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NON-CENTRAL PÓLYA-AEPPLI PROCESS AND RUIN PROBABILITY*

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Abstract

In this paper we introduce a stochastic process which is a sum of Pólya-Aeppli process and homogeneous Poisson process and call it a Non-central Pólya-Aeppli process (NPAP). The probability mass function, recursion formulas and some properties are derived. As application we consider a risk model with NPAP counting process. The joint distribution of the time to ruin and deficit at the time of ruin is derived. The differential equation of the ruin probability is given. As example we consider the case of exponentially distributed claims.
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1 Introduction

In this paper we consider a process which is a sum of two independent stochastic counting processes. The first one is the Pólya-Aeppli process (PAP) which is introduced by Minkova (2004), [3] and characterized by Chukova and Minkova (2013), [1]. It is a compound Poisson process with geometric compounding distribution and has the following probability mass function (PMF)

$$P(N_1(t)=i)=\begin{cases} e^{-\lambda_1 t}, & i=0\\ e^{-\lambda_1 t}\displaystyle\sum_{j=1}^{i}\binom{i-1}{j-1}\frac{[\lambda_1(1-\rho)t]^j}{j!}\rho^{i-j}, & i=1,2,\dots, \end{cases} \tag{1}$$

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where $\lambda_1 > 0$ and $\rho \in [0, 1)$ are parameters. We use the notation $N_1(t) \sim PA(\lambda_1, \rho)$.

The mean and the variance of the Pólya-Aeppli process are given by $E(N_1(t)) = \frac{\lambda_1}{1-\rho}t$ and $Var(N_1(t)) = \frac{\lambda_1(1+\rho)}{(1-\rho)^2}t$. For the Fisher index of dispersion we obtain

$$FI(N_1(t)) = \frac{Var(N_1(t))}{E(N_1(t))} = \frac{1+\rho}{1-\rho}.$$

The second process, denoted by $N_2(t)$ is the homogeneous Poisson process (PP) with parameter $\lambda_2 > 0$, and PMF

$$P(N_2(t) = i) = \frac{(\lambda_2 t)^i}{i!} e^{-\lambda_2 t}, \ i = 0, 1, \ldots \tag{2}$$

We suppose that $N_1(t)$ and $N_2(t)$ are independent and consider the process $N(t) = N_1(t) + N_2(t)$. The probability generating function of the process $N(t)$ is given by

$$\Psi_{N(t)}(s) = e^{-\lambda_1 t(1-\psi_1(s))} e^{-\lambda_2 t(1-s)}, \tag{3}$$

where

$$\psi_1(s) = \frac{(1-\rho)s}{1-\rho s} \tag{4}$$

is the probability generating function of the shifted geometric distribution with success probability $1 - \rho < 1$, denoted by $Ge_1(1 - \rho)$.

Definition 1 *The process $N(t)$, defined by the probability generating function (3) is called a Non-central Pólya-Aeppli process. We denote $N(t) \sim NPAP(\lambda_1, \lambda_2, \rho)$.*

In Ong and Lee (1979) [4] the Noncentral negative binomial distribution arises as a model in photon and neural counting, birth and death processes and mixture models. It is defined as a sum of two independent random variables, one that is negative binomial and another one, Pólya-Aeppli distributed. This motivated us to give the name Non-central Pólya-Aeppli process for the defined process $N(t)$. The dividend problem for the compound Poisson risk model was considered in [5].

The probability mass function of the Non-central Pólya-Aeppli process

$N(t)$ is given by

$$P(N(t)=i) = \begin{cases} e^{-(\lambda_1+\lambda_2)t}, \quad i=0 \\ e^{-(\lambda_1+\lambda_2)t}\left[\frac{(\lambda_2 t)^i}{i!} + \sum_{j=1}^{i} \frac{(\lambda_2 t)^{i-j}}{(i-j)!} \sum_{k=1}^{j} \binom{j-1}{k-1} \frac{[\lambda_1(1-\rho)t]^k}{k!} \rho^{j-k}\right]. \\ \quad i=1,2,\dots \end{cases} \tag{5}$$

The paper is organize as follows. In the next Section 2 we define the Noncentral Pólya-Aeppli process as a pure birth process. Some properties are given. In Section 3, the Non-central Pólya-Aeppli risk model is analyzed. A differential equation for the joint distribution of the time to ruin and the deficit at the time of ruin is derived. As a particular case we obtain the differential equation for the ruin probability. In Section 4 we consider the case of exponentially distributed claims.

