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ON SOME DICHOTOMY PROPERTIES OF DYNAMICAL SYSTEMS ON THE WHOLE LINE*

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Abstract

The aim of this paper is to present several dichotomy properties of nonautonomous systems in Banach spaces. We discuss the connections between discrete-time and continuous-time dichotomic behavior on the whole line. The key points of the method are the discrete asymptotic properties and the input-output techniques. As consequences, we present several applications and criteria for exponential dichotomy of evolution families on the whole line, which extend the previous results in this framework.

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1 Introduction

Exponential dichotomy was always among the most important asymptotic properties, playing a substantial role in the qualitative theory of dynamical systems. Starting with the iconic landmarks established by Massera and Schäffer [35], Coffman and Schäffer [19], Coppel [20], Daleckĭi and Kreĭn [21], the studies devoted to exponential dichotomy were developed around several central directions: the detection of an exponential dichotomy through various methods (see Aulbach and Minh [1], Barreira and Valls [3], Baskakov [7], [8], Chicone and Latushkin [17], Chow and Leiva [18], Huy and Minh [31], Huy [32], Latushkin, Randolph and Schnaubelt [34], Megan, Sasu and Sasu [36], [37], Minh, Räbiger and Schnaubelt [41], Minh and Huy [42], Minh [43], Palmer [45], [46], Sasu and Sasu [55]-[57], Sasu [58], Sasu [60], [61], Sasu and Sasu [62], Zhang [66], Zhou and Zhang [67], Zhou, Lu and Zhang [68]), the robustness of the exponential dichotomy subjected to certain classes of perturbations (see e.g. Barreira and Valls [2], [4], Dragičević and Zhang [24], Sasu [61], Zhou and Zhang [67]) and the qualitative properties of dynamical systems related to exponential dichotomy (see Barreira, Dragičević and Valls [6], Dragičević [23], Elaydi and Hájek [26], Elaydi and Janglajew [27], Henry [30], Mihiţ, Megan and Ceauşu [39], Mihiţ, Borlea and Megan [40], Palmer [44], [45], [47], [48], Pötzsche [52], [53], Zhou and Zhang [67] and the references therein).

A special class of methods in the study of the exponential dichotomy is represented by the Perron type methods. The theory has the origins in the pioneering work of Perron [49] and the method essentially consists in characterizing an asymptotic property of a dynamical system in terms of the solvability of a well-chosen associated input-output system or by using certain properties of some input-output operators. A substantial contribution in this framework was made in the sixties by Massera and Schäffer [35] and Coffman and Schäffer [19], by introducing the notion of admissibility. We also mention the remarkable monographs of Daleckĭi and Kreĭn [21] and Coppel [20]. Outstanding achievements in the study of the dichotomy were obtained in the works of Ben-Artzi, Gohberg [11], Ben-Artzi, Gohberg and Kaashoek [12], Palmer [44] and Zhang [66], where the dichotomy of nonautonomous systems was expressed in terms of invertibility or of Fredholm properties of the associated input-output operator. In the case of variational dynamical systems, important steps were made by Chow and Leiva [18]. Chronologically, one of the next notable steps was made by Aulbach and Minh [1] in the case of difference equations. A remarkable stage was marked by Minh, Räbiger and Schaubelt [41], where the authors obtained a unified study in terms of Perron methods for stability, expansivity and dichotomy of evolution families on the half-line. Their work was the starting point for many studies devoted to input-output techniques in the asymptotic theory of dynamical systems over the past twenty years (see e.g. Huy and Minh [31], Megan, Sasu and Sasu [36], [37], Minh and Huy [42], Sasu and Sasu [62] and the references therein). Notable results for nonautonomous systems on the whole line were obtained by Y. Latushkin, T. Randolph and R. Schnaubelt [34]. It should be mentioned also the monograph of Chicone and Latushkin [17] for a beautiful and extensive study on the Perron methods both for nonautonomous and variational systems, based on the theory of evolution semigroups.

The Perron methods, which are also known as input-output techniques, have had a strong impact to the development of the asymptotic theory of dynamical systems. A large number of papers in this area were focused on stability properties and their robustness in the presence of perturbations (Berezansky and Braverman [13], Braverman and Zhukovskiy [14], Braverman and Karabash [16], Pituk [50], Sasu and Sasu [54]). A substantial part of the studies based on Perron criteria were devoted to various dichotomic behaviors (Barreira and Valls [3], Barreira, Dragičević and Valls [6], Chow and Leiva [18], Henry [30], Huy and Minh [31], Huy [32], Megan, Sasu and Sasu [36], [37], Minh and Huy [42], Minh [43], Palmer [44], [46], Sasu and Sasu [55]-[57], Sasu [58], Sasu and Sasu [62], [64], Sasu [60], [61], [63]), but also to expansiveness properties (Megan, Sasu and Sasu [38], Palmer [46], Sasu [59]). Moreover the Perron methods have proved to be very important in the study of the most complex asymptotic property of dynamical systems - the exponential trichotomy (see e.g. Elaydi and Hajék [25], Elaydi and Janglajew [27], Sasu [63], Sasu and Sasu [64], [65] and the references therein).

In the last decades, the theory of discrete dynamical systems had an intensive development and represented the foundation for many notable answers to a large variety of open problems (see e.g. Elaydi [28], [29], Henry [30], Kloeden, Pötzsche and Rasmussen [33], Pötzsche [51]). In many interesting problems arising in the asymptotic theory of dynamical systems, their treatment in the framework of discrete dynamics led to spectacular conclusions and also to new applications (see Aulbach and Minh [1], Barreira and Valls [2], [4], [5], Baskakov [7]–[9], Baskakov and Kharitonov [10], Berezansky and Braverman [13], [15], Braverman and Zhukovskiy [14], Braverman and Karabash [16], Chow and Leiva [18], Dragičević [22], [23], Elaydi and Janglajew [27], Huy and Minh [31], Megan, Sasu and Sasu [37], Minh [43], Palmer [45], [47], [48], Pötzsche [52], [53], Sasu and Sasu [54]-[57], [62], [64], Sasu [58], Sasu [61], [63], Zhou and Zhang [67]). An interesting subject in this framework is to establish the connections between an asymptotic behavior of a dynamical system and the homologous behavior of the associated discrete-time system. If in the stability case the connections are clear, when the asymptotic properties become more complex, like expansiveness, dichotomy or even more in the trichotomy case, the techniques are more and more diverse and in certain situations more complicated (see Baskakov [8], Megan, Sasu and Sasu [37], [38], Palmer [45], Sasu [59], Sasu and Sasu [55], [62], [64] and the references therein).

