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A CORRESPONDING VECTORIAL FORM OF DERIVATIVE OF BIQUATERNIONIC FUNCTIONS*

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Abstract

In this paper, we give the notation and properties of the vectorial form of biquaternions. The differential operators and calculations result from a modified multiplication with the vectorial form.

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1 Introduction

The ordinary biquaternions are named by Hamilton [3] in 1844. Since then, more developments of theories of biquaternions. Kravchenko [7] gave a review of some results obtained with quaternionic analysis and quaternionic reformulations for electromagnetic fields and for Dirac's spinors. Ward [9] discovered significant uses for the quaternion and Cayley number algebra in physics and gave various representations of certain topics in particularly relativity. Buchheim [1] extended properties of biquaternions contained an outline of a calculus devised by Clifford's sketch for the analytical treatment of the theory of screws. Girard [2] showed various physical covariance groups such as the Lorentz group, the general theory of relativity group and the conformal group related to the quaternion group. Sangwine et al. [8]

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introduced the fundamental properties of biquaternions presented including several different representations, some of them new, and definitions of fundamental operations such as the scalar and vector parts and conjugates. Kim and Shon [4, 5] gave extended theories of hyperholomorphic functions in special quaternions and their calculations of derivatives. Also, Kim and Shon [6] researched equivalent forms of the corresponding Cauchy-Riemann system of regular functions of special quaternionic variables such as reduced quaternions and split quaternions.
The algebra of biquaternions can be considered as a tensor product $\mathbb{C} \otimes \mathbb{H}$ on $\mathbb{R}$ (the field of real numbers), where $\mathbb{C}$ is the field of complex numbers and $\mathbb{H}$ is the algebra of real quaternions. In the definition of quaternions, one supposes all components of quaternions are complex numbers. Then, one obtains the definition of extended quaternions which are called also biquaternions. Thus, a biquaternion P has the form $P = z_0 + z_1 i + z_2 j + z_3 k$, where z_r $(r = 0, 1, 2, 3)$ are complex numbers and i, j, k are the quaternionic imaginary units. The quaternions form a division algebra, that is, for each non-zero element there exists an inverse. While, biquaternions have the property of associativity but there exist zero divisors which do not have inverses.
In this paper, we introduce an additional and multiplicational law for biquaternions and we research the differential operators applied to a pure multiplication with the biquaternionic relative velocity and motions. Also, we investigate modified representations of formulas and forms of biquaternions with differentials.

2 Preliminaries

A quaternion is a number of the form :

$$Q = x_0 + x_1 i + x_2 j + x_3 k,$$

where $x_r \in \mathbb{R}$ $(r = 0, 1, 2, 3)$ and

$$i^2 = j^2 = k^2 = ijk = -1,$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

A biquaternion is a number of the form :

$$P = z_0 + z_1 i + z_2 j + z_3 k,$$

where $z_r \in \mathbb{C}$ $(r = 0, 1, 2, 3)$, the base i, j and k satisfy

$$i^2 = j^2 = k^2 = ijk = -1$$

and the remaining relations

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

The set of biquaternions is denoted by $\mathbb{BQ}$ and it is isomorphic to $\mathbb{C}^4$. A biquaternion is also expressed

$$\begin{aligned} P &= z_0 + z_1 i + z_2 j + z_3 k \\ &= (x_0 + y_0 i) + (x_1 + y_1 i)i + (x_2 + y_2 i)j + (x_3 + y_3 i)k. \end{aligned}$$

