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RISK MODELS OF ORDER K*

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Abstract

In this paper we consider two compound processes of order k, one that is a Poisson process of order k and another one, Pólya-Aeppli process of order k. We define a Poisson of order k risk model and we consider a Pólya-Aeppli of order k risk model. For these risk models we define an exponential martingales. The corresponding martingale approximations of the ruin probability for these processes are given. Finally, we compare these models in the case of exponentially distributed claims.

MSC: 60K10; 62P05.

keywords: Poisson process, Pólya-Aeppli process, ruin probability, risk model, martingale approximation

1 Introduction.

We consider the standard risk model $\{X(t),\ t \geq 0\}$, defined on the complete probability space $(\Omega, \mathcal{F}, P)$ and given by

$$X(t) = ct - \sum_{i=1}^{N(t)} Z_i, \quad \left(\sum_{1}^{0} = 0\right). \tag{1}$$

Here c is a positive real constant representing the risk premium rate and $\{N(t),\ t \geq 0\}$ is a counting process.

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In the classical risk model the process $N(t)$ is a homogeneous Poisson process, see Grandell (1991), [2] and Rolski et al. (1999), [18].

It is known that the Poisson process is one of the basic counting processes in the risk theory. For a given insurance company, the process $N(t)$ is interpreted as the number of the claims that arrive to the company during a time interval $[0, t]$. The sequence $\{Z_i\}_{i=1}^{\infty}$ of non-negative independent and identically distributed random variables is independent of the counting process $\{N(t),\ t \geq 0\}$. The random sum $\sum_{i=1}^{N(t)} Z_i$ represents the aggregated claim amount to the insurance company up to time t. The claim sizes $\{Z_i\}_{i=1}^{\infty}$ are distributed as the random variable Z with common distribution function $F,\ F(0) = 0$ and a mean value $\mu = EZ < \infty$.

For the Poisson process is known that its mean and variance are equal but in the insurance and financial data the variance is always greater than the mean. This fact makes the modeling with a Poisson process not so realistic and leads to different types of generalizations of the classical risk model. Many different generalizations of the Poisson process are derived in the years. The most famous of them are concerned with the compound Poisson process. Minkova (2004), see [12] defines a Pólya-Aeppli process which is a compound Poisson process with geometric compounding distribution. This process is characterized in detail by Chukova and Minkova (2013), see [5]. Other generalization of the Poisson process is the non-homogeneous Pólya-Aeppli process, where the intensity parameter is a function of the time, see Chukova and Minkova (2018), [7], Leonenko et al. (2017), [10] and Slimacek and Lindqvist (2016), [19]. Of course there are some modified birth process which are interesting and are given in Minkova (2001), [11].

The measure of the process's dispersion variability is given by the Fisher index of dispersion FI. It is a ratio of the variance to the mean, see Fisher (1934), [8]. The Fisher index of dispersion is equal to one, i.e. $FI(N(t)) = 1$, when $N(t)$ is a Poisson process. Such process is called equi-dispersed process.

The counting process $N(t)$ is over-dispersed when the Fisher index of dispersion is greater than one, i.e. $FI(N(t)) > 1$. The counting process $N(t)$ is under-dispersed when the Fisher index of dispersion is lower than one, i.e. $FI(N(t)) < 1$, see Xekalaki (2006), [21] and Minkova and Balakrishnan (2013), [14].

In the present paper we consider two cases. The first one is when the compounding random variable Y is a discrete uniformly distributed over

$k > 1$ points with probability mass function given by

$$P(Y = i) = \frac{1}{k}, \quad i = 1, 2, \ldots, k.$$

The second one is when the compounding random variable Y is truncated geometric distributed with probability mass function

$$P(Y = i) = \frac{1 - \rho}{1 - \rho^k} \rho^{i-1}, \quad i = 1, 2, \ldots, k.$$

This article is organized as follows. In Section 2 we consider two compound processes of order k, the first one is a Poisson process of order k, see [9] and the second one is a Pólya-Aeppli process of order k, see [6].