In this paper, we present connections between discrete dichotomy and continuous-time dichotomy of nonautonomous systems defined on the whole line. We consider a constructive method, starting from the discrete-time behavior, which will involve direct computations as well as input-output techniques based on a criteria for uniform exponential dichotomy. We show step by step how one can (re)build the family of dichotomy projections on the whole line beginning from the existence of the discrete-time projections for uniform exponential dichotomy. Thus, we expose a direct construction for all the dichotomic properties based on the initial discrete behavior and discuss technical aspects that present advantages compared with other approaches in this topic. As consequences, we present characterizations for uniform exponential dichotomy of nonautonomous systems, which extend the previous discrete input-output criteria for uniform exponential dichotomy of evolution families on the whole line.

2 Uniform exponential dichotomy of discrete systems

For the sake of clarity we begin with several basic notations and definitions.

Indeed, let X be a real or complex Banach space and let I_d be the identity operator on X. The norm on X and on $\mathcal{B}(X)$ - the space of all bounded linear operators on X - will be denoted by $||\cdot||$. Throughout this paper, $\mathbb{R}$ will denote the set of real numbers and $\mathbb{R}_+$ the set of positive real numbers. We denote by $\mathbb{Z}$ the set of real integers and by $\ell^\infty(\mathbb{Z}, X)$ the space of all bounded sequences $s : \mathbb{Z} \to X$, which is a Banach space with respect to the norm

$$||s||_\infty := \sup_{n\in\mathbb{N}} ||s(n)||.$$

Let $\{A(n)\}_{n\in\mathbb{Z}} \subset \mathcal{B}(X)$. We consider the discrete nonautonomous system

$$(A) \qquad x(n+1) = A(n)x(n), \quad n \in \mathbb{Z}.$$

Let $\Delta = \{(m,n) \in \mathbb{Z}\times\mathbb{Z} : m \geq n\}$. The discrete evolution family associated to (A) is $\Phi_A = \{\Phi_A(m,n)\}_{(m,n)\in\Delta}$ given by

$$\Phi_A(m,n) = \begin{cases} A(m-1)\dots A(n), & m > n \\ I_d, & m = n \end{cases}.$$

Remark 1. $\Phi_A = \{\Phi_A(m,n)\}_{(m,n)\in\Delta}$ satisfies the evolution property

$$\Phi_A(m,j)\Phi_A(j,n) = \Phi_A(m,n), \quad \forall (m,j),(j,n) \in \Delta.$$

Moreover, the discrete-time system (A) has uniformly bounded coefficients, i.e. $\sup_{n\in\mathbb{Z}} ||A(n)|| < \infty$ if and only if Φ_A has uniform exponential growth, i.e. there is $\omega \in \mathbb{R}$ such that

$$||\Phi_A(m,n)|| \leq e^{\omega(m-n)}, \quad \forall (m,n) \in \Delta.$$

We recall that an operator $P \in \mathcal{B}(X)$ is a *projection* if $P^2 = P$. Then $RangeP$ and $KerP$ are closed linear subspaces, P-invariant and $X = RangeP \oplus KerP$.

Definition 1. We say that the system (A) has a *uniform exponential dichotomy* if there exist a family of projections $\{P(n)\}_{n\in\mathbb{Z}}$ and two constants $N \geq 1, \nu > 0$ such that the following properties are satisfied:

(i) $A(n)P(n) = P(n+1)A(n)$, for all $n \in \mathbb{Z}$;

(ii) $||\Phi_A(m,n)x|| \leq Ne^{-\nu(m-n)}||x||$, for all $x \in Range\ P(n)$ and all $(m,n) \in \Delta$;

(iii) $||\Phi_A(m,n)y|| \geq \frac{1}{N}\, e^{\nu(m-n)}||y||$, for all $y \in Ker\ P(n)$ and all $(m,n) \in \Delta$;

(iv) for every $n \in \mathbb{Z}$, the restriction $A(n)_| : KerP(n) \to KerP(n+1)$ is an isomorphism.

Remark 2. From Definition 1 (i) it immediately follows that if (A) has a uniform exponential dichotomy with respect to the family of projections $\{P(n)\}_{n\in\mathbb{Z}}$, then $\Phi_A(m,n)P(n) = P(m)\Phi_A(m,n)$, for all $(m,n) \in \Delta$.

We associate to the system (A) the input-output system

$$(S_A) \qquad \gamma(n+1) = A(n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{Z}$$

with $s, \gamma \in \ell^\infty(\mathbb{Z}, X)$.

Definition 2. The pair $(\ell^\infty(\mathbb{Z}, X), \ell^\infty(\mathbb{Z}, X))$ is said to be *admissible* for the system (S_A) if for every $s \in \ell^\infty(\mathbb{Z}, X)$ there exists a unique $\gamma \in \ell^\infty(\mathbb{Z}, X)$ such that the pair (γ, s) satisfies the system (S_A).

The connections between the existence of the uniform exponential dichotomy and various admissibility properties with pairs of sequence spaces were established in [61]. There we discussed the axiomatic structures of the sequence spaces that could be considered in the admissible pair as input space and also as output space.

As a consequence of the main result in [61], we deduce the following:

Theorem 1. *The following assertions are equivalent:*

(i) if the pair $(\ell^\infty(\mathbb{Z}, X), \ell^\infty(\mathbb{Z}, X))$ is admissible for the system (S_A), then the system (A) has a uniform exponential dichotomy;

(ii) if $\sup_{n\in\mathbb{Z}} ||A(n)|| < \infty$, then the system (A) has a uniform exponential dichotomy if and only if the pair $(\ell^\infty(\mathbb{Z}, X), \ell^\infty(\mathbb{Z}, X))$ is admissible for (S_A).

Proof. This follows from Corollary 3.5 in [61] for $W(\mathbb{Z}, X) = \ell^\infty(\mathbb{Z}, X)$.
□

Remark 3. A distinct proof for (ii) was given in [55]. We also refer to Henry [30], for an approach based on Green functions.

In certain conditions, the projections for uniform exponential dichotomy on the whole line are uniformly bounded, uniquely determined and their structures can be expressed in various equivalent forms (see e.g. [55], [56], [61] and the references therein). A natural approach to the properties of the family of projections for uniform exponential dichotomy on the whole line will be presented in what follows.