For avoiding this confusion by renaming i, j,and k:

$$P = (x_0 + y_0 i) + (x_1 + y_1 i)e_1 + (x_2 + y_2 i)e_2 + (x_3 + y_3 i)e_3,$$

where $e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1$, $i = \sqrt{-1}$ and $ie_r = e_r i$ $(r = 1, 2, 3)$. So, P can be written as the complex combination of two quaternions:

$$\begin{aligned} P &= (x_0 + y_0 i) + (x_1 + y_1 i)e_1 + (x_2 + y_2 i)e_2 + (x_3 + y_3 i)e_3 \\ &= (x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3) + i(y_0 + y_1 e_1 + y_2 e_2 + y_3 e_3), \end{aligned}$$

where $x_r,\ y_r \in \mathbb{R}$ $(r = 0, 1, 2, 3)$. Then we have

$$P = p + iq,$$

where $p, q \in \mathbb{H}$ and the set of biquaternions is isomorphic to $\mathbb{H}^2$. Consider the matrices corresponded with e_1, e_2 and e_3:

$$e_1 \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 \leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Further,

$$e_1 e_2 = -e_2 e_1 = e_3$$

$$\leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

It is easy to prove the remaining identities. Then, each of these matrices has a square equal to the negative of the identity matrix. By using these matrices, we can represent a biquaternion $P = z_0 + z_1 e_1 + z_2 e_2 + z_3 e_3$. Then $\mathbb{BQ}$ is isomorphic to $M_{2\times 2}(C)$, that is, we give a linear transformation $f : \mathbb{BQ} \to M_{2\times 2}(C)$ which is one-to-one and onto, as follows:

$$f(z_0 + z_1 e_1 + z_2 e_2 + z_3 e_3) = \begin{pmatrix} z_0 + z_1 i & z_2 + z_3 i \\ -z_2 + z_3 i & z_0 - z_1 i \end{pmatrix},$$

where $z_r \in \mathbb{C}$ $(r = 0, 1, 2, 3)$.

Proposition 1. *Each formula for the addition and the multiplication of two biquaternions p and q is the same as for quaternions:*
The addition

$$P + Q = (z_0 + z_1e_1 + z_2e_2 + z_3e_3) + (w_0 + w_1e_1 + w_2e_2 + w_3e_3)$$

is closed, associative and commutative. The additive identity is

$$0 = 0 + 0e_1 + 0e_2 + 0e_3$$

and the opposite of P is

$$-P = -z_0 + (-z_1)e_1 + (-z_2)e_2 + (-z_3)e_3.$$

The multiplication

$$\begin{aligned} PQ &= (z_0 + z_1e_1 + z_2e_2 + z_3e_3)(w_0 + w_1e_1 + w_2e_2 + w_3e_3) \\ &= (z_0w_0 + z_0w_1e_1 + z_0w_2e_2 + z_0w_3e_3) \\ &\quad +(z_1w_0e_1 - z_1w_1 + z_1w_2e_3 - z_1w_3e_2) \\ &\quad +(z_2w_0e_2 - z_2w_1e_3 - z_2w_2 + z_2w_3e_1) \\ &\quad +(z_3w_0e_3 + z_3w_1e_2 - z_3w_2e_1 - z_3w_3) \end{aligned}$$

is closed, associative and not commutative. The multiplicative identity is

$$1 = (1 + 0i) + 0e_1 + 0e_2 + 0e_3.$$

Since there are zero divisors in $\mathbb{BQ}$, we consider the following elements of $\mathbb{BQ}$

(i) The biconjugate: $P^* = z_0 - z_1e_1 - z_2e_2 - z_3e_3$;
(ii) The complex conjugate: $P^\dagger = \overline{z_0} + \overline{z_1}e_1 + \overline{z_2}e_2 + \overline{z_3}e_3$;
(iii) The complex biconjugate: $P^\dagger = \overline{z_0} - \overline{z_1}e_1 - \overline{z_2}e_2 - \overline{z_3}e_3$,
where $\overline{z_r} = x_r - y_ri$ $(r = 0, 1, 2, 3)$. We note that

$$PP^* \in \mathbb{C},$$

$$PP^\dagger \in \mathbb{R} + \mathbb{C}e_1 + \mathbb{C}e_2 + \mathbb{C}e_3,$$

$$P\overline{P} \in \mathbb{R} + \mathbb{R}ie_1 + \mathbb{R}ie_2 + \mathbb{R}ie_3,$$

$$(PQ)^* = Q^*P^*, \quad (PQ)^\dagger = P^\dagger Q^\dagger, \quad \overline{(PQ)} = \overline{Q}\ \overline{P}.$$