2 Non-central Pólya-Aeppli process as a pure birth process

In this section we define the Non-central Pólya-Aeppli process as a pure birth process. Suppose that $\{N(t),\ t \geq 0\}$ is the number of times a certain event occurs in time interval $(0,t]$. The transition probabilities of the counting process $N(t)$ for every $m = 0,1,\dots$ are specified by the following postulates:

$$P(N(t+h)=n \mid N(t)=m) = \begin{cases} [1-(\lambda_1+\lambda_2)h]+o(h), & n=m \\ [\lambda_1(1-\rho)+\lambda_2]h+o(h), & n=m+1 \\ \lambda_1(1-\rho)\rho^{k-1}h+o(h), & n=m+k,\ k=1,2,\dots \end{cases}$$

where $o(h) \to 0$ as $h \to 0$. We denote the probabilities of $N(t)$ by $P_m(t) = P(N(t)=m),\ m=0,1,2,\dots$. Then the above postulates yield to the following Kolmogorov forward equations:

$$\begin{aligned} P_0'(t) &= -(\lambda_1+\lambda_2)P_0(t), \quad m=0 \\ P_m'(t) &= -(\lambda_1+\lambda_2)P_m(t)+(\lambda_1(1-\rho)+\lambda_2)P_{m-1}(t) \\ &\quad +\lambda_1(1-\rho)\sum_{k=2}^{m}\rho^{k-1}P_{m-k}(t),\ m=2,3,\ldots. \end{aligned} \tag{6}$$

with $\sum_2^1 = 0$. Suppose that the following initial conditions are fulfilled

$$P_0(0)=1 \quad \text{and} \quad P_m(0)=0, \quad m=1,2,\ldots. \tag{7}$$

From the equations of (6) we get the following differential equation for the probability generating function of the process $N(t)$

$$\frac{\partial \Psi_{N(t)}(s)}{\partial t} = -[\lambda_1(1-\psi_1(s))+\lambda_2(1-s)]\Psi_{N(t)}(s). \tag{8}$$

The solution of (8) with the initial condition $\Psi_{N(0)}(s)=1$ is given by (3). This leads to the second definition.

Definition 2 *The stochastic process, defined by the differential equations (6) with initial conditions (7) is called a Non-central Pólya-Aeppli process.*

2.1 Moments

By differentiation in (3) we obtain the mean and the variance of the Non-central Pólya-Aeppli process

$$E(N(t)) = \left(\frac{\lambda_1}{1-\rho}+\lambda_2\right)t \quad \text{and} \quad Var(N(t)) = \left[\lambda_1\frac{1+\rho}{(1-\rho)^2}+\lambda_2\right]t.$$

For the Fisher index of dispersion we obtain

$$FI(N(t)) = \frac{\lambda_2(1-\rho)^2+\lambda_1(1+\rho)}{(1-\rho)[\lambda_2(1-\rho)+\lambda_1]}.$$

It is easy to check that

$$FI(N(t)) = 1+\frac{2\lambda_1\rho}{(1-\rho)[\lambda_2(1-\rho)+\lambda_1]},$$

i.e the Non-central Pólya-Aeppli process is over-dispersed related to Poisson process, and

$$FI(N(t)) = \frac{1+\rho}{1-\rho} - \frac{2\lambda_2\rho}{\lambda_2(1-\rho)+\lambda_1} < \frac{1+\rho}{1-\rho},$$

i.e., the Non-central Pólya-Aeppli process is under-dispersed related to Pólya-Aeppli process.