For every $n \in \mathbb{Z}$ we consider the linear space

$$\mathcal{F}_n(\mathbb{Z}, X) := \{\varphi \in \ell^\infty(\mathbb{Z}, X) : \varphi(k) = A(k-1)\varphi(k-1), \quad \forall k \le n\}.$$

Theorem 2. (*The structure theorem) If the discrete system* (A) *has uniformly bounded coefficients and has a uniform exponential dichotomy with respect to the family of projections* $\{P(n)\}_{n\in\mathbb{Z}}$*, then:*

(i) $\sup\limits_{n\in\mathbb{Z}} ||P(n)|| < \infty$*;*

(ii) *Range* $P(n) = \{x \in X : \sup_{m\ge n} ||\Phi_A(m,n)x|| < \infty\}$*;*

(iii) *Ker* $P(n) = \{x \in X :$ *there exists* $\varphi \in \mathcal{F}_n(\mathbb{Z}, X)$ *with* $\varphi(n) = x\}$*.*

Proof. Let $N \ge 1, \nu > 0$ be two constants given by Definition 1. According to our hypothesis, there is $K > 0$ such that

$$||A(n)|| \le K, \quad \forall n \in \mathbb{Z}. \tag{1}$$

From (1) it follows that

$$||\Phi_A(m,n)|| \le K^{m-n}, \quad \forall (m,n) \in \Delta. \tag{2}$$

(i) Let $h \in \mathbb{N}^*$ be such that

$$e^{2\nu h} > N^2.$$

Let $n \in \mathbb{Z}$ and let $x \in X$. From Definition 1 (iii), (ii) and relation (2) we successively have that

$$\frac{1}{N} e^{\nu h} ||(I - P(n))x|| \le ||\Phi_A(n+h,n)(I-P(n))x|| \le$$

$$\le ||\Phi_A(n+h,n)x|| + ||\Phi_A(n+h,n)P(n)x|| \le K^h ||x|| + Ne^{-\nu h}||P(n)x|| \le$$
$$\le (K^h + N)||x|| + Ne^{-\nu h}||(I-P(n))x||$$

which implies that

$$\frac{e^{2\nu h} - N^2}{Ne^{\nu h}} ||(I-P(n))x|| \le (K^h + N)||x||. \tag{3}$$

Denoting by

$$\delta := \frac{(K^h + N)Ne^{\nu h}}{e^{2\nu h} - N^2}$$

we have that $\delta > 0$. In addition, from (3) we deduce that

$$||(I - P(n))x|| \leq \delta \ ||x||.$$

Since $(n, x) \in \mathbb{Z} \times X$ were arbitrary and δ doesn't depend on n or x, we obtain that

$$||(I - P(n))x|| \leq \delta \ ||x||, \quad \forall x \in X, \forall n \in \mathbb{N}.$$

This implies that

$$||I - P(n)|| \leq \delta, \quad \forall n \in \mathbb{N}$$

which shows that

$$||P(n)|| \leq 1 + \delta, \quad \forall n \in \mathbb{N}.$$

In what follows, we denote by $L := \sup_{n \in \mathbb{Z}} ||P(n)||$.

(ii) Let $n \in \mathbb{Z}$. Obviously, *Range* $P(n) \subset \{x \in X : \sup_{m \geq n} ||\Phi_A(m,n)x|| < \infty\}$. Conversely, let $x \in X$ with $\alpha_x := \sup_{m \geq n} ||\Phi_A(m,n)x|| < \infty$. Then, from Definition 1 (iii) and (i) we successively have that

$$\frac{1}{N} e^{\nu(m-n)} ||(I - P(n))x|| \leq ||\Phi_A(m,n)(I - P(n))x|| =$$

$$= ||(I - P(m))\Phi_A(m,n)x|| \leq (1 + L)\alpha_x, \quad \forall m \geq n$$

which implies that

$$||(I - P(n))x|| \leq (1 + L)\alpha_x N e^{-\nu(m-n)}, \quad \forall m \geq n. \tag{4}$$

For $m \to \infty$ in (4) we obtain that $x = P(n)x$, so $x \in RangeP(n)$.

(iii) Let $n \in \mathbb{Z}$. We consider the subspace $\Omega(n) := \{x \in X : \text{ there is } \varphi \in \mathcal{F}_n(\mathbb{Z}, X) \text{ with } \varphi(n) = x\}$.

Let $x \in KerP(n)$. From Definition 1 (iv) we deduce that $\Phi_A(m,n)_| : KerP(n) \to KerP(m)$ is invertible, for all $m \geq n$, and we denote by $\Phi_A(m,n)_|^{-1}$ its inverse. We consider the sequence

$$\varphi : \mathbb{Z} \to X, \quad \varphi(k) = \begin{cases} 0, & k \geq n+1 \\ x, & k = n \\ \Phi_A(n,k)_|^{-1} x, & k \leq n-1 \end{cases}.$$

Using Definition 1 (iii) we deduce that

$$||\varphi(k)|| \leq N e^{-\nu(n-k)} ||x||, \quad \forall k \leq n.$$

In particular, this shows that $\varphi \in \ell^\infty(\mathbb{Z}, X)$. Moreover, an easy computation shows that

$$\varphi(k) = A(k-1)\varphi(k-1), \quad \forall k \le n$$

so $\varphi \in \mathcal{F}_n(\mathbb{Z}, X)$. This shows that $x \in \Omega(n)$. Thus, we have that $KerP(n) \subset \Omega(n)$.

Conversely, let $x \in \Omega(n)$. Then there is $\delta \in \mathcal{F}_n(\mathbb{Z}, X)$ with $\delta(n) = x$. We successively have that

$$||P(n)x|| = ||P(n)\delta(n)|| = ||P(n)\Phi_A(n,k)\delta(k)|| = ||\Phi_A(n,k)P(k)\delta(k)|| \le$$
$$\le Ne^{-\nu(n-k)}||P(k)\delta(k)|| \le LN||\delta||_\infty \, e^{-\nu(n-k)}, \quad \forall k \le n. \tag{5}$$

For $k \to -\infty$ in (5) we have that $P(n)x = 0$, so $x \in KerP(n)$. We obtain that $\Omega(n) \subset KerP(n)$ and the proof is complete. □

Remark 4. Although the projections for uniform exponential dichotomy on the whole line are uniquely determined, their structure may be represented by using diverse methods (see also Proposition 2.2 in [55], Lemma 2.1 in [56]). The key-motivation for choosing a representation (or another) depends on the input-output methods involved.

3 Exponential dichotomy of nonautonomous systems

Let X be a real or complex Banach space and let I_d be the identity operator on X. First, we briefly recall some definitions, notations and basic properties.

