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SOLVING A FUZZY TRANSPORTATION PROBLEM BASED ON EXPONENTIAL MEMBERSHIP FUNCTIONS *

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Abstract

In the traditional transportation problem, it is assumed that decision makers are confident about the exact values of transportation costs, supply and demand of the product. When solving a transportation problem with inaccurately determined transportation costs, supply and demand quantities, a fuzzy approach is used. In this paper, a transportation problem based on statistical data is considered. The frequency distributions on the transportation costs and the supply and demand quantities in real-life transportation problems are used as the base for determining the parameters of exponential membership functions. An approach for solving fuzzy transportation problem using exponential fuzzy numbers is proposed. A numerical example is solved to illustrate the described approach.

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1 Introduction

The transportation problem is one of the earliest applications of linear programming problems and it has wide applications in logistics and supply chain for reducing the cost. It is concerned with shipping a commodity between a set of sources (e.g. manufacturing centers or supply centers) and a set of destinations (e.g. consumer centers or demand centers). The problem is to minimize the total transportation costs, while satisfies the limitations of the quantities of supply centers and demand centers.

In the traditional transportation problem, it is assumed that decision makers are confident about the exact values of transportation costs, supply and demand of the product. Often in real-life, however, these values may be uncertain due to uncontrollable factors. In this case, a fuzzy approach is needed.

A fuzzy transportation problem (FTP) is a transportation problem in which the transportation costs and/or the supply and demand quantities are fuzzy numbers. An important stage in solving a FTP is ranking of fuzzy numbers and the introduction of rules for comparison of two fuzzy numbers.

After Bellman and Zadeh [4] and Zadeh [25] introduced the concept of fuzziness, Zimmermann [26] proved that solutions obtained by fuzzy linear programming are always efficient. His publication provides a basis for developing several methods of solving a FTP and sets the beginning of the increasing interest of researchers on the subject.

Here we will indicate in chronological order a small part of the significant research in the scientific field and some of its results. OhEigeartaigh [18] proposed an algorithm for solving transportation problem where the supply and demand quantities are fuzzy numbers with linear or triangular membership functions. Chanas et al. [6] developed a FTP with crisp transportation costs and fuzzy supply and demand values. Chanas and Kuchta [7] described a method for solving a FTP by converting to a bicriterial transportation problem with crisp objective function which provides crisp solution. Liu and Kao [14] suggested a method based on the Zadehs extension principle, in which the transportation costs and the supply and demand quantities are fuzzy numbers.

Saad and Abbas [21] presented a parametric approach for solving a FTP. Gani and Razak [9] also used a parametric approach and obtained a fuzzy solution for a two stage FTP with trapezoidal fuzzy numbers. Pandian and Natarajan [19] proposed a fuzzy zero point method to find the fuzzy optimal solution and they also used trapezoidal fuzzy numbers. The trapezoidal fuzzy numbers were used also by Dinagar and Palanivel [8], Ahmed et al. [1], Sharma et al. [22], Muruganandam and Srinivasan [15]. Kaur and Kumar [12] described a new approach for solving a FTP using generalized trapezoidal fuzzy numbers. Thamaraiselvi and Santhi [23] used generalized Hexagonal fuzzy numbers. The object of this paper is transportation problem in which statistics on values of transportation costs, supply and demand of the product are available. Based on statistical data presented through time series, it is proposed to form fuzzy numbers with exponential membership function of a particular type and to solve the obtaining FTP by using an appropriate ranking function of the formed fuzzy numbers.

The rest of the paper is organized as follows: Section 2 consists of three parts. In 2.1, basic concepts of fuzzy sets theory are outlined briefly. In 2.2, the description of the problem and the input data are presented. In 2.3, the used designations in relation to the grouping of data in interval classes and frequency distributions are indicated. The exponential membership function and the calculation of its parameters based on the frequency distribution are discussed in Section 3. In Section 4, formulas for the α-cuts of the exponential membership function are obtained. Then a ranking method, which assign a crisp value to an exponential fuzzy number, is presented. In Section 5, the algorithm for solving a FTP is described and a numerical example illustrates its implementation. Some summaries are made in Section 6.

2 Preliminaries

2.1 Some basic concepts of fuzzy sets theory

The concepts and principles of fuzzy sets theory can be found in [3, 11, 20, 27]. Definitions of the some basic notions used in the following presentation will be briefly indicated here.

Let X is a collection of objects and $x \in X$.

A fuzzy set A in X is the set of ordered pairs $A = \{(x, \mu_A(x)) \mid x \in X\}$, where $\mu_A(x)\colon X \to T \subseteq [0,1]$ is called a membership function for the fuzzy set A.

If $sup\ \mu_A(x) = 1$, then A is called normal fuzzy set. If $sup\ \mu_A(x) < 1$, then A is subnormal.

The support of A is the subset of points of X at which $\mu_A(x)$ is positive, i.e. $support(A) = \{x \in X \mid \mu_A(x) > 0\}$.

The core of a fuzzy set A is the set of all points x in X such that $\mu_A(x) = 1$, i.e. $core(A) = \{x \in X \mid \mu_A(x) = 1\}$.

For any $\alpha > 0$, $\alpha \in T \subseteq [0,1]$, an α-cut or α-level of the fuzzy set A in X is the set $A_\alpha = \{x \mid x \in X, \mu_A(x) \geq \alpha\}$.

The fuzzy set A is convex if for any $x_1, x_2 \in X$ and any $\lambda \in [0,1]$ is fulfilled:

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq min\{\mu_A(x_1), \mu_A(x_2)\}. \tag{1}$$

Alternatively, the fuzzy set A is convex if all its α-cuts are convex.

A fuzzy set in the real line $\Re$ with the membership function $\mu_A(x)\colon \Re \to [0,1]$ that satisfies the conditions for normaly, convexity and piecewise continuity and $support(A)$ is bounded, is called a fuzzy number.

It is fulfilled that the fuzzy set A in $\Re$ is a fuzzy number if and only if A is normal and A_α is bounded, closed and convex for any $\alpha \in (0,1]$.

