) \geq 0\,.$$

Further on, we compute $X_1^{(1)}, \ldots, X_s^{(1)}$ applying the recursive equation (4). According to Lemma 3 (iv) we induce

$$\begin{aligned}\mathcal{R}_i({X_1}^{(1)}, \ldots, {X_s}^{(1)}) &= (X_i^{(1)} - Y_i^{(0)})(\gamma_i I_n - D + C_i Y_i^{(0)})\\ &+X_i^{(1)} C_i (X_i^{(1)} - Y_i^{(0)}) + \textstyle\sum_{j\neq i} e_{ij}(X_j^{(1)} - X_j^{(0)}) \ \geq 0\,,\\ & i = 1, \ldots, s\,,\end{aligned}$$

because $\gamma_i\, I_n - D_i \geq 0, X_i^{(1)} \geq Y_i^{(0)} \geq X_i^{(0)}, i = 1, \ldots, s$.

In order to prove $\hat{X}_i \geq X_i^{(1)}$ we consider equality Lemma 3 (v)

$$\begin{aligned}\mathcal{R}_i(\hat{X}_1, \ldots, \hat{X}_s) \ &= (Y_i^{(0)} - \hat{X}_i)(\gamma_i I_n + D_i)\\ &+(\gamma_i I_n - A_i + \hat{X}_i C_i)\hat{X}_i + \textstyle\sum_{j\neq i} e_{ij}\hat{X}_j \geq 0\,.\end{aligned}$$

Note that $\gamma_i I_n - A_i \geq 0$. We have

$$(Y_i^{(0)} - \hat{X}_i) = H_i^{(0)}\,(\gamma_i I_n + D_i)^{-1} \leq 0\,,$$

because

$$H_i^{(0)} := \mathcal{R}_i(\hat{X}_1, \ldots, \hat{X}_s) - (\gamma_i I_n - A_i + \hat{X}_i C_i)\hat{X}_i - \sum_{j\neq i} e_{ij}\hat{X}_j \ \leq 0\,,\ i = 1, \ldots, s\,.$$

Thus $\hat{X}_i \geq Y_i^{(0)} i = 1, \ldots, s$. Moreover, applying equality Lemma 3 (vi) we obtain

$$(\gamma_i I_n + A_i)(X_i^{(1)} - \hat{X}_i) = \mathcal{R}_i(\hat{X}_1, \ldots, \hat{X}_s) - (\hat{X}_i - Y_i^{(0)})(\gamma_i I_n - D_i + C_i Y_i^{(k)})$$
$$-\hat{X}_i C_i(\hat{X}_i - Y_i^{(0)}) - \textstyle\sum_{j \neq i} e_{ij}(\hat{X}_j - Y_j^{(0)}), \quad i = 1, \ldots, s.$$

We infer $\hat{X}_i \geq X_i^{(1)}, i = 1, \ldots, s$.

So, we have proved inequalities (i) - (ii) for $k = 0$.

We assume that the inequalities (i) - (ii) hold for $k = 0, 1, \ldots, r$. We know matrices $X_i^{(r+1)}, i = 1, \ldots, s$ satisfy the properties:

$$\hat{X}_i \geq X_i^{(r+1)} \geq Y_i^{(r)} \geq X_i^{(r)}, \quad \imath = 1, \ldots, s,$$

and

$$\mathcal{R}_i(X_1^{(r)}, \ldots, X_s^{(r)}) \geq 0, \quad \mathcal{R}_i(Y_1^{(r)}, \ldots, Y_s^{(r)}) \geq 0,$$
$$\mathcal{R}_i(X_1^{(r+1)}, \ldots, X_s^{(r+1)}) \geq 0, \quad i = i = 1, \ldots, s.$$

We will prove the inequalities (i) - (ii) for $k = r + 1$.