Definition 3. A family $\mathcal{U} = \{U(t,s)\}_{t \ge s} \subset \mathcal{B}(X)$ is called an *evolution family* if the following properties hold:

(i) $U(t,t) = I_d$, for all $t \in \mathbb{R}$;

(ii) $U(t,\tau)U(\tau,s) = U(t,s)$, for all $t \ge \tau \ge s$;

(iii) there exist $M \ge 1, \omega > 0$ such that $||U(t,s)|| \le Me^{\omega(t-s)}$, for all $t \ge s$.

Definition 4. We say that an evolution family $\mathcal{U} = \{U(t,s)\}_{t \ge s}$ *has a uniform exponential dichotomy* if there exist a family of projections $\{P(t)\}_{t\in\mathbb{R}}$ and two constants $N \ge 1, \nu > 0$ such that the following properties are satisfied:

(i) $U(t,s)P(s) = P(t)U(t,s)$, for all $t \geq s$;

(ii) $||U(t,s)x|| \leq Ne^{-\nu(t-s)}||x||$, for all $x \in RangeP(s)$ and all $t \geq s$;

(iii) $||U(t,s)y|| \geq \frac{1}{N}e^{\nu(t-s)}||y||$, for all $y \in KerP(s)$ and all $t \geq s$;

(iv) for every $t \geq s$, the restriction $U(t,s)_{|} : KerP(s) \to KerP(t)$ is an isomorphism.

Let $\mathcal{U} = \{U(t,s)\}_{t \geq s}$ be an evolution family on X. We associate to $\mathcal{U}$ the discrete nonautonomous system

$$(A_{\mathcal{U}}) \qquad x(n+1) = U(n+1,n)x(n), \quad \forall n \in \mathbb{Z}.$$

Remark 5. The discrete evolution family associated to the discrete system $(A_{\mathcal{U}})$ is $\Phi_{\mathcal{U}} = \{\Phi_{\mathcal{U}}(m,n)\}_{(m,n)\in\Delta}$, where

$$\Phi_{\mathcal{U}}(m,n) = U(m,n), \quad \forall (m,n) \in \Delta.$$

Remark 6. From Definition 3 (iii) we have that

$$||U(n+1,n)|| \leq Me^{\omega}, \quad \forall n \in \mathbb{Z}.$$

This shows that the system $(A_{\mathcal{U}})$ has uniformly bounded coefficients.

We associate to the system $(A_{\mathcal{U}})$ the input-output system

$$(S_{\mathcal{U}}) \qquad \gamma(n+1) = U(n+1,n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{Z}$$

with $s, \gamma \in \ell^{\infty}(\mathbb{Z}, X)$.

For every $t_0 \in \mathbb{R}$, we consider the linear subspace

$$\mathcal{F}_{t_0}(\mathbb{R}, X) := \{f : \mathbb{R} \to X : \sup_{t\in\mathbb{R}} ||f(t)|| < \infty \\ \text{and } f(t) = U(t,s)f(s), \text{ for all } s \leq t \leq t_0\}$$

We also consider

$$\mathcal{S}(t_0) := \{x \in X : \sup_{t \geq t_0} ||U(t,t_0)x|| < \infty\}$$

called *the stable subspace* at the moment t_0 and

$$\mathcal{U}(t_0) := \{x \in X : \text{ there is } f \in \mathcal{F}_{t_0}(\mathbb{R}, X) \text{ with } f(t_0) = x\}$$

called *the unstable subspace* at the moment t_0.

Remark 7. If $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$, then $f(t) \in \mathcal{U}(t)$, for all $t \le t_0$.

We begin with a technical property.

Lemma 1. *Let $t, t_0 \in \mathbb{R}$ with $t \ge t_0$. Then:*

(i) $U(t, t_0)\mathcal{S}(t_0) \subseteq \mathcal{S}(t)$;

(ii) $U(t, t_0)\mathcal{U}(t_0) = \mathcal{U}(t)$.

Proof. This is a classical proof based directly on the definition of the subspaces $\mathcal{S}(\cdot)$ and $\mathcal{U}(\cdot)$, but for the sake of clarity we will present all the details. Indeed, (i) immediately follows from the way how the subspaces $\mathcal{S}(\cdot)$ are defined.

(ii) Let $x \in \mathcal{U}(t_0)$. Then, there is $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ with $f(t_0) = x$. We consider the function

$$h : \mathbb{R} \to X, \quad h(s) = \begin{cases} 0, & s > t \\ U(s, t_0)x, & s \in [t_0, t] \\ f(s), & s < t_0 \end{cases} .$$

Using the fact that $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ it is easy to see that

$$h(\tau) = U(\tau, s)h(s), \quad \forall s \le \tau \le t.$$

In addition, we have that $\sup_{s \in \mathbb{R}} ||h(s)|| < \infty$, so $h \in \mathcal{F}_t(\mathbb{R}, X)$. This implies that $U(t, t_0)x = h(t) \in \mathcal{U}(t)$.

Conversely, let $y \in \mathcal{U}(t)$. Then, there is $\varphi \in \mathcal{F}_t(\mathbb{R}, X)$ with $\varphi(t) = y$. Using Remark 7 we have that $\varphi(t_0) \in \mathcal{U}(t_0)$. Then, denoting by $x = \varphi(t_0)$ we deduce that $y = \varphi(t) = U(t, t_0)x$. This shows that $y \in U(t, t_0)\mathcal{U}(t_0)$. It follows that $\mathcal{U}(t) \subset U(t, t_0)\mathcal{U}(t_0)$. □

Theorem 3. *If the system $(A_{\mathcal{U}})$ has a uniform exponential dichotomy with respect to the family of projections $\{P(n)\}_{n \in \mathbb{Z}}$, then:*

(i) *Range* $P(n) = \mathcal{S}(n)$, *for all* $n \in \mathbb{Z}$;

(ii) *Ker* $P(n) = \mathcal{U}(n)$, *for all* $n \in \mathbb{Z}$;

(iii) *there are two constants* $L, \nu > 0$ *such that:*

- *(a)* $||U(t, t_0)x|| \le L\, e^{-\nu(t-t_0)}||x||$, *for all* $x \in \mathcal{S}(t_0)$ *and all* $t \ge t_0$;
- *(b)* $||U(t, t_0)y|| \ge \frac{1}{L}\, e^{\nu(t-t_0)}||y||$, *for all* $y \in \mathcal{U}(t_0)$ *and all* $t \ge t_0$;

(iv) the restriction $U(t,t_0)_{|}: \mathcal{U}(t_0) \to \mathcal{U}(t)$ *is an isomorphism, for all* $t \geq t_0$.