Remark 1 *For a fuzzy number, the convexity defined by (1) means that the membership function is first monotonically increasing and then monotonically decreasing or that it is monotone.*

The LR representation of fuzzy number (see [21]) means that its membership function is composed of two parts - a monotonically increasing left part $L_A(x)$ and a monotonically decreasing right part $R_A(x)$:

$$\mu_A(x) = \begin{cases} L_A(x), x \leq m \\ R_A(x), x > m \end{cases} \tag{2}$$

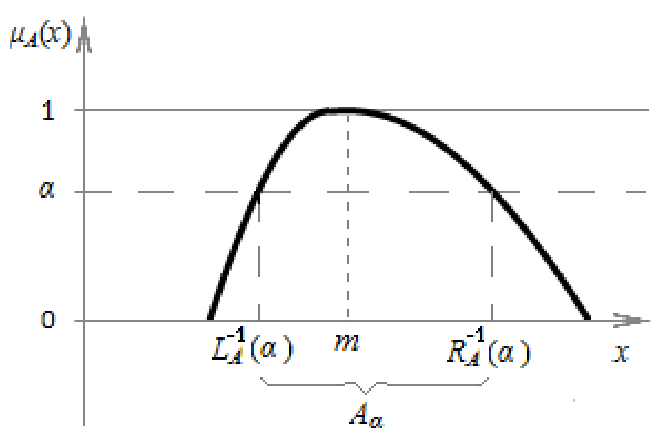

Figure 1: An α-cut A_α of the fuzzy number A with a membership function $\mu_A(x)$

Let $L_A^{-1}(\alpha)$ and $R_A^{-1}(\alpha)$ are inverse functions for increasing (left) and decreasing (right) sides of the membership function $\mu_A(x)$, respectively. Then for any $\alpha \in (0,1]$ the α-cut of the fuzzy number A (Figure 1) can be represented by:

$$A_\alpha = [L_A^{-1}(\alpha), R_A^{-1}(\alpha)] \subset (-\infty, +\infty).$$

2.2 Description of the problem and the input data

Lets consider a corporation with a fixed network of supply centers $S_i, i = \overline{1, n_S}$ and demand centers $D_j, j = \overline{1, n_D}$. Lets have the input data of N traditional transportation problems related to the N consecutive periods of the recent past. For the period $t, t = \overline{1, N}$, the costs $c_{ij}(t)$ of transporting one unit of the commodity from S_i to D_j, the supply quantities $a_i(t)$ and the demand quantities $b_j(t)$ are filled in Table 1.

	D_1	D_2	...	D_{n_D}	$a_i(t)$
S_1	$c_{11}(t)$	$c_{12}(t)$	...	$c_{1n_D}(t)$	$a_1(t)$
S_2	$c_{21}(t)$	$c_{22}(t)$	...	$c_{2n_D}(t)$	$a_2(t)$
...	...	...	...	...	...
S_{n_S}	$c_{n_S1}(t)$	$c_{n_S2}(t)$	...	$c_{n_Sn_D}(t)$	$a_{n_S}(t)$
$b_j(t)$	$b_1(t)$	$b_2(t)$	...	$b_{n_D}(t)$	

Table 1: A transportation table

Remark 2 *For fixed $i, i = \overline{1, n_S}$ and $j, j = \overline{1, n_D}$, the sequences of values $c_{ij}(t), a_i(t)$ and $b_j(t)$ corresponding to consecutive periods $t, t = \overline{1, N}$ can be considered as time series.*

The purpose of the research is to formulate the FTP based on statistical input data $c_{ij}(t), a_i(t), b_j(t)$ at $i = \overline{1, n_S}, j = \overline{1, n_D}$ and $t = \overline{1, N}$ and to solve through exponential membership functions. The transportation plan

$$X_{opt} = \begin{bmatrix} x_{11} & ... & x_{1n_D} \\ ... & ... & ... \\ x_{n_S1} & ... & x_{n_Sn_D} \end{bmatrix},$$

which is received as a solution of the transportation problem and refers to the next period $t = N + 1$, should be optimal according to the available data.

Remark 3 *By introducing appropriate weighting coefficients, the significance of the most recent input data (more up-to-date) can be increased.*

2.3 Grouped frequency distributions of input data

Let for the fixed $i, i = \overline{1, n_S}$ and $j, j = \overline{1, n_D}$, the values $c_{ij}(t), t = \overline{1, N}$ are grouped into $l = l_{ij}(l \geq 5$ class intervals of equal lengths and their frequencies are $f_k > 0, k = \overline{1, l}$. The modal class interval with the highest frequency

$$f_M = \max_{k=\overline{1,l}} \{f_k\}$$

and its midpoint $m = m_M$ are determined. The normalized frequencies $p_k = \frac{f_k}{f_M} \in (0, 1], k = \overline{1, l}$ are calculated and with that $p_M = 1$. The numerical data is filled in a table in the form of Table 2.

k	Class interval	Midpoint m_k	Frequency f_k	Normalized frequency p_k
1	$[x_0, x_1)$	$m_1 = \frac{x_0+x_1}{2}$	f_1	$p_1 = \frac{f_1}{f_M}$
2	$[x_1, x_2)$	$m_2 = \frac{x_1+x_2}{2}$	f_2	$p_2 = \frac{f_2}{f_M}$
...	...	...	...	...
l	$[x_{l-1}, x_l)$	$m_l = \frac{x_{l-1}+x_l}{2}$	f_l	$p_l = \frac{f_l}{f_M}$
	Total		N	

Table 2: Frequency table with the symbols used

A series of points $P_k(m_k, p_k), k = \overline{1, l}$ with evenly distributed abscissas are formed (Figure 2). The point $P_M(m_M, p_M)$ is divided the series of points into two groups:

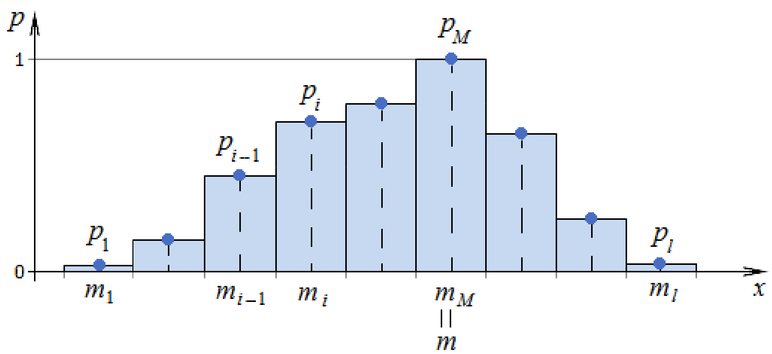

Figure 2: A histogram of the frequency distribution and used parameter markers

- left $P_k(m_k, p_k), k = \overline{1, M-1}$, containing $(M-1)$ points;
- right $P_k(m_k, p_k), k = \overline{M+1, l}$, containing $(l-M)$ points,

where $M-1 > 1$ and $l-M > 1$.