2.2 Recursions

For the PMF $P_m(t) = P(N(t) = m),\ m = 0, 1, \dots$ we have the following recursion formulas

$$\begin{aligned} P_1(t) &= [\lambda_1(1-\rho)+\lambda_2]tP_0(t), \\ P_m(t) &= \left[2\rho + \tfrac{[(\lambda_1(1-\rho)+\lambda_2)t-2\rho]}{m}\right] P_{m-1}(t) \\ &- \left[\rho^2 + 2\tfrac{\rho\lambda_2 t-\rho^2}{m}\right] P_{m-2}(t) + \tfrac{\lambda_2 t\rho^2}{m} P_{m-3}(t) \quad m = 2, 3, \dots, \end{aligned}$$

with $P_{-1}(t) = 0$ and $P_0(t) = e^{-(\lambda_1+\lambda_2)t}$.

3 Application to Risk Models

As application of the Non-central Pólya-Aeppli process we consider the standard risk model $\{X(t),\ t \geq 0\}$, defined on the complete probability space $(\Omega, \mathcal{F}, P)$ and given by

$$X(t) = ct - \sum_{i=1}^{N(t)} Z_i, \quad \left(\sum_{1}^{0} = 0\right). \tag{9}$$

Here c is a positive real constant representing the risk premium rate. The sequence $\{Z_i\}_{i=1}^{\infty}$ of non-negative independent and identically distributed random variables is independent of the counting process $\{N(t),\ t \geq 0\}$ and represents the claim sizes to the insurance company. The claim sizes $\{Z_i\}_{i=1}^{\infty}$ are distributed as the random variable Z with distribution function $F,\ F(0) = 0$ and mean μ.

We consider the risk model (9), where $N(t)$ is a Non-central Pólya-Aeppli process and we call this process Non-central Pólya-Aeppli risk model. The interpretation of the counting process is the following. Suppose that the successive claims are of two types, such that the first type of claims are counted by the Pólya-Aeppli process and the counting process of the second type of claims is the Poisson process. Our interest is in counting all the claims in total. Then, the number of the claims has a Non-central Pólya-Aeppli distribution.

The relative safety loading θ for the risk model in (9), is given by

$$\theta = \frac{EX(t)}{E\sum_{i=1}^{N(t)} Z_i} = \frac{c(1-\rho)}{\mu[\lambda_1 + \lambda_2(1-\rho)]} - 1.$$

In the case of positive safety loading $\theta > 0$, the premium income per unit time c should satisfy the following inequality

$$c > \left(\frac{\lambda_1}{1-\rho} + \lambda_2\right)\mu.$$

Let $\tau = \inf\{t : X(t) < -u\}$ with the convention of $\inf \emptyset = \infty$ be the time to ruin of an insurance company having initial capital $u \geq 0$. We denote by $\Psi(u) = P(\tau < \infty)$ the ruin probability and by $\Phi(u) = 1 - \Psi(u)$ the non-ruin probability. The main in the application is to analyze for this model the joint probability distribution $G(u, y)$ of the time to ruin τ and the deficit at the time of ruin $D = |u + X(\tau)|$. The function $G(u, y)$ is given by

$$G(u, y) = P(\tau < \infty, D \leq y), \quad y \geq 0, \tag{10}$$

see Klugman et al. (2004), [2]. It is clear that

$$\lim_{y \longrightarrow \infty} G(u, y) = \Psi(u) \tag{11}$$

Using the postulates for a sufficiently small h, we get

$$\begin{aligned} G(u, y) &= [1 - (\lambda_1 + \lambda_2)h]G(u + ch, y) \\ &+[\lambda_1(1-\rho) + \lambda_2]h \left[\int_0^{u+ch} G(u + ch - x, y)dF(x)\right. \\ &+ F(u + ch + y) - F(u + ch)] \\ &+\lambda_1(1-\rho)\sum_{k=2}^{\infty} \rho^{k-1} h \left[\int_0^{u+ch} G(u + ch - x, y)dF^{\star k}(x)\right. \\ &+ \left. F^{\star k}(u + ch + y) - F^{\star k}(u + ch)\right] + o(h), \end{aligned}$$

where $F^{\star k}(x), \ \ k = 1, 2, \ldots$ is the distribution function of $Z_1 + Z_2 + \ldots + Z_k$.