Proof. Let $M \geq 1$ and $\omega > 0$ be such that

$$||U(t,s)|| \leq Me^{\omega(t-s)}, \quad \forall t \geq s. \tag{6}$$

Let $N \geq 1, \nu > 0$ be the dichotomy constants given by Definition 1 for $(A_{\mathcal{U}})$. We set

$$L := M^2 N e^{2(\omega+\nu)}. \tag{7}$$

(i) For every $n \in \mathbb{Z}$ we consider the subspace

$$X_1(n) = \{x \in X : \sup_{m \geq n} ||U(m,n)x|| < \infty\}.$$

From Remark 5, Remark 6 and Theorem 2 we have that

$$Range\ P(n) = X_1(n), \quad \forall n \in \mathbb{Z}. \tag{8}$$

Let $n \in \mathbb{Z}$. It is obvious that $\mathcal{S}(n) \subset X_1(n)$.

Conversely, let $x \in X_1(n)$ and let $\delta_x = \sup_{m \geq n} ||U(m,n)x||$. Then, using relation (6) we deduce that

$$||U(t,n)x|| \leq ||U(t,[t])||\ ||U([t],n)x|| \leq Me^{\omega}\delta_x, \quad \forall t \geq n$$

which implies that $x \in \mathcal{S}(n)$.

It follows that $X_1(n) = \mathcal{S}(n)$. From (8) we obtain that

$$Range\ P(n) = \mathcal{S}(n), \quad \forall n \in \mathbb{Z}.$$

(ii) For every $n \in \mathbb{Z}$, we consider the subspace

$$X_2(n) = \{x \in X : \text{ there exists } \varphi \in \ell^\infty(\mathbb{Z}, X) \text{ with } \varphi(n) = x$$
$$\text{and } \varphi(k) = U(k,k-1)\varphi(k-1), \quad \forall k \leq n\}.$$

From Remark 5, Remark 6 and Theorem 2 we have that

$$Ker\ P(n) = X_2(n), \quad \forall n \in \mathbb{Z}. \tag{9}$$

We easily observe that $\mathcal{U}(n) \subset X_2(n)$. Conversely, let $x \in X_2(n)$. Then, there exists $\varphi \in \ell^\infty(\mathbb{Z}, X)$ with $\varphi(n) = x$ and

$$\varphi(k) = U(k,k-1)\varphi(k-1), \quad \forall k \leq n. \tag{10}$$

We consider the function

$$f : \mathbb{R} \to X, \quad f(t) = U(t, [t])\varphi([t]).$$

Then $\sup_{t\in\mathbb{R}} ||f(t)|| < \infty$. Moreover, from (10) we deduce that

$$f(t) = U(t, [t])U([t], [\tau])\varphi([\tau]) = U(t, \tau)f(\tau), \quad \forall \tau \le t \le n.$$

It follows that $f \in \mathcal{F}_n(\mathbb{R}, X)$. Since $f(n) = x$ we obtain that $x \in \mathcal{U}(n)$. So $X_2(n) \subset \mathcal{U}(n)$.

It follows that $X_2(n) = \mathcal{U}(n)$. Using (9) we deduce that

$$Ker\ P(n) = \mathcal{U}(n), \quad \forall n \in \mathbb{Z}.$$

(iii) Let $t_0 \in \mathbb{R}$.

(a) Let $x \in \mathcal{S}(t_0)$. Let $t \ge [t_0] + 1$. Using Lemma 1 and (ii) we have that $U([t_0]+1, t_0)x \in \mathcal{S}([t_0]+1) = Range\ P([t_0]+1)$. Using the asymptotic behavior of $(A_{\mathcal{U}})$ on $\{Range\ P(n)\}_{n\in\mathbb{Z}}$, the connections given by (i) and relation (6), we successively have that

$$||U([t], t_0)x|| \le Ne^{-\nu([t]-[t_0]-1)}||U([t_0] + 1, t_0)x|| \le NMe^{\omega+2\nu}e^{-\nu(t-t_0)}||x||. \tag{11}$$

Then, from (6), (7) and (11), we deduce that

$$||U(t, t_0)x|| \le ||U(t, [t])||\ ||U([t], t_0)x|| \le Le^{-\nu(t-t_0)}||x||, \quad \forall t \ge [t_0] + 1. \tag{12}$$

In addition, from (6) we immediately obtain that

$$||U(t, t_0)x|| \le Le^{-\nu(t-t_0)}||x||, \quad \forall t \in [t_0, [t_0] + 1]. \tag{13}$$

Finally, from (12) and (13) we have that

$$||U(t, t_0)x|| \le Le^{-\nu(t-t_0)}||x||, \quad \forall x \in \mathcal{S}(t_0), \forall t \ge t_0.$$

(b) Let $y \in \mathcal{U}(t_0)$. Then there exists $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ with $f(t_0) = y$. Let $z = f([t_0])$. Since $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ we have that

$$y = f(t_0) = U(t_0, [t_0])f([t_0]) = U(t_0, [t_0])z.$$

Using relation (6) it follows that

$$||y|| \le Me^{\omega}||z||. \tag{14}$$

Since $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ we have in particular that $f \in \mathcal{F}_{[t_0]}(\mathbb{R}, X)$. Based on (ii), this implies that $z = f([t_0]) \in \mathcal{U}([t_0]) = Ker P([t_0])$.

Let $t \geq t_0$. Using the asymptotic behavior of $(A_{\mathcal{U}})$ on $\{Ker\ P(n)\}_{n\in\mathbb{Z}}$ and relation (14), we successively deduce that

$$||U([t]+1, t_0)y|| = ||U([t]+1, [t_0])z|| \geq \frac{1}{N} e^{\nu([t]+1-[t_0])}||z|| \geq$$

$$\geq \frac{1}{N} e^{\nu(t-t_0)}||z|| \geq \frac{1}{NMe^{\omega}}||y||. \tag{15}$$

Then, from relations (6) and (15) we successively deduce that

$$||U(t, t_0)y|| \geq \frac{1}{Me^{\omega}}||U([t]+1, t_0)y|| \geq$$

$$\geq \frac{1}{NM^2e^{2\omega}} e^{\nu(t-t_0)}||y|| \geq \frac{1}{L} e^{\nu(t-t_0)}||y||.$$

It follows that

$$||U(t, t_0)y|| \geq \frac{1}{L} e^{\nu(t-t_0)}||y||, \quad \forall y \in \mathcal{U}(t_0), \forall t \geq t_0. \tag{16}$$

(iii) Let $t \geq t_0$. From Lemma 1 we have that $U(t, t_0)_{|} : \mathcal{U}(t_0) \to \mathcal{U}(t)$ is surjective. Moreover, from relation (16) we deduce that it is also injective, so the restriction $U(t, t_0)_{|}$ is an isomorphism. □

Theorem 4. *If the discrete system $(A_{\mathcal{U}})$ has a uniform exponential dichotomy, then:*

(i) $\mathcal{S}(t_0) \cap \mathcal{U}(t_0) = \{0\}$*;*

(ii) $\mathcal{S}(t_0)$ *is a closed linear subspace, for all* $t_0 \in \mathbb{R}$*;*

(iii) $\mathcal{U}(t_0)$ *is a closed linear subspace, for all* $t_0 \in \mathbb{R}$*.*

Proof. Let $L, \nu > 0$ be given by Theorem 3 (iii).