In Section 3 we define a Poisson of order k risk model and we consider a Pólya-Aeppli of order k risk model, see [6]. We also define exponential martingales related to these risk models and obtain the corresponding martingale approximations. In Section 4 we compare these models in the case of exponentially distributed claims. Some concluding remarks are given in Section 5.

2 Counting processes of order k.

Let the counting process $N(t)$ is a compound Poisson process with a discrete compounding distribution, i.e.

$$N(t) = Y_1 + Y_2 + \ldots + Y_{N_1(t)}, \tag{2}$$

where $Y_i,\ i = 1, 2, \ldots$ are independent and identically distributed as Y random variables independent of the process $N_1(t)$. The counting process $N_1(t)$ is a Poisson process with intensity $\lambda > 0$. We denote $N_1(t) \sim Po(\lambda t)$.

Let Y is the compounding random variable with probability generating function given by $\psi_Y(s) = E(s^Y),\ s \in (0, 1)$.

The probability mass function and the probability generating function of the process $N_1(t)$ are as follows

$$P(N_1(t) = i) = \frac{(\lambda t)^i e^{-\lambda t}}{i!}, \quad i = 0, 1, \ldots \tag{3}$$

and

$$\psi_{N_1(t)}(s) = e^{-\lambda t(1-s)}. \tag{4}$$

The probability generating function of the process $N(t)$ in (2) is given by

$$\psi_{N(t)}(s) = e^{-\lambda t[1-\psi_Y(s)]}. \tag{5}$$

Definition 1. *If the compounding random variable Y is a discrete distributed, truncated at 0 and from the right away from $k+1$, then the random variable N has a distribution of order k, see [4].*

The discrete distributions of order k were introduced in the early eighties by Philippou et al. (1983), [16] and Philippou and Makri (1986), [17]. It is proved that the discrete distributions of order k can be represented as a Compound Generalized Powers Series Distributions, where the compounding distribution is a discrete distribution over $k \geq 1$ points, see Aki et al. (1984), [1] and Charalambides (1986), [4]. A good reference for these distributions is the book of Balakrishnan and Koutras (2002), [3].

Analogously if the compounding random variable Y has a distribution over $k \geq 1$ points, then the counting process $N(t)$ is called a compound process of order k.

2.1 Poisson process of order k.

The Poisson process of order k is defined by Philippou (1983), [15] and Charalambides (1986), [4] as a compound Poisson process. It is also a compound birth process, see Kostadinova and Minkova (2013), [9].

Let the compounding random variable Y in (2) be a discrete uniformly distributed over $k > 1$ points with probability mass function

$$P(Y = i) = \frac{1}{k}, \quad i = 1, 2, \ldots, k. \tag{6}$$

The probability generating function of Y, $\psi_Y(s) = E(s^Y)$, $s \in (0, 1)$ is given by

$$\psi_Y(s) = \frac{s}{k}\frac{1-s^k}{1-s}. \tag{7}$$

Definition 2. *The stochastic process, defined by the probability generating function (5) and compounding distribution, defined by (7) is called a Poisson process of order k with parameter λ. We use the notation $N(t) \sim Po_k(\lambda t)$.*

Remark 1. *If $k = 1$, the discrete uniform distribution in (6) degenerates at point 1 and the process $N(t)$ is a homogeneous Poisson process.*

Remark 2. *The mean and the variance of the Poisson process of order k are given by*

$$E(N(t)) = \frac{k+1}{2}k\lambda t \quad and \quad Var(N(t)) = \frac{(k+1)(2k+1)}{6}k\lambda t.$$

For the Fisher index of dispersion we get

$$FI(N(t)) = \frac{Var(N(t))}{E(N(t))} = 1 + \frac{2}{3}(k-1) > 1,$$

i.e. the Poisson process of order k is over-dispersed related to the Poisson process.

This makes the Poisson process of order k suitable for financial data.

2.2 Pólya-Aeppli process of order k.

The Pólya-Aeppli distribution of order k is introduced by Minkova (2010), [13]. It is a compound Poisson distribution with truncated geometric compounding distribution and probability generating function given by (5). Chukova and Minkova (2015) defined the corresponding process as a pure birth process and give an application to risk theory, see [6].