We have $PP^* = z_0^2 + z_1^2 + z_2^2 + z_3^2$. If $PP^* = 0$, then P is a zero divisor. If $x_r \neq 0$ $(r = 0, 1, 2, 3)$, $PP^* \neq 0$ and $PP^* \in \mathbb{C}$. Then, the analogous inverse, called bi-inverse, is defined as follows:

$$P^{-1} = P^*(PP^*)^{-1} = \frac{P^*}{PP^*} \qquad (x_r \neq 0, \quad r = 0, 1, 2, 3).$$

3 Hyperholomorphic function of biquaternionic variables

Consider the following analogues of the Fueter operator

$$D_F^* := \frac{\partial}{\partial z_0} + e_1\frac{\partial}{\partial z_1} + e_2\frac{\partial}{\partial z_2} - e_3\frac{\partial}{\partial z_3}$$

and

$$D_F = \frac{\partial}{\partial z_0} - e_1\frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial z_2} + e_3\frac{\partial}{\partial z_3},$$

where $\frac{\partial}{\partial z_r} = \frac{\partial}{\partial x_r} - i\frac{\partial}{\partial y_r}$ $(r = 0, 1, 2, 3)$. Then the complex Laplacian in $\mathbb{C}^4$ is

$$\Delta_{\mathbb{C}^4} = D_F D_F^* = \frac{\partial^2}{\partial z_0^2} + \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \frac{\partial^2}{\partial z_3^2}.$$

Let $\Omega \subset \mathbb{C}^4$ and let F be a function as

$$\begin{cases} F : \Omega \ \to \ \mathbb{BQ}, \\ F(P) = f_0 + f_1e_1 + f_2e_2 + f_3e_3, \end{cases}$$

where $f_r = f_r(z_0, z_1, z_2, z_3)$ $(r = 0, 1, 2, 3)$ are complex-valued functions.

Definition 1. *Let Ω be an open set in $\mathbb{BQ}$. A function F is said to be hyperholomorphic in Ω if the following limit*

$$F'(P) = \lim_{\triangle P \to 0} (\triangle P)^{-1}\{F(P + \triangle P) - F(P)\}, \tag{1}$$

exists, where $\triangle P$ is different from P to a point P_0, that is, $\triangle P = (\triangle z_0) + (\triangle z_1)e_1 + (\triangle z_2)e_2 + (\triangle z_3)e_3$ with $\triangle z_r = \triangle x_r + i\triangle y_r$. Also, $\triangle P \to 0$ implies that each component approaches to zero and $\triangle x_r \neq 0$ $(r = 0, 1, 2, 3)$. Moreover, the limit is said to be the derivative of F, denoted by $F'(P)$.

The limit (1) is calculated as follows:

$$\begin{aligned} F'(P) &= \lim_{\triangle P \to 0} (\triangle P)^{-1}\{F(P + \triangle P) - F(P)\} \\ &= \lim_{\triangle P \to 0} \frac{(\triangle P)^*\{F(P + \triangle P) - F(P)\}}{(\triangle P)(\triangle P)^*} \\ &= \lim_{\substack{\triangle z_t \to 0 \\ (t=0,1,2,3)}} \frac{(\triangle z_0 - \sum_{r=1}^3 (\triangle z_r)e_r)(\sum_{r=1}^3 (\triangle f_r)e_r)}{\sum_{r=1}^3 (\triangle z_r)^2}. \end{aligned}$$