Remark 4 *If $M-1 > 1$ or $l-M > 1$ is not fulfilled, then an extra point is added to the left (at $M-1 \leq 1$) or right (at $l-M \geq 1$). The abscissa of the new point must maintain the even distribution of the abscissas and the ordinate must be a positive number close to zero, for example 0.001.*

3 Parameters of an exponential membership function

In their research of bispectral index data sets, Nasibov and Ulutagay [17] applied a fuzzy approach. They used fuzzy numbers with LR-type exponential membership function of the species:

$$\mu_A(x) = \begin{cases} e^{-(\frac{m-x}{\sigma_L})^{\beta_L}}, x \leq m \\ e^{-(\frac{x-m}{\sigma_R})^{\beta_R}}, x > m \end{cases} \tag{3}$$

and obtained formulas for the unknown parameters $\sigma_L, \beta_L, \sigma_R, \beta_R$ by least squares approximation of the normalized frequency distribution of the data.

In this paper, we use the results obtained in [17] with another goal and first we will present the formulas for the unknown parameters and their output.

Let us interpolate separately the left and right groups of points of the frequency distribution described in 2.3. The left group of points $P_k(m_k, p_k), k = \overline{1, M-1}$ is interpolated by the increasing left part $L_A(x) = e^{-(\frac{m-x}{\sigma_L})^{\beta_L}}, x \leq m$ of a function (3). The right group of points $P_k(m_k, p_k), k = \overline{M+1, l}$ is interpolated by the decreasing right part $R_A(x) = e^{-(\frac{x-m}{\sigma_R})^{\beta_R}}, x > m$ of a function (3).

The unknown parameters are determined so that it is satisfied:

$$\begin{cases} e^{-(\frac{m-x}{\sigma_L})^{\beta_L}} \approx p_k, k = \overline{1, M-1} \\ e^{-(\frac{x-m}{\sigma_R})^{\beta_R}} \approx p_k, k = \overline{M+1, l} \end{cases} \tag{4}$$

After double logarithmization on both sides of (4) and forming the squares of differences between the left and right sides, the object functions

E_L and E_R are obtained:

$$E_L(\sigma_L, \beta_L) = \sum_{k=1}^{M-1} \left[\beta_L \ln \left(\frac{m - m_k}{\sigma_L} \right) - \ln(-\ln p_k) \right]^2 \to \min$$

$$E_R(\sigma_R, \beta_R) = \sum_{k=M+1}^{l} \left[\beta_R \ln \left(\frac{m_k - m}{\sigma_R} \right) - \ln(-\ln p_k) \right]^2 \to \min$$

The solutions of the systems:

$$\begin{cases} \frac{\partial E_L}{\partial \sigma_L} = -2\frac{\beta_L}{\sigma_L} \sum\limits_{k=1}^{M-1} \left[\beta_L \ln(\frac{m-m_k}{\sigma_L}) - \ln(-\ln p_k) \right] = 0 \\ \frac{\partial E_L}{\partial \beta_L} = -2 \sum\limits_{k=1}^{M-1} \left[\left(\beta_L \ln(\frac{m-m_k}{\sigma_L}) - \ln(-\ln p_k) \right) \ln(\frac{m-m_k}{\sigma_L}) \right] = 0 \end{cases}$$

and

$$\begin{cases} \frac{\partial E_R}{\partial \sigma_R} = -2\frac{\beta_R}{\sigma_R} \sum\limits_{k=M+1}^{l} \left[\beta_R \ln(\frac{m_k-m}{\sigma_R}) - \ln(-\ln p_k) \right] = 0 \\ \frac{\partial E_R}{\partial \beta_R} = -2 \sum\limits_{k=M+1}^{l} \left[\left(\beta_R \ln(\frac{m_k-m}{\sigma_R}) - \ln(-\ln p_k) \right) \ln(\frac{m_k-m}{\sigma_R}) \right] = 0 \end{cases}$$

are:

$$\sigma_L = \exp \left(\frac{A_L.B2_L - AB_L.B_L}{A_L.B_L - (M-1).AB_L} \right); \tag{5}$$

$$\beta_L = \frac{A_L}{B_L - (M-1).\ln \sigma_L} \tag{6}$$

and

$$\sigma_R = \exp \left(\frac{A_R.B2_R - AB_R.B_R}{A_R.B_R - (l-M).AB_R} \right); \tag{7}$$

$$\beta_R = \frac{A_R}{B_R - (l-M).\ln \sigma_R} \tag{8}$$

where:

$A_L = \sum\limits_{k=1}^{M-1} \ln(-\ln(p_k);$ $\quad$ $A_R = \sum\limits_{k=M+1}^{l} \ln(-\ln(p_k);$

$B_L = \sum\limits_{k=1}^{M-1} \ln(m - m_k);$ $\quad$ $B_R = \sum\limits_{k=M+1}^{l} \ln(m_k - m);$

$B2_L = \sum\limits_{k=1}^{M-1} \ln^2(m - m_k);$ $\quad$ $B2_R = \sum\limits_{k=M+1}^{l} \ln^2(m_k - m);$

$AB_L = \sum\limits_{k=1}^{M-1} \ln(-\ln(p_k).\ln(m - m_k);$ $\quad$ $AB_R = \sum\limits_{k=M+1}^{l} \ln(-\ln(p_k).\ln(m_k - m)$

The formulas (5)-(8) are valid when the conditions:

$$(M-1).AB_L \neq A_L.B_L; \qquad (l-M).AB_R \neq A_R.B_R;$$
$$(M-1).\ln\sigma_L \neq B_L; \qquad (l-M).\ln\sigma_R \neq B_R$$

are fulfilled.

Remark 5 *The requirements* $L_A^{-1}(0) = \lim\limits_{\alpha\to 0} L_A^{-1}(\alpha) > -\infty$ *and* $R_A^{-1}(0) = \lim\limits_{\alpha\to 0} R_A^{-1}(\alpha) < +\infty$ *shall be considered satisfied where* $L_A^{-1}(\alpha)$ *and* $R_A^{-1}(\alpha)$ *are the inverse functions of the increasing left and the decreasing right side of the membership function (3), respectively.*

Remark 6 *The frequency distribution may have two or more adjacent class intervals with maximum frequencies* $f_{M_L} = f_{M_R}$ *instead of one. Then, for interpolation of the two groups of points* $P_k(m_k, p_k), k = \overline{1, M_L - 1}$ *and* $P_k(m_k, p_k), k = \overline{M_R + 1, l}$*, the membership function:*

$$\mu_A(x) = \begin{cases} e^{-(\frac{m_L - x}{\sigma_L})^{\beta_L}}, & x < m_L \\ 1, & m_L \leq x \leq m_R \\ e^{-(\frac{x - m_R}{\sigma_R})^{\beta_R}}, & m_R < x \end{cases} \tag{9}$$

is more appropriate. Here $m_L = m_{M_L}$ *and* $m_R = m_{M_R}$ *are the midpoints of the class intervals with maximum frequencies.*