Suppose that the compounding random variable Y given in (2) has a truncated geometric distributed with success probability $1-\rho$.

The probability mass function and the probability generating function of the compounding variable are given by

$$P(Y = i) = \frac{1-\rho}{1-\rho^k}\rho^{i-1}, \quad i = 1, 2, \dots, k \tag{8}$$

and

$$\psi_Y(s) = \frac{(1-\rho)s}{1-\rho^k}\frac{1-\rho^k s^k}{1-\rho s}, \tag{9}$$

where $k \geq 1$ is a fixed integer number.

Definition 3. *The stochastic process, defined by the probability generating function (5) and compounding distribution, defined by (8) and (9) is called a Pólya-Aeppli process of order k with parameters $\lambda > 0$ and $\rho \in [0,1)$. We use the notation $N(t) \sim PA_k(\lambda t, \rho)$.*

Remark 3. *In the case of $k \to \infty$, the Pólya-Aeppli process of order k approaches to the Pólya-Aeppli process, defined by Minkova (2004), [12]. If $\rho = 0$ it is a homogeneous Poisson process, see [13].*

Remark 4. *The mean and the variance of the Pólya-Aeppli process of order k are given by*

$$E(N(t)) = \frac{1+\rho+\ldots+\rho^{k-2}+\rho^{k-1}-k\rho^k}{1-\rho^k}\lambda t$$

and

$$Var(N(t)) = \frac{1+3\rho+5\rho^2+7\rho^3+\ldots+(2k-3)\rho^{k-2}+(2k-1)\rho^{k-1}-k^2\rho^k}{1-\rho^k}\lambda t.$$

For the Fisher index of dispersion we get

$$\begin{aligned} FI(N(t)) &= \frac{Var(N(t))}{E(N(t))} \\ &= 1 + \frac{2\rho+\ldots+2(k-2)\rho^{k-2}+2(k-1)\rho^{k-1}+k\rho^k(1-k)}{1+\rho+\ldots+\rho^{k-2}+\rho^{k-1}-k\rho^k} > 1, \end{aligned}$$

i.e. the Pólya-Aeppli process of order k is over-dispersed related to the Poisson process.

This makes the Pólya-Aeppli process of order k suitable for financial data also.

3 Risk models of order k.

3.1 Poisson of order k risk model.

Let the counting process $N(t)$ in the risk model (1) is a Poisson process of order k. We call this model a Poisson of order k risk model. The interpretation of the counting process is the following. If the insurance policies are separated into independent groups then the number of the groups has a Poisson distribution. We suppose that the groups are homogeneous and identically distributed. The number of the policies in each group has a discrete uniform distribution over k points.

For this risk model the relative safety loading θ is given by

$$\theta = \frac{EX(t)}{E(Z_1+Z_2+\ldots+Z_{N(t)})} = \frac{2c}{\lambda\mu(k+1)} - 1.$$

Let $\tau = \inf\{t : X(t) < -u\}$ with the convention of $\inf\emptyset = \infty$ be the time to ruin of an insurance company having initial capital $u \geq 0$. We denote by $\Psi(u) = P(\tau < \infty)$ the ruin probability. We assume that $\theta > 0$. When $\theta < 0,\ \Psi(u) = 0$. In the case of positive safety loading $\theta > 0$, the premium income per unit time c should satisfy the following inequality

$$c > \frac{\lambda\mu(k+1)}{2}. \tag{10}$$

The condition given in (10) is called a net profit condition.

Remark 5. *If k=1, the net profit condition given in (10) coincides with the net profit condition of the Poisson process.*

3.2 Pólya-Aeppli of order k risk model.

Let the counting process $N(t)$ in the risk model (1) is a Pólya-Aeppli process of order k. Chukova and Minkova (2015) called this model a Pólya-Aeppli of order k risk model, see [6]. The interpretation of the counting process is the following. If the insurance policies are separated into independent groups, then the number of the groups has a Poisson distribution. We suppose that the groups are homogeneous and identically distributed. The number of the policies in each of the groups has a truncated geometric distribution.