If the limit (1) exists, the limit satisfies the following equations:

$$\frac{\partial f_0}{\partial z_0}=\frac{\partial f_1}{\partial z_1}=\frac{\partial f_2}{\partial z_2}=\frac{\partial f_3}{\partial z_3},\quad \frac{\partial f_1}{\partial z_0}=-\frac{\partial f_0}{\partial z_1}=-\frac{\partial f_3}{\partial z_2}=\frac{\partial f_2}{\partial z_3},$$

$$\frac{\partial f_2}{\partial z_0}=\frac{\partial f_3}{\partial z_1}=-\frac{\partial f_0}{\partial z_2}=-\frac{\partial f_1}{\partial z_3},\quad \frac{\partial f_3}{\partial z_0}=-\frac{\partial f_2}{\partial z_1}=\frac{\partial f_1}{\partial z_2}=-\frac{\partial f_0}{\partial z_3},$$

where $\frac{\partial}{\partial z_r}$ $(r=0,1,2,3)$ are differential operators in complex analysis. The above system is called the corresponding Cauchy-Riemann system of biquaternionic variables.

Theorem 1. *Let* Ω *be an open set in* $\mathbb{BQ}$ *and let* f_r $(r=0,1,2,3)$ *are continuously differential functions in* Ω*. A function* F *is hyperholomorphic in* Ω *if and only if each* f_r $(r=0,1,2,3)$ *is a continuously differential function such that* $\frac{\partial f_r}{\partial z_t}$ $(r,t=0,1,2,3)$ *are continuous in* Ω *and a function* F *satisfies* $D_F^*F=0$ *in* Ω*.*

Proof. Suppose that a function F is hyperholomorphic in Ω. Then F satisfies the corresponding Cauchy-Riemann system of biquaternionic variables. Hence, by using the corresponding Cauchy-Riemann system, we have f_r $(r=0,1,2,3)$ are continuously differential functions and

$$\begin{aligned}
D_F^*F &= \Big(\sum_{r=0}^{2} e_r\frac{\partial}{\partial z_r}-e_3\frac{\partial}{\partial z_3}\Big)\Big(\sum_{r=0}^{3} f_re_r\Big)\\
&= \frac{\partial f_0}{\partial z_0}-\frac{\partial f_1}{\partial z_1}-\frac{\partial f_2}{\partial z_2}+\frac{\partial f_3}{\partial z_3}+\Big(\frac{\partial f_0}{\partial z_1}+\frac{\partial f_1}{\partial z_0}+\frac{\partial f_3}{\partial z_2}+\frac{\partial f_2}{\partial z_3}\Big)e_1\\
&\quad+\Big(\frac{\partial f_2}{\partial z_0}+\frac{\partial f_0}{\partial z_2}-\frac{\partial f_3}{\partial z_1}-\frac{\partial f_1}{\partial z_3}\Big)e_2+\Big(\frac{\partial f_3}{\partial z_0}-\frac{\partial f_0}{\partial z_3}+\frac{\partial f_2}{\partial z_1}-\frac{\partial f_1}{\partial z_2}\Big)e_3\\
&= 0.
\end{aligned}$$

Conversely, the equation $D_F^*F=0$ is equivalent to the following system:

$$\begin{cases}
\dfrac{\partial f_0}{\partial z_0}-\dfrac{\partial f_1}{\partial z_1}-\dfrac{\partial f_2}{\partial z_2}+\dfrac{\partial f_3}{\partial z_3}=0,\\
\dfrac{\partial f_0}{\partial z_1}+\dfrac{\partial f_1}{\partial z_0}+\dfrac{\partial f_3}{\partial z_2}+\dfrac{\partial f_2}{\partial z_3}=0,\\
\dfrac{\partial f_2}{\partial z_0}+\dfrac{\partial f_0}{\partial z_2}-\dfrac{\partial f_3}{\partial z_1}-\dfrac{\partial f_1}{\partial z_3}=0,\\
\dfrac{\partial f_3}{\partial z_0}-\dfrac{\partial f_0}{\partial z_3}+\dfrac{\partial f_2}{\partial z_1}-\dfrac{\partial f_1}{\partial z_2}=0.
\end{cases}\tag{2}$$