(i) Let $t_0 \in \mathbb{R}$ and let $x \in \mathcal{S}(t_0) \cap \mathcal{U}(t_0)$. Then, from Theorem 3 (iii) (a) and (b) we obtain that

$$\frac{1}{L} e^{\nu(t-t_0)}||x|| \leq ||U(t, t_0)x|| \leq Le^{-\nu(t-t_0)}||x||, \quad \forall t \geq t_0. \tag{17}$$

Relation (17) implies that

$$||x|| \leq L^2 e^{-2\nu(t-t_0)}||x||, \quad \forall t \geq t_0. \tag{18}$$

From relation (18) it follows that $x = 0$. Thus $\mathcal{S}(t_0) \cap \mathcal{U}(t_0) = \{0\}$.

(ii) Let $t_0 \in \mathbb{R}$. Let $(x_n) \subset \mathcal{S}(t_0)$ with $x_n \underset{n\to\infty}{\longrightarrow} x$. From Theorem 3 (iii) (a) we obtain that

$$||U(t,t_0)x_n|| \le Le^{-\nu(t-t_0)}||x_n||, \quad \forall n \in \mathbb{N}, \forall t \ge t_0. \tag{19}$$

For $n \to \infty$ in (19) we deduce that

$$||U(t,t_0)x|| \le Le^{-\nu(t-t_0)}||x||, \quad \forall t \ge t_0. \tag{20}$$

From (20) it follows that

$$\sup_{t\ge t_0} ||U(t,t_0)x|| \le L\ ||x||$$

so $x \in \mathcal{S}(t_0)$. This shows that $\mathcal{S}(t_0)$ is closed.

(iii) Let $t_0 \in \mathbb{R}$. Let $(y_n) \subset \mathcal{U}(t_0)$ with $y_n \underset{n\to\infty}{\longrightarrow} y$. For every $n \in \mathbb{N}$ let $f_n \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ with $f_n(t_0) = y_n$.

According to Remark 7 we have that $f_n(t) \in \mathcal{U}(t)$, for all $t \le t_0$ and $n \in \mathbb{N}$. Then, from Theorem 3 (iii) (b) we have that

$$||f_n(t) - f_m(t)|| \le L\ e^{-\nu(t_0-t)}||U(t_0,t)(f_n(t) - f_m(t))|| =$$

$$= L\ e^{-\nu(t_0-t)}||f_n(t_0) - f_m(t_0)|| \le L\ ||y_n - y_m||, \quad \forall n, m \in \mathbb{N}, \forall t \le t_0. \tag{21}$$

Relation (21) implies that for every $t \le t_0$ the sequence $(f_n(t))$ is convergent in X. Thus, it makes sense to define

$$f : \mathbb{R} \to X, \quad f(t) = \begin{cases} y, & t > t_0 \\ \lim\limits_{n\to\infty} f_n(t), & t \le t_0 \end{cases}.$$

Using Theorem 3 (iii) (b) we have that

$$||y_n|| = ||f_n(t_0)|| = ||U(t_0,t)f_n(t)|| \ge \frac{1}{L}e^{\nu(t_0-t)}||f_n(t)||, \quad \forall n \in \mathbb{N}, \forall t \le t_0. \tag{22}$$

From relation (22) it follows that

$$||f_n(t)|| \le L\ ||y_n||, \quad \forall n \in \mathbb{N}, \forall t \le t_0. \tag{23}$$

For $n \to \infty$ in (23), we obtain that $||f(t)|| \le L||y||$, for all $t \le t_0$. This shows that $\sup_{t\in\mathbb{R}} ||f(t)|| < \infty$. In addition, from $f_n \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ we have that

$$f_n(t) = U(t,s)f_n(s), \quad \forall s \le t \le t_0, \forall n \in \mathbb{N}. \tag{24}$$

For $n \to \infty$ in (24) it follows that

$$f(t) = U(t,s)f(s), \quad \forall s \le t \le t_0.$$

This implies that $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$, so $y = f(t_0) \in \mathcal{U}(t_0)$. It follows that $\mathcal{U}(t_0)$ is closed and the proof is complete. $\square$

Theorem 5. *Let $\mathcal{U} = \{U(t,s)\}_{t\ge s}$ be an evolution family on X. Then $\mathcal{U}$ has a uniform exponential dichotomy if and only if the discrete system $(A_{\mathcal{U}})$ associated to $\mathcal{U}$ has a uniform exponential dichotomy.*

Proof. Necessity. From Definition 1 and Definition 4 it follows that if $\mathcal{U}$ has a uniform exponential dichotomy with respect to the family of projections $\{P(t)\}_{t\in\mathbb{R}}$, then $(A_{\mathcal{U}})$ has a uniform exponential dichotomy with respect to the family of projections $\{P(n)\}_{n\in\mathbb{Z}}$.

Sufficiency. According to Theorem 4, we have that for every $t_0 \in \mathbb{R}$ the subspaces $\mathcal{S}(t_0)$ and $\mathcal{U}(t_0)$ are closed and

$$\mathcal{S}(t_0) \cap \mathcal{U}(t_0) = \{0\}. \tag{25}$$

Step 1. We prove that $\mathcal{S}(t_0) + \mathcal{U}(t_0) = X$, for all $t_0 \in \mathbb{R}$.