For this risk model, see [6]. The relative safety loading θ is given by

$$\theta = \frac{EX(t)}{E(Z_1 + Z_2 + \ldots + Z_{N(t)})} = \frac{c(1-\rho^k)}{\lambda\mu(1+\rho+\rho^2+\ldots+\rho^{k-1}-k\rho^k)} - 1.$$

In the case of positive safety loading $\theta > 0$, the premium income per unit time c should satisfy the following inequality

$$c > \frac{\lambda\mu(1+\rho+\rho^2+\ldots+\rho^{k-1}-k\rho^k)}{1-\rho^k}. \tag{11}$$

4 Martingale approximation.

4.1 Martingales for the risk models of order k.

Let us denote by $(\mathcal{F}_t^X)$ the natural filtration generated by any stochastic process $X(t)$. $(\mathcal{F}_t^X)$ is the smallest complete filtration to which the process $X(t)$ is adapted. As the ruin times are first entrance times to some interval we need a complete filtration in order to assure that the ruin times are stopping times.

We denote by $LS_Z(r) = \int\limits_0^\infty e^{-rx} dF_Z(x)$ the Laplace-Stieltjes transform ($LS - transform$) of any random variable Z with distribution function $F_Z(x)$.

From the martingale theory [20] we get the following lemmas:

Lemma 1. *For the risk model of order k we have the equality*

$$Ee^{-rX(t)} = e^{g(r)t},$$

where

$$g(r) = -cr - \lambda[1 - \psi_Y(LS_Z(-r))]. \tag{12}$$

Proof. Let $S(t) = \sum\limits_{i=1}^{N(t)} Z_i$ be the sum of the aggregated claims up to time t, where $N(t)$ is a compound process of order k, independent of $Z_i,\ i = 1, 2, \ldots$

The LS-transform of the sum $S(t)$ is given by

$$\begin{aligned} LS_{S(t)}(r) &= E[e^{-rS(t)}] = E[e^{-r(Z_1+\ldots+Z_{N(t)})}] \\ &= \sum_{m=0}^{\infty} E(e^{-r(Z_1+\ldots+Z_{N(t)})}|N(t) = m)P(N(t) = m) \\ &= \sum_{m=0}^{\infty} E(e^{-r(Z_1+\ldots+Z_m)})P(N(t) = m) \\ &= \sum_{m=0}^{\infty} P(N(t) = m)(LS_Z(r))^m \\ &= \psi_{N(t)}(LS_Z(r)) = e^{-\lambda t[1-\psi_Y(LS_Z(r))]}. \end{aligned}$$

Analogously the Laplace-Stieltjes transform of the process $X(t)$ is given by

$$\begin{aligned} LS_{X(t)}(r) &= E[e^{-rX(t)}] = E[e^{-r[ct-S(t)]}] = e^{-rct}Ee^{rS(t)} \\ &= e^{-rct}\psi_{N(t)}(LS_Z(-r)) \\ &= e^{-rct}e^{-\lambda t[1-\psi_Y(LS_Z(-r))]} = e^{g(r)t}, \end{aligned}$$

where $g(r)$ is given by (12). As the counting process $N(t)$ is a compound process of order k, its corresponding probability generating function is given by $\psi_{N(t)}(s) = e^{-\lambda t(1-\psi_Y(s))}$.

Lemma 2. *For all $r \in \mathbb{R}$ the process*

$$M(t) = e^{-rX(t)-g(r)t}, \ t \geq 0 \tag{13}$$

is an $\mathcal{F}_t^X$-martingale, provided that $LS_Z(-r) < \infty$.