and by referring [9] and combining each above equation, we have

$$\begin{aligned}\frac{\triangle F}{\triangle P} &= \frac{\sum_{r=0}^{3}\triangle z_r\triangle f_r}{\sum_{r=0}^{3}(\triangle z_r)^2} + \frac{\triangle z_0\triangle f_1 - \triangle z_1\triangle f_0 - \triangle z_2\triangle f_3 + \triangle z_3\triangle f_2}{\sum_{r=0}^{3}(\triangle z_r)^2} \\ &\quad + \frac{\triangle z_0\triangle f_2 + \triangle z_1\triangle f_3 - \triangle z_2\triangle f_0 - \triangle z_3\triangle f_1}{\sum_{r=0}^{3}(\triangle z_r)^2} \\ &\quad + \frac{\triangle z_0\triangle f_3 - \triangle z_1\triangle f_2 + \triangle z_2\triangle f_1 - \triangle z_3\triangle f_0}{\sum_{r=0}^{3}(\triangle z_r)^2}\end{aligned}$$

and

$$\begin{aligned}&\Big|\frac{(\triangle z_0 - \sum_{r=1}^{3}(\triangle z_r)e_r)(\sum_{r=1}^{3}(\triangle f_r)e_r)}{\sum_{r=1}^{3}(\triangle z_r)^2} \\ &-\Big(\frac{\partial f_0}{\partial z_0}dz_0 + \frac{\partial f_1}{\partial z_1}dz_1 + \frac{\partial f_2}{\partial z_2}dz_2 + \frac{\partial f_3}{\partial z_3}dz_3\Big)\Big| \to 0\end{aligned}$$

as $\triangle z_r \to 0$ $(r = 0, 1, 2, 3)$. Thus, we obtain that the limit (1) exists. □

Theorem 2. *Let Ω be an open set in $\mathbb{BQ}$. If a function F is hyperholomorphic in Ω, then $D_F F = \frac{\partial F}{\partial z_0}$.*

Proof. Since F is hyperholomorphic in Ω, F satisfies $D_F^* F = 0$, that is, F satisfies the system (2). Then we have the following equations:

$$\begin{aligned}D_F F &= \frac{\partial f_0}{\partial z_0} + \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} - \frac{\partial f_3}{\partial z_3} + e_1\Big(-\frac{\partial f_0}{\partial z_1} + \frac{\partial f_1}{\partial z_0} - \frac{\partial f_3}{\partial z_2} - \frac{\partial f_2}{\partial z_3}\Big) \\ &\quad + e_2\Big(\frac{\partial f_2}{\partial z_0} - \frac{\partial f_0}{\partial z_2} + \frac{\partial f_3}{\partial z_1} + \frac{\partial f_1}{\partial z_3}\Big) + e_3\Big(\frac{\partial f_3}{\partial z_0} + \frac{\partial f_0}{\partial z_3} - \frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial z_2}\Big) \\ &= \frac{\partial f_0}{\partial z_0} + \frac{\partial f_1}{\partial z_0}e_1 + \frac{\partial f_2}{\partial z_0}e_2 + \frac{\partial f_3}{\partial z_0}e_3 = \frac{\partial F}{\partial z_0}.\end{aligned}$$

Thus, we obtain that $D_F F = \frac{\partial F}{\partial z_0}$. □

Since an element of $\mathbb{BQ}$ is

$$\begin{aligned}P &= z_0 + z_1e_1 + z_2e_2 + z_3e_3 \\ &= (x_0 + y_0 i) + (x_1 + y_1 i)e_1 + (x_2 + y_2 i)e_2 + (x_3 + y_3 i)e_3 \\ &= [x_0 + i(y_1e_1 + y_2e_2 + y_3e_3) + iy_0 + (x_1e_1 + x_2e_2 + x_3e_3)],\end{aligned}$$

generally a biquaternion is grouped into two vectors

$$P = p + q = \begin{pmatrix} x_0 \\ i\vec{x} \end{pmatrix} + \begin{pmatrix} iy_0 \\ \vec{y} \end{pmatrix}. \qquad (3)$$