4 Ranking exponential fuzzy numbers

4.1 Determination of α-cuts

The formulas for the left and right side of the α-cuts of (3) can be obtained from $\mu_A\left(L_A^{-1}(\alpha)\right) = \alpha$ and $\mu_A\left(R_A^{-1}(\alpha)\right) = \alpha$ (Figure 1). The solutions of the equations are:

$$L_A^{-1}(\alpha) = m - \sigma_L \sqrt[\beta_L]{\ln\frac{1}{\alpha}}; \quad R_A^{-1}(\alpha) = m + \sigma_R \sqrt[\beta_R]{\ln\frac{1}{\alpha}}. \tag{10}$$

Therefore, for any $\alpha \in (0, 1]$ the α-cut of the fuzzy number A with the membership function (3) is:

$$A_\alpha = \left[m - \sigma_L \sqrt[\beta_L]{\ln\frac{1}{\alpha}}\,, \quad m + \sigma_R \sqrt[\beta_R]{\ln\frac{1}{\alpha}}\right].$$

4.2 A ranking index of a fuzzy number

A method for ranking fuzzy numbers is first described by Jain [10], and then different ideas are offered to solve this problem. One of these ideas belongs to Yager [24]. He defined the function $F(A) = \int_0^{\alpha_{\max}} M(A_\alpha)\, d\alpha$, where $M(A_\alpha)$ is the mean of the α-cut of the fuzzy number. The ranking index of a fuzzy number, which is introduced and used in some studies [5, 16, 13], is analogous to the function $F(A)$.

Further, under a ranking index of the fuzzy number A, we will understand:

$$r(A) = \int_0^1 \frac{L_A^{-1}(\alpha) + R_A^{-1}(\alpha)}{2}\, d\alpha, \qquad (11)$$

where the mean value of the α-cut is $M(A_\alpha) = \frac{L_A^{-1}(\alpha)+R_A^{-1}(\alpha)}{2}$ and $\alpha_{\max} = 1$.

According to (10) and (11), the ranking index of the fuzzy number A with the membership function (3) is:

$$\begin{aligned} r(A) = & \int_0^1 \tfrac{1}{2}\left[\left(m - \sigma_L \sqrt[\beta_L]{\ln \tfrac{1}{\alpha}}\right) + \left(m + \sigma_R \sqrt[\beta_R]{\ln \tfrac{1}{\alpha}}\right)\right] d\alpha = \\ = & \; m\int_0^1 d\alpha + \tfrac{\sigma_R}{2}\int_0^1 \sqrt[\beta_R]{\ln \tfrac{1}{\alpha}}\, d\alpha - \tfrac{\sigma_L}{2}\int_0^1 \sqrt[\beta_L]{\ln \tfrac{1}{\alpha}}\, d\alpha \end{aligned}$$

and it is obtained:

$$r(A) = m + \frac{1}{2}\left[\sigma_R\, \Gamma\left(1 + \frac{1}{\beta_R}\right) - \sigma_L\, \Gamma\left(1 + \frac{1}{\beta_L}\right)\right], \qquad (12)$$

where $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$ is the Gamma function.

Remark 7 *The substitution* $\ln \frac{1}{\alpha} = t$ *is used to calculate the integral:*

$$\int_0^1 \sqrt[\beta]{\ln \frac{1}{\alpha}}\, d\alpha = \int_{+\infty}^0 \sqrt[\beta]{t}\, (-e^{-t}) dt = \int_0^{+\infty} t^{\frac{1}{\beta}}\, e^{-t} dt = \Gamma\left(1 + \frac{1}{\beta}\right),$$

by which the result (12) is obtained.

The result (12) is also valid under the conditions of Remark 6 using $m = \frac{m_L + m_R}{2}$.

5 Solving a fuzzy transportation problem

5.1 A Brief description

Depending on the method used, the optimal solution of the FTP can be represented by fuzzy numbers or crisp values. The both approaches can be found in the literature [19]. Some authors are of the opinion that useful information is lost when the fuzzy numbers are replaced with crisp values in the process of solving. This is valid in the general case, but not for the transportation problem we are considering. The decision makers must organize the transportation of goods in specified quantities and if the resulting optimal solution is expressed by fuzzy numbers, then the fuzzy numbers must first be converted to crisp values. Therefore, we chose an approach in which the optimal solution is expressed by crisp values.

Without describing in detail the traditional algorithm to solve a transportation problem, we will present the sequence of the steps to solve the FTP formulated in 2.2:

1. The time series $c_{ij}(t), a_i(t), b_j(t), t = \overline{1, N}$ from each cell in Table 1 are considered separately. For the time series $w(t), t = \overline{1, N}$, where w is denoted c_{ij}, a_i or $b_j, i = \overline{1, n_S}, j = \overline{1, n_D}$, the following is performed:

   (a) The frequency distribution is built.

   (b) The modal class interval and its midpoint are determined.

   (c) The normalized frequencies are calculated.

   (d) Points in a two-dimensional coordinate system are considered, so their abscisses are the midpoints of the class intervals and their ordinates are the normalized frequencies. A series of points obtained are formed.

   (e) The series of points is divided into left and right groups relative to the point of the modal class interval.

   (f) The parameters $\sigma_L^w, \beta_L^w, \sigma_R^w, \beta_R^w$ of the membership function (3) are calculated using formulas (5)-(8).

   (g) The ranking index $r(A^w)$ of the fuzzy number A^w is calculated by formula (12).

2. A traditional transportation problem is constructed, in which the transportation costs, the supply and demand quantities of commodity are determined by the ranking indices of the fuzzy numbers.

3. The transportation problem is solved by one of the known methods.

4. For the next period $t = N + 1$, the commodity is transported in accordance with the resulting solution.

5.2 Numerical example

We will illustrate the solution of the formulated FTP with export data of a company located in Bulgaria. The commodity is transported by trucks from supply centers in Sofia (S_1), Plovdiv (S_2) and Varna (S_3) to demand centers in Bucharest (D_1), Craiova (D_2), Nis (D_3) and Skopje (D_4). The quantity of commodity is measured in tonnes. The transportation costs for the carriage of a tonne of commodity from the supply centers to the demand centers are indicated in euro.