Proof. From Lemma 1 for $0 \leq v \leq t$ we have

$$\begin{aligned} E(M(t)|\mathcal{F}_v^X) &= E\left[e^{-rX(t)-g(r)t}|\mathcal{F}_v^X\right] \\ &= E[e^{-rX(v)-g(r)v}e^{-r(X(t)-X(v))-g(r)(t-v)}|\mathcal{F}_v^X] \\ &= M(v)E[e^{-r(X(t)-X(v))}e^{-g(r)(t-v)}] \\ &= M(v)e^{g(r)(t-v)}e^{-g(r)(t-v)} = M(v) \end{aligned}$$

and then (13).

4.2 Martingale approach to the risk models of order k.

Using the martingale properties of the process $M(t)$, given in Lemma 2, we give some useful inequalities for the ruin probability. For the martingale approach in the classical case see Schmidli (1996), [20].

Proposition 1. *Let $r > 0$. For the ruin probabilities of the risk model of order k we have the following results*
(i) $\Psi(u,t) \leq e^{-ru} \sup_{0\leq s\leq t} e^{g(r)s}$ *for* $0 \leq t < \infty$
(ii) $\Psi(u) \leq e^{-ru} \sup_{s\geq 0} e^{g(r)s}$
(iii) If the Lundberg's exponent R exist, then R is the largest positive real solution of the equation

$$cr + \lambda[1 - \psi_Y(LS_Z(-r))] = 0 \tag{14}$$

and

$$\Psi(u) \leq e^{-Ru}. \tag{15}$$

Proof. Let the process $M(t)$ be a martingale relative the σ - algebras $\mathcal{F}^X$, generated by the process $X(t)$ and $\tau = inf\{t \geq 0 : X(t) < 0\}$ be the time to ruin for an insurance company.

Applying the martingale stopping time theorem on the martingale $M(t)$ we obtain that

$$\begin{aligned} 1 &= M(0) = EM(t_0 \wedge \tau) = E[M(t_0 \wedge \tau), \tau \leq t_0] + E[M(t_0 \wedge \tau), \tau > t_0] \\ &\geq E[M(t_0 \wedge \tau), \tau \leq t_0] = E[e^{-rX(\tau)-g(r)\tau} | \tau \leq t_0] P(\tau \leq t_0) \\ &\geq e^{ru} E[e^{-g(r)\tau} | \tau \leq t_0] P(\tau \leq t_0). \end{aligned}$$

The process $X(\tau) \leq 0$ for $\tau < \infty$. Then the inequality $e^{-ru} \geq 1$ holds. Since t_0 in the above calculations was arbitrary selected, then for every t we have

$$P(\tau \leq t) \leq \frac{e^{-ru}}{E[e^{-g(r)\tau} | \tau \leq t]} \leq e^{-ru}. \tag{16}$$

The statement (i) of the proposition follows from the above relation. By letting $t \to \infty$ in statement (i) we obtain statement (ii).

According to the interpretation of the Lundberg exponent, see Rolski et al. (1999), [18] the constant R is such that the process $E(e^{-RX(t)})$ is a martingale. Following Lemma 1 we see that the constant R is a positive root of the equation $g(r) = 0$, where $g(r)$ is given in (12).

Taking the condition (14) we obtain the inequality (15) from (16).

Remark 6. *The condition (14) is known as Cramér condition and (15) as Lundberg inequality.*

5 Exponentially distributed claims.

Let us suppose that the claim sizes $\{Z_i\}_{i=1}^{\infty}$ are exponentially distributed, i.e. $Z \sim exp(\mu)$ with distribution function $F_Z(x) = 1 - e^{-\frac{x}{\mu}},\ x \geq 0,\ \mu > 0$ and mean value $EZ = \mu < \infty$.

Then for the LS-transform of the exponentially distributed claims we have the following expression

$$LS_Z(-r) = E(e^{rZ}) = \int_0^\infty e^{rx} dF_Z(x) = \int_0^\infty e^{rx} d(1 - e^{-\frac{x}{\mu}}) = \frac{1}{1 - \mu r},$$

where $r < \frac{1}{\mu}$.

It is known that $M_Z(r) = \frac{1}{1-\mu r}$ is the moment generating function of the exponential distribution.