Let $t_0 \in \mathbb{R}$. Let $x \in X$ and let $h = [t_0]$. We define the sequence

$$s : \mathbb{Z} \to X, \quad s(n) = \begin{cases} -U(h+1, t_0)x, & n = h+1 \\ 0, & n \ne h+1 \end{cases}.$$

Since the system $(A_{\mathcal{U}})$ has a uniform exponential dichotomy, from Remark 6 and Theorem 1 (ii) we deduce that the pair $(\ell^\infty(\mathbb{Z}, X), \ell^\infty(\mathbb{Z}, X))$ is admissible for the input-output system $(S_{\mathcal{U}})$. Then, there is $\gamma \in \ell^\infty(\mathbb{Z}, X)$ such that the pair (γ, s) satisfies the system $(S_{\mathcal{U}})$. This implies that

$$\gamma(h+1) = U(h+1, h)\gamma(h) - U(h+1, t_0)x \tag{26}$$

and

$$\gamma(n+1) = U(n+1, n)\gamma(n), \quad \forall n \ge h+1. \tag{27}$$

From (27), inductively, it follows that

$$\gamma(n) = U(n, h+1)\gamma(h+1), \quad \forall n \ge h+1. \tag{28}$$

From (26) we have that

$$\gamma(h+1) = U(h+1, t_0)U(t_0, h)\gamma(h) - U(h+1, t_0)x =$$

$$= U(h+1, t_0)\,[U(t_0, h)\gamma(h) - x]\,. \tag{29}$$

Let $y := U(t_0, h)\gamma(h) - x$. From (28) and (29) we obtain that

$$\gamma(n) = U(n, t_0)y, \quad \forall n \geq h+1. \tag{30}$$

Let $M \geq 1$ and $\omega > 0$ be such that

$$||U(t, s)|| \leq Me^{\omega(t-s)}, \quad \forall t \geq s. \tag{31}$$

Let $t \geq h+1$. Using relations (30) and (31) we deduce that

$$||U(t, t_0)y|| = ||U(t, [t])U([t], t_0)y|| \leq Me^{\omega}||U([t], t_0)y|| =$$

$$= Me^{\omega}||\gamma([t])|| \leq Me^{\omega}||\gamma||_{\infty}. \tag{32}$$

If $t \in [t_0, h+1) = [t_0, [t_0]+1)$, then we have that

$$||U(t, t_0)y|| \leq Me^{\omega}||y||. \tag{33}$$

From relations (32) and (33) it follows that $\sup_{t \geq t_0} ||U(t, t_0)y|| < \infty$, so $y \in \mathcal{S}(t_0)$.

Since γ is the solution of $(S_{\mathcal{U}})$ corresponding to the input s we have that

$$\gamma(n) = U(n, n-1)\gamma(n-1), \quad \forall n \leq h.$$

Inductively, we deduce that

$$\gamma(n) = U(n, j)\gamma(j), \quad \forall j \leq n \leq h. \tag{34}$$

We consider the function

$$f : \mathbb{R} \to X, \quad f(t) = U(t, [t])\gamma([t]).$$

Then, using relation (34) we deduce that

$$f(t) = U(t, [t])U([t], [\tau])\gamma([\tau]) = U(t, [\tau])\gamma([\tau]) =$$

$$= U(t, \tau)f(\tau), \quad \forall \tau \leq t \leq h. \tag{35}$$

In addition, from relation (31) it follows that

$$||f(t)|| \leq Me^{\omega}||\gamma||_{\infty}, \quad \forall t \in \mathbb{R}. \tag{36}$$

From relations (35) and (36) we obtain that $f \in \mathcal{F}_h(\mathbb{R}, X)$, so $\gamma(h) = f(h) \in \mathcal{U}(h)$. From Lemma 1 (ii) it follows that $z := U(t_0, h)\gamma(h) \in \mathcal{U}(t_0)$.

Thus, we deduce that $x = -y + U(t_0, h)\gamma(h) = -y + z \in \mathcal{S}(t_0) + \mathcal{U}(t_0)$. Finally, taking into account that $x \in X$ and $t_0 \in \mathbb{R}$ were arbitrary, it follows that

$$\mathcal{S}(t_0) + \mathcal{U}(t_0) = X, \quad \forall t_0 \in \mathbb{R}. \tag{37}$$

Step 2. We prove that $\mathcal{U}$ has a uniform exponential dichotomy.

Using relations (25) and (37) we obtain that

$$X = \mathcal{S}(t) \oplus \mathcal{U}(t), \quad \forall t \in \mathbb{R}.$$

For every $t \in \mathbb{R}$, let $P(t)$ be the projection with *Range* $P(t) = \mathcal{S}(t)$ and *Ker* $P(t) = \mathcal{U}(t)$. Then, from Lemma 1 we immediately deduce that

$$U(t, s)P(s) = P(t)U(t, s), \quad t \geq s.$$

Finally, from Theorem 3 (iii) and (iv) we obtain that $\mathcal{U}$ has a uniform exponential dichotomy. □

Remark 8. Using a distinct approach, an equivalent result to Theorem 5 was obtained in [55]. A different proof for Theorem 5 was given in [64], where the result was obtained as a consequence of a more general property - the uniform exponential trichotomy.

Remark 9. For other interesting connections between discrete dichotomies and continuous time dichotomies we also refer to Palmer in [45] for the case of differential equations and respectively to Megan, Sasu and Sasu [37], Sasu and Sasu [62] for dynamical systems on the half-line. A distinct approach based on the theory of evolution semigroups was developed by Baskakov in [7], [8]. It should be mentioned that the philosophy of the methods considered in the study of the exponential dichotomy on the half-line is distinct compared with those considered on the whole line since a dynamical system defined on the half-line may be dichotomic with respect to an infinite number of families of projections. On the half-line every study devoted to exponential dichotomy begins with the assumption that the initial stable subspace is closed and complemented, one chooses a complement for it - the initial unstable subspace - and the entire construction depends on that (see [62] and the references therein).

4 Applications: input-output characterizations for uniform exponential dichotomy of nonautonomous systems

In this section, using Theorem 5, we present characterizations for uniform exponential dichotomy of nonautonomous systems in Banach spaces, modeled by evolution families.

We begin by recalling some definitions and basic properties from the theory of Banach sequence spaces. For a description of some fundamental properties of Banach sequence spaces we refer to [61].

We denote by $\mathcal{S}(\mathbb{Z},\mathbb{R})$ the linear space of all sequences $s : \mathbb{Z} \to \mathbb{R}$.

Definition 5. A linear subspace $B \subset \mathcal{S}(\mathbb{Z},\mathbb{R})$ is said to be a *normed sequence space* if there is a mapping $|\cdot|_B : B \to \mathbb{R}_+$ with the following properties:

(*i*) $|s|_B = 0$ if and only if $s = 0$;

(*ii*) $|cs|_B = |c|\ |s|_B$, for all $(c,s) \in \mathbb{R} \times B$;

(*iii*) $|u+s|_B \le |u|_B + |s|_B$, for all $u, s \in B$;

(*iv*) if $|u(k)| \le |s(k)|$, for all $k \in \mathbb{Z}$, and $s \in B$, then also $u \in B$ and $|u|_B \le |s|_B$.