For each cell in the transportation table, we have 26 values that reflect the relevant indicator over the past 26 weeks.

Let us look at the costs of transporting one unit of the commodity from Sofia (S_1) to Bucharest (D_1). By observing the pattern of Table 2, Table 3 presents the frequency distribution of the sample $c_{11}(t), t = \overline{1, 26}$ and some necessary calculations. The middle of the class interval with the highest frequency $f_6 = 5$ is $m = m_6 = 37.5$. Using the normalized frequencies for the ordinates, the point $(37.5, 1)$ divides the sequence of points $(m_k, p_k), k = \overline{1, 10}$ into two groups: left at $k = \overline{1, 5}$ and right at $k = \overline{7, 10}$.

k	Interval	m_k	f_k	p_k	(A_L)	(B_L)	$(B2_L)$	(AB_L)
1	[32, 33)	32.5	1	0.2	0.47589	1.60944	2.59029	0.76591
2	[33, 34)	33.5	2	0.4	-0.08742	1.38629	1.92181	-0.12119
3	[34, 35)	34.5	4	0.8	-1.49994	1.09861	1.20695	-1.64785
4	[35, 36)	35.5	3	0.6	-0.67173	0.69315	0.48045	-0.46561
5	[36, 37)	36.5	4	0.8	-1.49994	0.00000	0.00000	0.00000
6	[37, 38)	37.5	5	1.0	(A_R)	(B_R)	$(B2_R)$	(AB_R)
7	[38, 39)	38.5	3	0.6	-0,67173	0.00000	0.00000	0.00000
8	[39, 40)	39.5	2	0.4	-0.08742	0.69315	0.48045	-0.06060
9	[40, 41)	40.5	1	0.2	0.47589	1.09861	1.20695	0.52281
10	[41, 42)	41.5	1	0.2	0.47589	1.38629	1.92181	0.65972

Table 3: Frequency distribution of the sample $c_{11}(t), t = \overline{1, 26}$

The last four columns in Table 3 contain the components from which the algebraic sums are obtained:

$$
\begin{array}{ll}
A_L = -3.28314; & A_R = 0.19262; \\
B_L = 4.78749; & B_R = 3.17805; \\
B2_L = 6.19950; & B2_R = 3.60921; \\
AB_L = -1.46874 & AB_R = 1.12193.
\end{array}
$$

The parameters of the left and right parts of the membership function (3) are calculated by formulas (5)-(8). The results are:

$$
\begin{array}{ll}
\sigma_L^{c_{11}} = 4.90792; & \sigma_R^{c_{11}} = 1.03675; \\
\beta_L^{c_{11}} = 2.09725; & \beta_R^{c_{11}} = 0.89364.
\end{array}
$$

The resulting exponential membership function is shown in Figure 3.

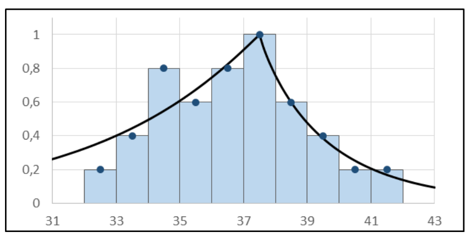

Figure 3: The membership function of $A^{c_{11}}$

According to formula (12), the ranking index of the fuzzy number corresponding to the transportation costs $c_{11}(t), t = \overline{1, 26}$, is:

$$
r(A^{c_{11}}) = 37.5 + \frac{1}{2}\left[1.03675\ \Gamma\left(1 + \frac{1}{0.89364}\right) - 4.90792\ \Gamma\left(1 + \frac{1}{2.09725}\right)\right]
$$

and it is obtained $r(A^{c_{11}}) = 34.87845$.

The sample $c_{12}(t), t = \overline{1, 26}$ of transportation costs from Sofia (S_1) to Craiova (D_2) is represented by its frequency distribution in Table 4. The middle of the class interval with the highest frequency is $m = m_3 = 26.5$. The left group of points contains only two points and the right group contains six points. The parameters of the membership function of the fuzzy number $A^{c_{12}}$ (Figure 4) and its ranking index are calculated in the same way as the previous ones. The results are:

$$
\begin{array}{ll}
\sigma_L^{c_{12}} = 1.53758; & \sigma_R^{c_{12}} = 2.78713; \\
\beta_L^{c_{12}} = 2.53189; & \beta_R^{c_{12}} = 1.34689; \\
r(A^{c_{12}}) = 27.69214. &
\end{array}
$$

k	Interval	m_k	f_k	p_k
1	[24, 25)	24.5	1	0.14286
2	[25, 26)	25.5	5	0.71429
3	[26, 27)	26.5	7	1.00000
4	[27, 28)	27.5	6	0.85714
5	[28, 29)	28.5	2	0.28571
6	[29, 30)	29.5	2	0.28571
7	[30, 31)	30.5	1	0.14286
8	[31, 32)	31.5	1	0.14286
9	[32, 33)	32.5	1	0.14286

Table 4: Frequency distribution of the sample $c_{12}(t), t = \overline{1, 26}$

k	Interval	m_k	f_k	p_k
1	[14, 15)	14.5	1	0.16667
2	[15, 16)	15.5	2	0.33333
3	[16, 17)	16.5	5	0.83333
4	[17, 18)	17.5	6	1.00000
5	[18, 19)	18.5	6	1.00000
6	[19, 20)	19.5	5	0.83333
7	[20, 21)	20.5	1	0.16667

Table 5: Frequency distribution of the sample $c_{13}(t), t = \overline{1, 26}$

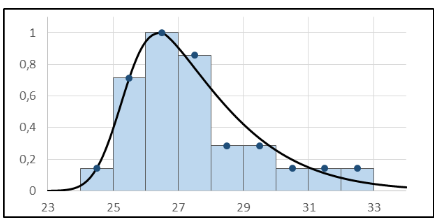

Figure 4: The membership function of $A^{c_{12}}$

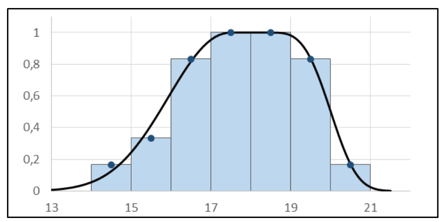

Figure 5: The membership function of $A^{c_{13}}$

The frequency distribution of costs $c_{13}(t), t = \overline{1, 26}$ for transportation of commodity from Sofia (S_1) to Nis (D_3) has two adjacent maximum frequency class intervals (Table 5 and Figure 5). The parameters of the left part $L_{A^{c_{13}}}(x)$ of the membership function (9) are calculated at $m_L = m_4 = 17.5$, and the parameters of the right part $R_{A^{c_{13}}}(x)$ at $m_R = m_5 = 18.5$. The results are:

$$\sigma_L^{c_{13}} = 2.13238; \qquad \sigma_R^{c_{13}} = 1.67573;$$
$$\beta_L^{c_{13}} = 2.13509; \qquad \beta_R^{c_{13}} = 3.29682.$$

The arithmetic mean $m = \frac{m_L + m_R}{2} = 18$ is substituted in formula (12) and for the ranking index of the fuzzy number is obtained $r(A^{c_{13}}) = 17.61461$.