Our main goal in this section is to compare the Poisson of order k risk model and Pólya-Aeppli of order k risk model. For the purpose we take arbitrary values of the parameters $k,\ \lambda,\ \mu$ and ρ. The value of the premium income per unit time c is chosen so that the safety loading θ is positive.

5.1 Model 1: Poisson of order k risk model.

Let the counting process $N(t)$ in the Poisson of order k risk model has an intensity λ. Then from the equation $g(r) = 0$ for this risk model we obtain the following equation

$$cr + \lambda \left[1 - \frac{1-(1-\mu r)^k}{k\mu r(1-\mu r)^k}\right] = 0. \tag{17}$$

The above equation's solution is called Crámer-Lundberg's exponent. Our interest is in finding the maximum positive real root R of the equation (17).

5.2 Model 2: Pólya-Aeppli of order k risk model.

Let the counting process $N(t)$ in the Pólya-Aeppli of order k risk model has an intensity λ and parameter ρ. Then from the equation $g(r) = 0$ for this risk model we obtain the following equation

$$cr + \lambda \left[1 - \frac{(1-\rho)((1-\mu r)^k - \rho^k)}{(1-\rho^k)(1-\mu r)^k(1-\mu r-\rho)}\right] = 0. \tag{18}$$

Since the value of the Lundberg's exponent R is a measure of the dangerousness of the risk business, we search the maximum positive real root R of the equation (18).

5.3 Model's Risk business comparison.

Example 1: Let $k = 3,\ \lambda = 2,\ \mu = 1$. For the Pólya-Aeppli of order k risk model we have an intensity λ and a parameter $\rho \in [0, 1)$. We give five values for the parameter ρ, i.e. $\rho = 0.1,\ 0.3,\ 0.5,\ 0.7$ and $\rho = 0.9$. We take $c = 4$ under the conditions (10) and (11).

For these values of the parameters we compare the Poisson of order k risk model and the Pólya-Aeppli of order k risk model. For the insurance company the model with the greater R is preferred. Taking this values of the parameter we obtain that the greater root R of the equation (17) has the form

$$R = \frac{2c - \lambda\mu + \sqrt{\lambda\mu(2c + \lambda\mu)}}{2c\mu} \qquad (19)$$

For the Poisson of order k risk model the root R has value 0.123.

The following table shows the results for the Lundberg's exponent of the Pólya-Aeppli of order k risk model

$\rho = 0.1$	$R = 0.505$
$\rho = 0.3$	$R = 0.361$
$\rho = 0.5$	$R = 0.262$
$\rho = 0.7$	$R = 0.193$
$\rho = 0.9$	$R = 0.143$

For this case the obtained data above show that the Pólya-Aeppli of order k risk model is better than the Poisson of order k risk model.

Example 2: Let $k = 5,\ \lambda = 2,\ \mu = 1$. Again we give five values for the parameter ρ, i.e. $\rho = 0.1,\ 0.3,\ 0.5,\ 0.7$ and $\rho = 0.9$. We take $c = 7$ under the conditions (10) and (11).

In this case for the largest positive real root R of the equation (17) we obtain $R = 0.063$.

The following table shows the results for the Lundberg's exponent of the Pólya-Aeppli of order k risk model

$\rho = 0.1$	$R = 0.9$
$\rho = 0.3$	$R = 0.432$
$\rho = 0.5$	$R = 0.284$
$\rho = 0.7$	$R = 0.3$
$\rho = 0.9$	$R = 0.1$

For this case the obtained data above show that the Pólya-Aeppli of order k risk model is better than the Poisson of order k risk model.

6 Concluding remarks.

In this paper we have introduced two risk models of order k. The first one is a Poisson of order k risk model and the second one is a Pólya-Aeppli of order k risk model. For these two models we obtained a martingale approach and defined exponential martingales. For different values of the parameters we made some calculations for the Lundberg's exponent as a measure of the business risk. Finally in the case of exponentially distributed claims, we consider two examples: for $k = 3$ and $k = 5$. For these examples we compared the introduced models and gave some conclusions. Another conclusions can be done for arbitrary k.

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