Rearranging the terms leads to

$$\begin{aligned}\frac{G(u+ch,y)-G(u,y)}{ch} &= \frac{\lambda_1+\lambda_2}{c}G(u+ch,y)\\
-\frac{\lambda_1(1-\rho)+\lambda_2}{c}&\left[\int_0^{u+ch} G(u+ch-x,y)dF(x)\right.\\
&\left.+F(u+ch+y)-F(u+ch)\right]\\
-\frac{\lambda_1(1-\rho)}{c}\sum_{k=2}^{\infty}\rho^{k-1}&\left[\int_0^{u+ch} G(u+ch-x,y)dF^{\star k}(x)\right.\\
&\left.+F^{\star k}(u+ch+y)-F^{\star k}(u+ch)\right]+\frac{o(h)}{h}.\end{aligned} \tag{12}$$

Let us denote by

$$H(x)=\frac{\lambda_1}{\lambda_1+\lambda_2}(1-\rho)\sum_{k=1}^{\infty}\rho^{k-1}F^{*k}(x)+\frac{\lambda_2}{\lambda_1+\lambda_2}F(x) \tag{13}$$

the probability distribution function of the aggregated claims. It follows from (13), that $H(0)=0$ and $H(\infty)=1$, i.e., $H(x)$ is a proper distribution function. By letting $h\to 0$ in (12), we obtain the following differential equation

$$\frac{\partial G(u,y)}{\partial u}=\frac{\lambda_1+\lambda_2}{c}\left[G(u,y)-\int_0^u G(u-x,y)dH(x)-[H(u+y)-H(u)]\right]. \tag{14}$$

In the following theorem we obtain the initial condition.

Theorem 1 *The function $G(0,y)$ is given by*

$$G(0,y)=\frac{\lambda_1+\lambda_2}{c}\int_0^y[1-H(u)]du. \tag{15}$$

Proof. Integrating (14) from 0 to ∞ with $G(\infty,y)=0$ leads to

$$\begin{aligned}-G(0,y) &= \frac{\lambda_1+\lambda_2}{c}\left[\int_0^{\infty}G(u,y)du\right.\\
&\left.-\textstyle\int_0^{\infty}\int_0^u G(u-x,y)dH(x)du-\int_0^{\infty}(H(u+y)-H(u))du\right]\end{aligned}$$

The change of the variables in the double integral and simple calculations yield to

$$G(0,y)=\frac{\lambda_1+\lambda_2}{c}\int_0^{\infty}[H(u+y)-H(u)]du$$

and then (15).

3.1 Ruin probability

Theorem 2 *For $u \geq 0$, the ruin probability $\Psi(u)$ satisfies the equation*

$$\frac{\partial \Psi(u)}{\partial u} = \frac{\lambda_1 + \lambda_2}{c} \left[\Psi(u) - \int_0^u \Psi(u-x) dH(x) - [1 - H(u)] \right]. \qquad (16)$$

Proof. The result follows from (14) and (11).

Remark 1 *The nonruin probability $\Phi(u)$ satisfies the equation*

$$\frac{\partial \Phi(u)}{\partial u} = \frac{\lambda_1 + \lambda_2}{c} \left[\Phi(u) - \int_0^u \Phi(u-x) dH(x) \right].$$

Theorem 3 *The ruin probability with no initial capital is given by*

$$\Psi(0) = \frac{\mu}{c} \left(\frac{\lambda_1}{1-\rho} + \lambda_2 \right). \qquad (17)$$

Proof. According to (11) and (15) we get

$$\Psi(0) = \lim_{y \to \infty} G(0, y) = \frac{\lambda_1 + \lambda_2}{c} \int_0^\infty [1 - H(u)] du.$$

If X is a random variable with distribution function $H(x)$, then by the definition of $H(x)$ and $EZ = \mu$ we obtain

$$EX = \left(\frac{\lambda_1}{(\lambda_1 + \lambda_2)(1-\rho)} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \right) \mu.$$

Using the fact that $EX = \int\limits_0^\infty [1 - H(x)] dx$ we obtain (17).