The parameters of the membership functions and the ranking indices of the fuzzy numbers in other cells on the transportation table are calculated analogously. The frequency distributions can be found in Tables 7-8 of the Appendix section. The histograms and graphs of the membership functions are shown in Figures 7-22. The results of the calculations are summarized in Table 9.

The resulting transportation problem (Table 6) is unbalanced and therefore a fictitious demand center D_5 has been added with a fictitious demand quantity:

$$b_5 = \sum_{i=1}^{3} a_i - \sum_{j=1}^{4} b_j = 16.104.$$

	Bucharest D_1	Craiova D_2	Nis D_3	Skopje D_4	Fict. D_5	a_i
Sofia S_1	34.878	27.692	17.615	26.805	0	38.499
Plovdiv S_2	39.977	37.342	34.401	40.499	0	31.912
Varna S_3	29.262	39.587	60.627	73.406	0	32.271
b_j	9.445	24.470	26.190	26.472	16.104	102.682

Table 6: Input data of the transportation problem

We solve the transportation problem with one of the known methods. It is assumed that the minimum transportation costs for the next period are 2570.115 euro at:

$$X_{opt} = \left[\begin{array}{cccc|c} 0 & 0 & 26.190 & 12.309 & 0 \\ 0 & 17.749 & 0 & 14.163 & 0 \\ 9.445 & 6.721 & 0 & 0 & 16.104 \end{array} \right],$$

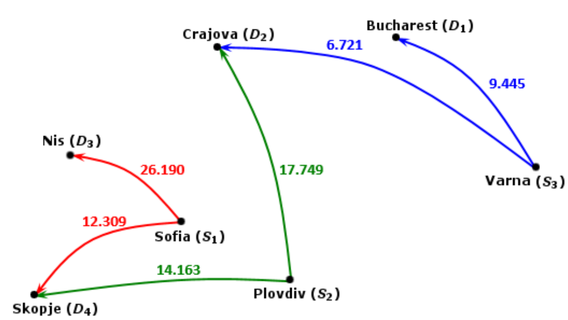

Figure 6: Optimal quantities of the commodity (in tons) and shipping destinations for the next period

The sheme of optimal shipments for the next period is shown in Figure 6.

6 Conclusion

The most important achievement in this research is the obtaining of the ranking index of the exponential fuzzy numbers with membership function (3).

The function (3) has some advantages over the traditionally used trapezoidal and generalized trapezoidal membership functions. The formulas obtained in this study make it possible the membership function (3) to be used to solve a wide range of problems where a fuzzy approach based on frequency distributions is required.

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References

[1] N. U. Ahmed, A. R. Khan, Md. S. Uddinl. Solution of Mixed Type Transportation Problem: A Fuzzy Approach. *Buletinul Institutului Politehnic din Iai: Automatic Control and Computer Science Section* LXI (LXV), 2:19-32, 2015.

[2] A. Alim, F. T. Johora, S. Babu, A. Sultana. Elementary Operations on L-R Fuzzy Number. *Advances in Pure Mathematics* 5(3):131-136, 2015.

[3] L. C. de Barros, R. C. Bassanezi, W. A. Lodwick. *A First Course in Fuzzy Logic, Fuzzy Dynamical Systems, and Biomathematics, Studies in Fuzziness and Soft Computing.* Springer-Verlag Berlin Heidelberg, 2017.

[4] R. E. Bellman, L. A. Zadeh. Decision-making in a fuzzy environment. *Management Science* 17(4):141-164, 1970.

[5] M. Brunelli, J. Mezei. How different are ranking methods for fuzzy numbers? A numerical study. *International Journal of Approximate Reasoning* 54(5):627-639, 2013.

[6] S. Chanas, W. Kolodziejckzy, A. Machaj. A fuzzy approach to the transportation problem. *Fuzzy Sets and Systems* 13(3):211-221, 1984.

[7] S. Chanas, D. Kuchta. A concept of the optimal solution of the transportation problem with fuzzy cost coefficients. *Fuzzy Sets and Systems* 82(3):299-305, 1996.

[8] D. S. Dinagar, K. Palanivel. The transportation problem in fuzzy environment. *International Journal of Algorithms Computing and Mathematics* 2: 65-71, 2009.

[9] A. N. Gani, K. A. Razak. Two stage fuzzy transportation problem. *Journal of Physical Sciences* 10:63-69, 2006.

[10] R. Jain. Decision-making in the presence of fuzzy variables. *IEEE Transactions on Systems, Man and Cybernetics* 6(10):698-703, 1976.

[11] J.-S. R. Jang, C.-T. Sun, E. Mizutani. *Neuro-Fuzzy and Soft Computing: A Computational Approach to Learning and Machine Intelligence.* Prentice-Hall, 1997.

[12] A. Kaur, A. Kumar. A New Approach for Solving Fuzzy Transportation Problems Using Generalized Trapezoidal Fuzzy Numbers. *Applied Soft Computing* 12(3):1201-1213, 2012.

[13] A. Kumar, P. Singh, A. Kaur. Ranking of Generalized Exponential Fuzzy Numbers using Integral Value Approach. *International Journal Advances in Soft Computing and Its Applications* 2(2):1-10, 2010.

[14] S.-T. Liu, C. Kao. Solving fuzzy transportation problems based on extension principle. *European journal of Operation Research* 153(3):661-674, 2004.

[15] S. Muruganandam, R. Srinivasan. A New Algorithm for Solving Fuzzy Transportation Problems with Trapezoidal Fuzzy Numbers. *International Journal of Recent Trends in Engineering and Research (IJRTER)* 2(3):428-437, 2016.

[16] R. Nagarajan, A. Solairaju. Computing Improved Fuzzy Optimal Hungarian Assignment Problems with Fuzzy Costs under Robust Ranking Techniques. *International Journal of Computer Applications (0975 8887)* 6(4):6-13, 2010.