Both terms p and q are called reduced biquaternions. For two biquaternions

$$P = p + q = \begin{pmatrix} a_0 \\ i\vec{a} \end{pmatrix} + \begin{pmatrix} ib_0 \\ \vec{b} \end{pmatrix} \quad \text{and} \quad Q = a + b = \begin{pmatrix} x_0 \\ i\vec{x} \end{pmatrix} + \begin{pmatrix} iy_0 \\ \vec{y} \end{pmatrix},$$

the multiplication of two biquaternions is

$$\begin{aligned} PQ &= \begin{pmatrix} a_0x_0 - b_0y_0 + \vec{x}\cdot\vec{a} - \vec{b}\cdot\vec{y} \\ i(a_0\vec{x} + x_0\vec{a} + b_0\vec{y} + y_0\vec{b} + \vec{a}\times\vec{y} - \vec{b}\times\vec{x}) \end{pmatrix} \\ &\quad + \begin{pmatrix} i(a_0y_0 + b_0x_0 - \vec{a}\cdot\vec{y} - \vec{b}\cdot\vec{x}) \\ a_0\vec{y} + x_0\vec{b} - b_0\vec{x} - y_0\vec{a} - \vec{a}\times\vec{x} + \vec{b}\times\vec{y} \end{pmatrix}. \end{aligned}$$

We give the inner product on $\mathbb{BQ}$ as:

$$P \cdot Q = a_0x_0 - \vec{a}\cdot\vec{x} - (b_0y_0 - \vec{b}\cdot\vec{y}).$$

Thus, the modulus of a biquaternion is

$$|P| = \sqrt{P\cdot P} = \sqrt{a_0^2 - \vec{a}\cdot\vec{a} - (b_0^2 - \vec{b}\cdot\vec{b})}.$$

In a special case, let the second of the two modified biquaternions be zero, that is, $q = b = 0$. Then the multiplication of two biquaternions is

$$PQ = \begin{pmatrix} a_0x_0 + \vec{x}\cdot\vec{a} \\ i(a_0\vec{x} + x_0\vec{a}) \end{pmatrix} + \begin{pmatrix} 0 \\ -\vec{a}\times\vec{x} \end{pmatrix}.$$

From the representation of the vectorial form (3), we have

$$F = \begin{pmatrix} u_0 \\ i\vec{u} \end{pmatrix} + \begin{pmatrix} iv_0 \\ \vec{v} \end{pmatrix} \quad \text{and} \quad D_F^* = \begin{pmatrix} \dfrac{\partial}{\partial x_0} \\ i\nabla_x^\dagger \end{pmatrix} + \begin{pmatrix} -i\dfrac{\partial}{\partial y_0} \\ -\nabla_y^\dagger \end{pmatrix},$$

where $\vec{u} = u_1e_1 + u_2e_2 + u_3e_3$, $\vec{v} = v_1e_1 + v_2e_2 + v_3e_3$ with $u_r,\ v_r$ $(r = 1, 2, 3)$ are real-valued functions and

$$\nabla_x^\dagger = -e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3} \quad \text{and} \quad \nabla_y^\dagger = -e_1\frac{\partial}{\partial y_1} - e_2\frac{\partial}{\partial y_2} + e_3\frac{\partial}{\partial y_3}.$$