We compute $Y_i^{(r+1)}, i = 1, \ldots, s$ via (4), i.e.

$$Y_i^{(r+1)} = [(\gamma_i I_n - A_i + X_i^{(r+1)} C_i) X_i^{(r+1)} + B_i + \sum_{j \neq i} e_{ij} X_j^{(r+1)}]\,(\gamma_i I_n + D_i)^{-1} \geq 0.$$

According to Lemma 3 (i) we extract

$$Y_i^{(r+1)} - X_i^{(r+1)} = \mathcal{R}_i({X_1}^{(r+1)}, \ldots, {X_s}^{(r+1)})(\gamma_i I_n + D_i)^{-1} \geq 0, \quad i = 1, \ldots, s.$$

From Lemma 3 (ii), we conclude

$$\mathcal{R}_i({Y_1}^{(r+1)}, \ldots, {Y_s}^{(r+1)}) = (\gamma_i I_n - A_i + X_i^{(r+1)} C_i)(Y_i^{(r+1)} - X_i^{(r+1)})$$
$$+(Y_i^{(r+1)} - X_i^{(r+1)}) C_i Y_i^{(r+1)} + \textstyle\sum_{j \neq i} e_{ij}(Y_j^{(r+1)} - X_j^{(r+1)}) \geq 0,$$
$$i = 1, \ldots, , s.$$

We compute $X_i^{(r+2)}, i = 1, \ldots, s$ via the second equation of (4). Consider the equality (iii) of Lemma 3 for $k = r + 1$. We write down:

$$X_i^{(r+2)} - Y_i^{(r+1)} = (\gamma_i I_n + A_i)^{-1}\, \mathcal{R}_i({Y_1}^{(r+1)}, \ldots, {Y_s}^{(r+1)}) \geq 0, i = 1, \ldots, s.$$

Next, we apply of Lemma 3 (iv) for

$$\begin{aligned}&\mathcal{R}_i({X_1}^{(r+2)}, \dots, {X_s}^{(r+2)}) = (X_i^{(r+2)} - Y_i^{(r+1)})(\gamma_i I_n - D_i + C_i Y_i^{(r+1)})\\ &\qquad X_i^{(r+2)} C_i (X_i^{(r+2)} - Y_i^{(r+1)}) + \textstyle\sum_{j\neq i} e_{ij}(X_j^{(r+2)} - X_j^{(r+1)}) \geq 0\,,\\ &\qquad\quad i = 1, \dots, s\,.\end{aligned}$$

Thus $\mathcal{R}_i({X_1}^{(r+2)}, \dots, {X_s}^{(r+2)}) \geq 0\,,\ i = 1, \dots, s\,.$

In order to prove $\hat{X}_i \geq X_i^{(r+2)}$ we consider equality Lemma 3 (v)

$$\begin{aligned}\mathcal{R}_i(\hat{X}_1, \dots, \hat{X}_s) \;&= (Y_i^{(r+1)} - \hat{X}_i)(\gamma_i I_n + D_i)\\ &+(\gamma_i I_n - A_i + \hat{X}_i C_i)(\hat{X}_i - X_i^{(r+1)}) - (\hat{X}_i - X_i^{(r+1)}) C_i X_i^{(r+1)}\\ &+ \textstyle\sum_{j\neq i} e_{ij}(\hat{X}_j - X_j^{(r+1)})\,, \quad i = 1, \dots, s\,.\end{aligned}$$

Then

$$Y_i^{(r+1)} - \hat{X}_i = H_i^{(r+1)}\,(\gamma_i I_n + D_i)^{-1} \leq 0\,,$$

because $H_i^{(r+1)} \leq 0$, and

$$\begin{aligned}H_i^{(r+1)} &:= \mathcal{R}_i(\hat{X}_1, \dots, \hat{X}_s) - (\gamma_i I_n - A_i + \hat{X}_i C_i)(\hat{X}_i - X_i^{(r+1)})\\ &\quad -(\hat{X}_i - X_i^{(r+1)}) C_i X_i^{(r+1)} - \textstyle\sum_{j\neq i} e_{ij}(\hat{X}_j - X_j^{(r+1)})\,, \quad i = 1, \dots, s\,.\end{aligned}$$

Thus $\hat{X}_i \geq Y_i^{(r+1)} i = 1, \dots, s\,.$

Further on, taking into account Lemma 3 (vi) we obtain

$$X_i^{(r+2)} - \hat{X}_i = (\gamma_i I_n + A_i)^{-1}\, T_i^{(r+1)} \leq 0\,,$$

because $T_i^{(r+1)} \leq 0$, and

$$\begin{aligned}T_i^{(r+1)} &:= \mathcal{R}_i(\hat{X}_1, \dots, \hat{X}_s) - (\hat{X}_i - Y_i^{(r+1)})(\gamma_i I_n - D_i + C_i Y_i^{(r+1)})\\ &\quad -\hat{X}_i C_i (\hat{X}_i - Y_i^{(r+1)}) - \textstyle\sum_{j\neq i} e_{ij}(\hat{X}_j - Y_j^{(r+1)})\,, \quad i = 1, \dots, s\,.\end{aligned}$$