[17] E. Nasibov, S. Peker. Exponential Membership Function Evaluation based on Frequency. *Asian Journal of Mathematics and Statistics* 4(1): 8-20, 2011.

[18] M. OhEigeartaigh. A fuzzy transportation algorithm. *Fuzzy Sets and Systems* 8(3):235-243, 1982.

[19] P. Pandian, G. Natarajan. A new algorithm for finding a fuzzy optimal solution for fuzzy transportation problems. *Applied Mathematical Sciences* 4(2):79-90, 2010.

[20] T. J. Ross. *Fuzzy Logic with Engineering Applications,* Second edition. John Wiley and Sons Ltd, 2004.

[21] O. M. Saad, S. A. Abbas. A parametric study on transportation problem under fuzzy environment. *European journal of Operation Research* 11(1):115-124, 2003.

[22] G. Sharma, S. H. Abbas, V. K. Guptal. A New Approach to solve Fuzzy Transportation Problem for Trapezoidal Number. *Journal of Progressive Research in Mathematics (JPRM)* 4(3):386-392, 2015.

[23] A. Thamaraiselvi, R. Santhi. Solving Fuzzy Transportation problem with Generalized Hexagonal Fuzzy Numbers. *IOSR Journal of Mathematics (IOSR-JM)* 11(5, Ver.II):8-13, 2015.

[24] R. Yager. A procedure for ordering fuzzy subsets of the unit interval. *Information Sciences* 24(2):143-161, 1981.

[25] L. A. Zadeh. Fuzzy sets as a basic for a theory of possibility. *Fuzzy Sets and Systems* 1(1):3-28, 1978.

[26] H.-J. Zimmermann. Fuzzy programming and linear programming with several objective functions. *Fuzzy Sets and Systems* 1(1):45-55, 1978.

[27] H.-J. Zimmermann. Fuzzy set theory. *WIREs Computational Statistics* 2(3):317-332, 2010.

A Appendix

$c_{14}(t)$		$c_{21}(t)$		$c_{22}(t)$		$c_{23}(t)$		$c_{24}(t)$	
m_k	f_k	m_k	f_k	m_k	f_k	m_k	f_k	m_k	f_k
22.5	1	36.5	1	36.5	2	28.5	1	36.5	1
23.5	2	37.5	5	37.5	4	29.5	1	37.5	5
24.5	3	38.5	6	38.5	4	30.5	2	38.5	6
25.5	6	39.5	5	39.5	5	31.5	3	39.5	4
26.5	5	40.5	4	40.5	6	32.5	5	40.5	3
27.5	4	41.5	2	41.5	4	33.5	4	41.5	3
28.5	2	42.5	1	42.5	1	34.5	4	42.5	2
29.5	2	43.5	1			35.5	3	43.5	1
30.5	1	44.5	1			36.5	2	44.5	1
						37.5	1		
$c_{31}(t)$		$c_{32}(t)$		$c_{33}(t)$		$c_{34}(t)$			
m_k	f_k	m_k	f_k	m_k	f_k	m_k	f_k		
23.5	1	37.5	1	56.5	1	68.5	1		
24.5	2	38.5	3	57.5	3	69.5	3		
25.5	4	39.5	4	58.5	5	70.5	5		
26.5	5	40.5	4	59.5	4	71.5	5		
27.5	3	41.5	5	60.5	6	72.5	4		
28.5	4	42.5	5	61.5	5	73.5	2		
29.5	3	43.5	3	62.5	1	74.5	3		
30.5	2	44.5	1	63.5	1	75.5	2		
31.5	1					76.5	1		
32.5	1								

Table 7: Frequency distributions of the transportation costs

$a_1(t)$		$a_2(t)$		$a_3(t)$			
m_k	f_k	m_k	f_k	m_k	f_k		
34.5	1	28.5	1	27.5	1		
35.5	1	29.5	1	28.5	1		
36.5	2	30.5	2	29.5	2		
37.5	4	31.5	4	30.5	1		
38.5	4	32.5	7	31.5	2		
39.5	7	33.5	8	32.5	4		
40.5	6	34.5	2	33.5	5		
41.5	1	35.5	1	34.5	6		
				35.5	3		
				36.5	1		
$b_1(t)$		$b_2(t)$		$b_3(t)$		$b_4(t)$	
m_k	f_k	m_k	f_k	m_k	f_k	m_k	f_k
6.5	1	20.5	1	22.5	1	22.5	1
7.5	4	21.5	5	23.5	3	23.5	3
8.5	9	22.5	6	24.5	5	24.5	5
9.5	8	23.5	5	25.5	6	25.5	6
10.5	2	24.5	3	26.5	4	26.5	4
11.5	1	25.5	3	27.5	3	27.5	3
12.5	1	26.5	2	28.5	1	28.5	1
		27.5	1	29.5	2	29.5	2
				30.5	1	30.5	1