Then we have

$$\begin{aligned} D_F^* F &= \left\{ \begin{pmatrix} \dfrac{\partial}{\partial x_0} \\ i\nabla_x^{\dagger} \end{pmatrix} + \begin{pmatrix} -i\dfrac{\partial}{\partial y_0} \\ -\nabla_y^{\dagger} \end{pmatrix} \right\} \left\{ \begin{pmatrix} u_0 \\ i\vec{u} \end{pmatrix} + \begin{pmatrix} iv_0 \\ \vec{v} \end{pmatrix} \right\} \\ &= \begin{pmatrix} \dfrac{\partial u_0}{\partial x_0} + \nabla_x^{\dagger} \cdot \vec{u} + \dfrac{\partial v_0}{\partial y_0} + \nabla_y^{\dagger} \cdot \vec{v} \\ i(\nabla_x^{\dagger} u_0 + \dfrac{\partial \vec{u}}{\partial x_0} + \nabla_x^{\dagger} \times \vec{v} - \nabla_y^{\dagger} \times \vec{u} - \dfrac{\partial \vec{v}}{\partial y_0} - \nabla_y^{\dagger} v_0) \end{pmatrix} \\ &\quad + \begin{pmatrix} i(\dfrac{\partial v_0}{\partial x_0} - \nabla_x^{\dagger} \cdot \vec{v} - \dfrac{\partial u_0}{\partial y_0} + \nabla_y^{\dagger} \cdot \vec{u}) \\ -\nabla_x^{\dagger} \times \vec{u} + \dfrac{\partial \vec{v}}{\partial x_0} - \nabla_x^{\dagger} v_0 + \dfrac{\partial \vec{u}}{\partial y_0} - \nabla_y^{\dagger} u_0 - \nabla_y^{\dagger} \times \vec{v} \end{pmatrix}. \end{aligned}$$

Hence, if a function F is hyperholomorphic in Ω, then we obtain the corresponding vectorial Cauchy-Riemann system as follows:

$$\begin{cases} \dfrac{\partial u_0}{\partial x_0} + \dfrac{\partial v_0}{\partial y_0} + \nabla_x^{\dagger} \cdot \vec{u} + \nabla_y^{\dagger} \cdot \vec{v} = 0, \\ \dfrac{\partial v_0}{\partial x_0} - \dfrac{\partial u_0}{\partial y_0} - \nabla_x^{\dagger} \cdot \vec{v} + \nabla_y^{\dagger} \cdot \vec{u} = 0, \\ \dfrac{\partial \vec{u}}{\partial x_0} - \dfrac{\partial \vec{v}}{\partial y_0} + \nabla_x^{\dagger} u_0 - \nabla_y^{\dagger} v_0 + \nabla_x^{\dagger} \times \vec{v} - \nabla_y^{\dagger} \times \vec{u} = 0, \\ \dfrac{\partial \vec{v}}{\partial x_0} + \dfrac{\partial \vec{u}}{\partial y_0} - \nabla_x^{\dagger} v_0 - \nabla_y^{\dagger} u_0 - \nabla_x^{\dagger} \times \vec{u} - \nabla_y^{\dagger} \times \vec{v} = 0. \end{cases} \tag{4}$$

Theorem 3. *Let Ω be an open set in $\mathbb{BQ}$ and the differential operator be*

$$D_F = \begin{pmatrix} \dfrac{\partial}{\partial x_0} \\ -i\nabla_x^{\dagger} \end{pmatrix} + \begin{pmatrix} -i\dfrac{\partial}{\partial y_0} \\ \nabla_y^{\dagger} \end{pmatrix}.$$

If F is hyperholomorphic in Ω, then we have $\frac{\partial}{\partial z_0} F$.