We infer $\hat{X}_i \geq X_i^{(r+2)}\,, i = 1, \dots, s\,.$

Hence, the induction process has been completed. Thus the matrix sequences $\{X_1^{(k)}, \dots, X_s^{(k)}\}_{k=0}^{\infty}$ are nonnegative, monotonically increasing and bounded from above by $(\hat{X}_1, \dots, \hat{X}_s)$ (in the elementwise ordering). We denote $\lim_{k\to\infty}(X_1^{(k)}, \dots, X_s^{(k)}) = (\tilde{X}_1, \dots, \tilde{X}_s)$. By taking the limits in (4) it follows that $(\tilde{X}_1, \dots, \tilde{X}_s)$ is a solution of $\mathcal{R}_i(X_1, \dots, X_s) = 0, i = 1, \dots, s$ with the property $\tilde{X}_i \leq \hat{X}_i, i = 1, \dots, s$.

Assume there is another solution $(\tilde{Z}_1, \ldots, \tilde{Z}_s)$ with $\tilde{Z}_i \leq \tilde{X}_i$. There exists sufficiently large index r such that $X_i^{(r+1)} \geq \tilde{Z}_i \geq Y_i^{(r)} \geq X_i^{(r)}$, $i = 1, \ldots, s$.

Applying Lemma 3 (vi) for $\hat{X}_i = \tilde{Z}_i$, $i = 1, \ldots, s$, we get

$$\begin{aligned} 0 = (\gamma_i I_n + A_i)(X_i^{(r+1)} - \tilde{Z}_i) + (\tilde{Z}_i - Y_i^{(r)})(\gamma_i I_n - D_i + C_i Y_i^{(r)}) \\ + \tilde{Z}_i C_i (\tilde{Z}_i - Y_i^{(r)}) + \textstyle\sum_{j\neq i} e_{ij}(\tilde{Z}_j - Y_j^{(r)}) \,, \quad i = 1, \ldots, s \,. \end{aligned}$$

We rewrite

$$(\gamma_i I_n + A_i)(X_i^{(r+1)} - \tilde{Z}_i) = Q_i^{(r)} \,.$$

The matrix $Q_i^{(r)}$ is nonpositive because $\tilde{Z}_i \geq Y_i^{(r)}, i = 1, \ldots, s$. Thus $X_i^{(r+1)} - \tilde{Z}_i$ is nonpositive, which is a contradiction with the assumption $X_i^{(r+1)} \geq \tilde{Z}_i, i = 1, \ldots, s$. We infer the solution $(\tilde{X}_1, \ldots, \tilde{X}_s)$ is the minimal one.

We know $\tilde{X}_i \leq \hat{X}_i \,, i = 1, \ldots, s$ and thus $A_i - \hat{X}_i C_i \leq A_i - \tilde{X}_i C_i$, $i = 1, \ldots, s$. In addition $A_i - \hat{X}_i C_i$ is a nonsingular M-matrix and $A_i - \tilde{X}_i C_i$ is a Z-matrix. Applying Lemma 2 , we conclude that $A_i - \tilde{X}_i C_i$ is a nonsingular M-matrix for $i = 1, \ldots, s$ and moreover $-A_i + \tilde{X}_i C_i, i = 1, \ldots, s$ is c-stable. In similar way we refer that $D_i - C_i \tilde{X}_i \,, i = 1, \ldots, s$ are nonsingular M-matrices.

The theorem is proved.

4 Numerical Examples

We apply the proposed new iterations (4) and (5) and ALI method for computing the minimal nonnegative solution to (1). We will show the effectiveness of the proposed new iterations. We compare the numerical behaviour of the new iterations with the ALI iteration method (ALI) in [17]. Two hundred runs are executed for each example for a fixed value of n (the size on matrix coefficients). All iterative methods are executed in MATLAB (version R2014a) on a personal computer. The iterations stop when the current iterative step satisfies $RES_i \leq 1.0e-12$, where RES_i is defined as [17]:

$$RES_i := \frac{\|\mathcal{R}_i({X_1}^{(k)}, \ldots, {X_s}^{(k)})\|}{\|\mathcal{R}_i({X_1}^{(0)}, \ldots, {X_s}^{(0)})\|} \,,$$

$i = 1, \ldots, s$.