Table 8: Frequency distributions of the supply and demand quantities

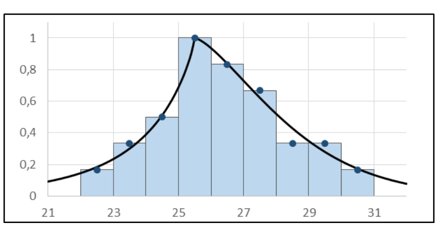

Figure 7: The membership function of $A^{c_{14}}$

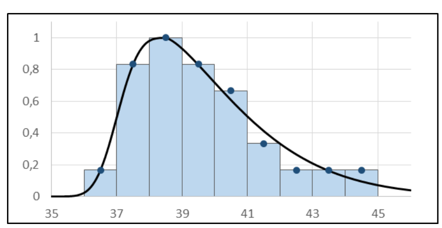

Figure 8: The membership function of $A^{c_{21}}$

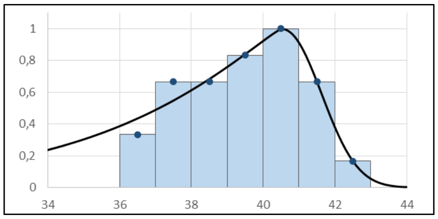

Figure 9: The membership function of $A^{c_{22}}$

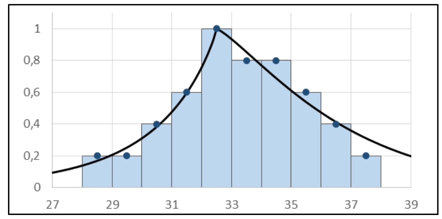

Figure 10: The membership function of $A^{c_{23}}$

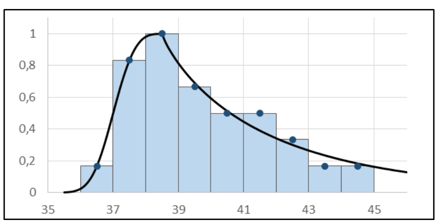

Figure 11: The membership function of $A^{c_{24}}$

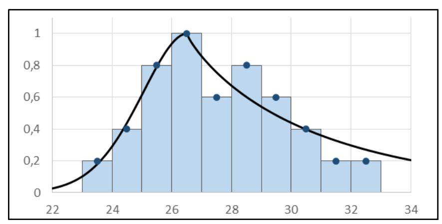

Figure 12: The membership function of $A^{c_{31}}$

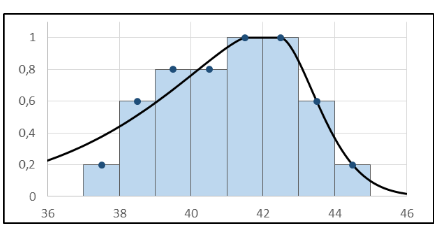

Figure 13: The membership function of $A^{c_{32}}$

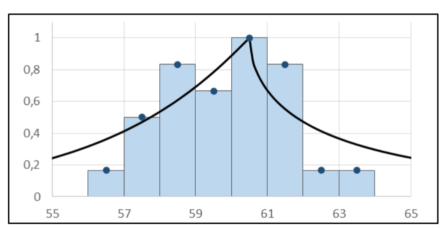

Figure 14: The membership function of $A^{c_{33}}$

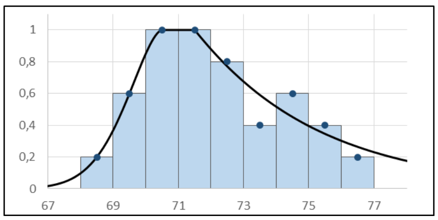

Figure 15: The membership function of $A^{c_{34}}$

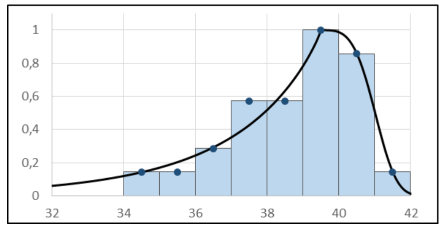

Figure 16: The membership function of A^{a_1}

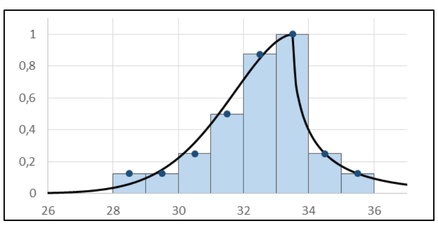

Figure 17: The membership function of A^{a_2}

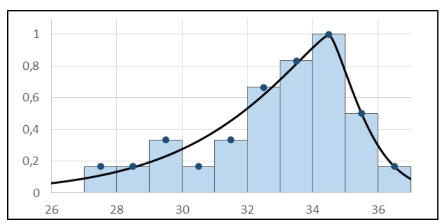

Figure 18: The membership function of A^{a_3}

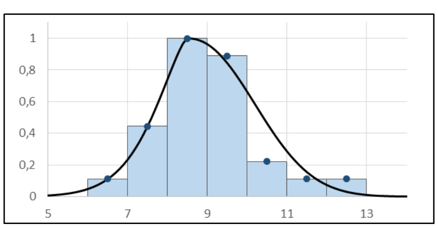

Figure 19: The membership function of A^{b_1}

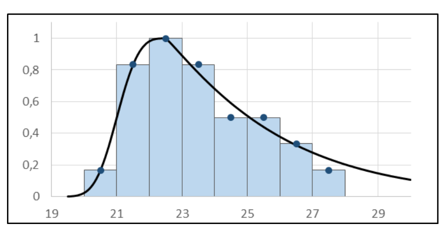

Figure 20: The membership function of A^{b_2}

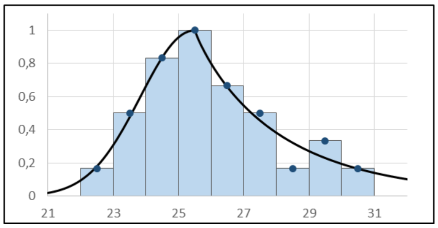

Figure 21: The membership function of A^{b_3}

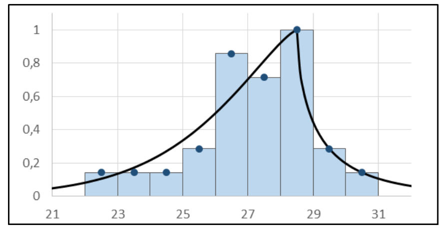

Figure 22: The membership function of A^{b_4}

w	m_L	m_R	m	σ_L	β_L	σ_R	β_R	$r(A^w)$
c_{11}			37.5	4.90792	1.03675	2.09725	0.89364	34.87845
c_{12}			26.5	1.53758	2.53189	2.78713	1.34689	27.69214
c_{13}	17.5	18.5	18	2.13238	2.13509	1.67573	3.29682	17.61461
c_{14}			25.5	1.60701	0.84293	3.36968	1.42501	26.80535
c_{21}			38.5	1.67573	3.29682	3.27479	1.41390	39.97709
c_{22}			40.5	4.71075	1.12976	1.52363	2.14373	37.34228
c_{23}			32.5	2.09725	0.89364	4.41528	1.24462	34.40106
c_{24}			38.5	1.67573	3.29682	3.22208	0.85119	40.49933
c_{31}			26.5	2.22631	1.82425	4.39174	0.86079	29.26247
c_{32}	41.5	42.5	42	4.07162	1.31074	1.50038	1.65565	39.58700
c_{33}			60.5	3.93803	1.03616	2.48511	0.57027	60.62678
c_{34}	70.5	71.5	71	1.50038	1.65565	3.76838	1.01348	73.40617
a_1			39.5	2.38677	0.90410	1.66722	3.65803	38.49853
a_2			33.5	2.78432	1.77162	0.57213	0.58496	31.91244
a_3			34.5	3.65487	1.21883	1.30669	1.37014	32.27063
b_1			8.5	1.15689	1.43803	2.25341	2.17970	9.44544
b_2			22.5	1.67573	3.29682	3.59257	1.09392	24.46982
b_3			25.5	2.30984	2.06355	2.61023	0.90627	26.18999
b_4			28.5	3.28056	1.34199	0.70139	0.63533	26.47201

Table 9: Parameter values and ranking indices of the fuzzy numbers