4 Exponentially distributed claims

Let us consider the case of exponentially distributed claim sizes with mean μ, i.e. $F(x) = 1 - e^{-\frac{x}{\mu}}, \ x \geq 0, \ \mu > 0$. In this case, for the distribution function $H(x)$ we obtain

$$H(x) = 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\frac{(1-\rho)x}{\mu}} - \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\frac{x}{\mu}},$$

i.e., it is a mixture of two exponentially distributed variables.

For the function $G(0,y)$ in (15) we obtain

$$G(0,y) = \frac{\mu}{c}\left(\frac{\lambda_1}{1-\rho}+\lambda_2\right)\left[1-\frac{\lambda_1}{\lambda_1+\lambda_2(1-\rho)}e^{-\dfrac{(1-\rho)y}{\mu}}-\frac{\lambda_2(1-\rho)}{\lambda_1+\lambda_2(1-\rho)}e^{-\dfrac{y}{\mu}}\right].$$

By differentiation of (16), in the case of exponentially distributed claims, we obtain the following differential equation for the ruin probability

$$\begin{aligned}\frac{\partial^3\Psi(u)}{\partial u^3} &+ \left(\frac{2-\rho}{\mu}-\frac{\lambda_1+\lambda_2}{c}\right)\frac{\partial^2\Psi(u)}{\partial u^2} \\ &+ \left(\frac{1-\rho}{\mu^2}-\frac{\lambda_1+\lambda_2(1-\rho)}{c\mu}\right)\frac{\partial\Psi(u)}{\partial u} = 0\,.\end{aligned} \tag{18}$$

In the case of positive safety loading, the characteristic equation of (18) has a root zero and two negative roots. The solution has the form $\Psi(u) = C_1 + C_2 e^{R_1 u} + C_3 e^{R_2 u}$, where R_1 and R_2 are the nonzero roots of the characteristic equation and $C_i, i = 1,2,3$ are constants. According to the condition $\Psi(\infty) = 0$, it follows that $C_1 = 0$ and the ruin probability is given by

$$\Psi(u) = C_2 e^{R_1 u} + C_3 e^{R_2 u},$$

where $C_2 + C_3 = \frac{\mu}{c}\left(\frac{\lambda_1}{1-\rho}+\lambda_2\right)$.

5 Concluding remarks

In this paper we have introduced a new stochastic process, which is a sum of two well known processes, Pólya-Aeppli and Poisson process. The defined process is a pure birth process and it can be applied as a counting process in risk model. For the risk model we obtain an equation for the joint distribution of the time to ruin and the deficit at the time of ruin. In the case of exponentially distributed claims, the equation for the ruin probability is solved.

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References

[1] Chukova, S., Minkova, L. D. Characterization of the Pólya-Aeppli process. Stochastic Analysis and Applications, **31**, 590–599 (2013).

[2] Klugman, S. A., Panjer, H., Willmot, G. Loss Models. From Data to Decisions. John Wiley & Sons, Inc., 2nd ed. (2004).

[3] Minkova, L. D. The Pólya-Aeppli process and ruin problems. J. Appl. Math. Stoch. Analysis, **3**, 221–234 (2004).

[4] Ong, S. H., Lee, P. A. The non-central negative binomial distribution. Biom. J., **21**, 611–628 (1979).

[5] Shen, Y., Yin, Ch.. Optimal Dividend Problem for a Compound Poisson Risk Model. Applied Mathematics, **5**, 1496-1502 (2014).