In the experiments, we choose the parameters γ_i as defined in (2). We take $X_1^{(0)} = \ldots = X_s^{(0)} = 0$ for all examples and all iterative methods. Thus $\mathcal{R}_i({X_1}^{(0)}, \ldots, {X_s}^{(0)}) = B_i$.

Example 1. We introduce an example with the matrix coefficients with different values of n: $B_1 = 0.75\, I_n$, $B_2 = B_1$, $B_3 = B_1$, $C_1 = 0.92\, I_n$, $C_2 = C_1$, $C_3 = C_1$, where I_n is an identity matrix order n. The matrices $A_i, D_i, i = 1, 2, 3$ are given in Matlab terminology as follows:

A_1=eye(n,n);
for i=1:n-1, A_1(i,i+1)=-1; A_1(i+1,i)=-0.1; end
for i=1:n-2, A_1(i,i+2)=-0.2; A_1(i+2,i)=-0.25; end
A_2=A_1; A_3=A_1;
D_1=A_1/5; D_2=4A_2/3; D_3=3A_3/2;
for i=1:n, A_1(i,i)=4; A_2(i,i)=3; A_3(i,i)=2; end
for i=1:n, D_1(i,i)=2; D_2(i,i)=4; D_3(i,i)=6; end

We take $X_1^{(0)} = X_2^{(0)} = X_3^{(0)} = 0$ and thus $\mathcal{R}(X_1^{(0)}, X_2^{(0)}, X_3^{(0)}) = B_i \geq 0$, (i.e. the matrix is nonnegative). We take an $s \times s$ matrix ($s = 3$)

$$E = (e_{ij}) = \begin{pmatrix} 0.0661 & 0.4512 & 0.8887 \\ 0.4965 & 0.3156 & 0.8780 \\ 0.6542 & 0.8914 & 0.1947 \end{pmatrix}.$$

Table 1.

	(3)		(4)		(5)	
n	*It*	CPU	*It*	CPU	*It*	CPU
12	33	1.6s	34	1.4s	36	1.4s
18	35	2.8s	37	2.0s	39	2.5s
36	39	10.4s	43	7.8s	44	9.3s
48	40	17.2s	43	13.5s	46	15.4s
55	41	23.4s	43	17.4s	46	21.0s

Example 2. We introduce an example with the matrix coefficients with different values of n: $B_1 = 0.75\, I_n$, $B_2 = B_1$, $B_3 = B_1$, $C_1 = 0.92\, I_n$, $C_2 = C_1$, $C_3 = C_1$, where I_n is an identity matrix order n. The matrices $A_i, D_i, i = 1, 2, 3$ are given in Matlab terminology as follows:

A_1=eye(n,n);
for i=1:n-1, A_1(i,i+1)=-0.5; A_1(i+1,i)=-0.03; end
for i=1:n-2, A_1(i,i+2)=-0.25; A_1(i+2,i)=-0.9; end
A_1(1,n) = -0.05; A_1(n,1) = -0.4;
A_2=A_1; A_2(1,n) = -0.8; A_2(n,1) = -0.06;
A_3=A_1; A_3(1,n) = -0.7; A_3(n,1) = -0.09;
D_1=A_1/5; D_2=4*A_2/3; D_3=3*A_3/2;
for i=1:n, A_1(i,i)=4; A_2(i,i)=3; A_3(i,i)=2; end
for i=1:n, D_1(i,i)=2; D_2(i,i)=4; D_3(i,i)=6; end

Table .2.

	(3)		(4)		(5)	
n	It	CPU	It	CPU	It	CPU
12	41	1.9s	44	1.5s	39	1.0s
18	45	3.7s	48	2.6s	51	3.2s
36	50	13.2s	54	10.0s	56	11.7s
48	52	22.0s	55	18.0s	58	28.1s
55	52	31.6s	56	23.1s	58	26.9s

5 Conclusion

We have made numerical experiments for computing the minimal nonnegative solution to a set of nonsymmetric Riccati equations (1). The numerical results are compared. These results confirm the effectiveness of the proposed new modifications of the decoupled iterations. In addition, the new proposed decoupled iterations have inside possibilities for improving their implementation.

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