Proof. By the corresponding vectorial Cauchy-Riemann system (4), we have

$$\begin{aligned} D_F F &= \begin{pmatrix} \dfrac{\partial u_0}{\partial x_0} - \nabla_x^\dagger \cdot \vec{u} + \dfrac{\partial v_0}{\partial y_0} - \nabla_y^\dagger \cdot \vec{v} \\ i\Big(-\nabla_x^\dagger u_0 + \dfrac{\partial \vec{u}}{\partial x_0} - \nabla_x^\dagger \times \vec{v} + \nabla_y^\dagger \times \vec{u} - \dfrac{\partial \vec{v}}{\partial y_0} + \nabla_y^\dagger v_0\Big) \end{pmatrix} \\ &\quad + \begin{pmatrix} i\Big(\dfrac{\partial v_0}{\partial x_0} + \nabla_x^\dagger \cdot \vec{v} - \dfrac{\partial u_0}{\partial y_0} - \nabla_y^\dagger \cdot \vec{u}\Big) \\ \nabla_x^\dagger \times \vec{u} + \dfrac{\partial \vec{v}}{\partial x_0} + \nabla_x^\dagger v_0 + \dfrac{\partial \vec{u}}{\partial y_0} + \nabla_y^\dagger u_0 + \nabla_y^\dagger \times \vec{v} \end{pmatrix} \\ &= \frac{\partial}{\partial x_0}\left\{\begin{pmatrix} u_0 \\ i\vec{u} \end{pmatrix} + \begin{pmatrix} iv_0 \\ \vec{v} \end{pmatrix}\right\} - i\frac{\partial}{\partial y_0}\left\{\begin{pmatrix} u_0 \\ i\vec{u} \end{pmatrix} + \begin{pmatrix} iv_0 \\ \vec{v} \end{pmatrix}\right\}. \end{aligned}$$

Therefore, we obtain

$$D_F F = \Big(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial y_0}\Big)\left\{\begin{pmatrix} u_0 \\ i\vec{u} \end{pmatrix} + \begin{pmatrix} iv_0 \\ \vec{v} \end{pmatrix}\right\} = \frac{\partial}{\partial z_0} F.$$

□

Example 1. *Let Ω be an open set in $\mathbb{BQ}$. Then, for some $A = \mu_0 + \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3$ and $B = \lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ with $\mu_r, \lambda_r \in \mathbb{C}$, by the vectorial form of A and B such that*

$$A = \begin{pmatrix} \alpha_0 \\ i\vec{\alpha} \end{pmatrix} + \begin{pmatrix} i\beta_0 \\ \vec{\beta} \end{pmatrix} \qquad \textit{and} \qquad B = \begin{pmatrix} a_0 \\ i\vec{a} \end{pmatrix} + \begin{pmatrix} ib_0 \\ \vec{b} \end{pmatrix},$$

the function $F = A + PB$ (also, $F = A + BP$) which is defined and continuously differential in Ω has the following result:

$$\begin{aligned} D_F F(P) &= \left\{\begin{pmatrix} \dfrac{\partial}{\partial x_0} \\ -i\nabla_x^\dagger \end{pmatrix} + \begin{pmatrix} -i\dfrac{\partial}{\partial y_0} \\ \nabla_y^\dagger \end{pmatrix}\right\} \\ &\qquad \left\{\begin{pmatrix} \alpha_0 + a_0 x_0 - b_0 y_0 + \vec{x} \cdot \vec{a} - \vec{b} \cdot \vec{y} \\ i(\vec{\alpha} a_0 \vec{x} + x_0 \vec{a} + b_0 \vec{y} + y_0 \vec{b} + \vec{a} \times \vec{y} - \vec{b} \times \vec{x}) \end{pmatrix}\right. \\ &\qquad \left. + \begin{pmatrix} i(\beta_0 + a_0 y_0 + b_0 x_0 - \vec{a} \cdot \vec{y} - \vec{b} \cdot \vec{x}) \\ \vec{\beta} + a_0 \vec{y} + x_0 \vec{b} - b_0 \vec{x} - y_0 \vec{a} - \vec{a} \times \vec{x} + \vec{b} \times \vec{y} \end{pmatrix}\right\} \\ &= \begin{pmatrix} 4a_0 \\ 0 \end{pmatrix} + \begin{pmatrix} i4\beta_0 \\ 0 \end{pmatrix} = 4\lambda_0, \end{aligned}$$

where $\lambda_0 \in \mathbb{C}$*. Therefore,* $D_F F(P)$ *is the first component of* B*.*

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