Moreover, if $(B, |\cdot|_B)$ is complete, then B is called *Banach sequence space.*

Definition 6. A Banach sequence space $(B, |\cdot|_B)$ is said to be *invariant under translations* if for every $s \in B$ and every $m \in \mathbb{Z}$, the sequence $s_m : \mathbb{Z} \to \mathbb{R}, s_m(k) = s(k-m)$ belongs to B and $|s_m|_B = |s|_B$.

We denote by χ_A the characteristic function of a set $A \subset \mathbb{Z}$. We denote by $\mathcal{T}(\mathbb{Z})$ the class of all Banach sequence spaces which are invariant under translations and contain at least a sequence which is not identically zero.

Definition 7. If $B \in \mathcal{T}(\mathbb{Z})$, then the mapping

$$F_B : \mathbb{N}^* \to \mathbb{R}_+, \quad F_B(n) = |\chi_{\{0,\ldots,n-1\}}|_B$$

is called *the fundamental function* of the space B.

Considering $\ell^1(\mathbb{Z},\mathbb{R}) := \{s \in \mathcal{S}(\mathbb{Z},\mathbb{R}) : \sum\limits_{n=-\infty}^{\infty} |s(n)| < \infty\}$, we have that

$$\ell^1(\mathbb{Z},\mathbb{R}) \subset B \subset \ell^\infty(\mathbb{Z},\mathbb{R}), \quad \forall B \in \mathcal{T}(\mathbb{Z}) \tag{38}$$

For details see Lemma 2.1 in [61]. Moreover, for examples of Banach sequence spaces that belong to the class $\mathcal{T}(\mathbb{Z})$ we refer Section 2 in [61].

Two key subclasses of $\mathcal{T}(\mathbb{Z})$ were used in [61] in the study of the exponential dichotomy. More precisely, we denote by:

- $\mathcal{W}(\mathbb{Z})$ - the class of all Banach sequence spaces $B \in \mathcal{T}(\mathbb{Z})$ with the property that $\sup\limits_{n\in\mathbb{N}} F_B(n) = \infty$;
- $\mathcal{H}(\mathbb{Z})$ - the class of all Banach sequence spaces $B \in \mathcal{T}(\mathbb{Z})$ with $\ell^1(\mathbb{Z},\mathbb{R}) \subsetneqq B$.

Let $(X, ||\cdot|)$ be a real or complex Banach space. For every $s : \mathbb{Z} \to X$ we consider the function

$$N_s : \mathbb{Z} \to \mathbb{R}_+, \quad N_s(k) = ||s(k)||$$

For each $B \in \mathcal{T}(\mathbb{Z})$ we denote by $B(\mathbb{Z},X) := \{s : \mathbb{Z} \to X : N_s \in B\}$. Then $B(\mathbb{Z},X)$ is a Banach space with respect to the norm $||s||_{B(\mathbb{Z},X)} := |N_s|_B$.

Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on X. We associate with $\mathcal{U}$ the input-output system

$$(S_{\mathcal{U}}) \qquad \gamma(n+1) = U(n+1,n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{Z}$$

with $s, \gamma \in \ell^\infty(\mathbb{Z},X)$.

Definition 8. Let $I, O \in \mathcal{T}(\mathbb{Z})$. The pair $(O(\mathbb{Z},X), I(\mathbb{Z},X))$ is said to be *admissible for the system* $(S_{\mathcal{U}})$ if for every sequence $s \in I(\mathbb{Z},X)$ the system $(S_{\mathcal{U}})$ has a unique solution $\gamma_s \in O(\mathbb{Z},X)$.

Theorem 6. *Let $I, O \in \mathcal{T}(\mathbb{Z})$ with $I \subset O$ and $O \in \mathcal{W}(\mathbb{Z})$ or $I \in \mathcal{H}(\mathbb{Z})$. Then, the evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ has a uniform exponential dichotomy if and only if the pair $(O(\mathbb{Z},X), I(\mathbb{Z},X))$ is admissible for the system $(S_{\mathcal{U}})$.*

Proof. We associate with $\mathcal{U}$ the discrete nonautonomous system

$$(A_{\mathcal{U}}) \qquad x(n+1) = U(n+1,n)x(n), \quad \forall n \in \mathbb{Z}$$

with $x \in \mathcal{S}(\mathbb{Z}, X)$.

Necessity. Suppose that $\mathcal{U}$ has a uniform exponential dichotomy. Then, the system $(A_{\mathcal{U}})$ has a uniform exponential dichotomy. Taking into account that the system $(A_{\mathcal{U}})$ has uniformly bounded coefficients, by applying Theorem 3.5 (ii) from [61], we obtain that the pair $(O(\mathbb{Z}, X), I(\mathbb{Z}, X))$ is admissible for the input-output system $(S_{\mathcal{U}})$.

Sufficiency. Suppose that the pair $(O(\mathbb{Z}, X), I(\mathbb{Z}, X))$ is admissible for the system $(S_{\mathcal{U}})$. Then, from Theorem 3.5 (i) from [61], it follows that the discrete system $(A_{\mathcal{U}})$ has a uniform exponential dichotomy. By applying Theorem 5 we deduce that the evolution family $\mathcal{U}$ has a uniform exponential dichotomy and the proof is complete. $\square$

Remark 10. Based on the results obtained in Section 3 in [61] (see Example 3.1 and Remark 3.4 in [61]) and on Theorem 5, we deduce that the result presented in Theorem 6 is the most general in this framework.

Remark 11. The particular case when $O, I \in \{c_0(\mathbb{Z}, \mathbb{R}), \ell^\infty(\mathbb{Z}, \mathbb{R})\}$ was studied in [55]. The particular case when $O, I \in \{\ell^p(\mathbb{Z}, X) : \ p \in [1, \infty)\}$ was treated in [56].

Corollary 1. *Let* $T \in \mathcal{T}(\mathbb{Z})$*. Then, an evolution family* $\mathcal{U} = \{U(t,s)\}_{t \geq s}$ *has a uniform exponential dichotomy if and only if the pair* $(T(\mathbb{Z}, X), T(\mathbb{Z}, X))$ *is admissible for the system* $(S_{\mathcal{U}})$*.*

Proof. From relation (38) we have that either $T \in \mathcal{H}(\mathbb{Z})$ or $T = \ell^1(\mathbb{Z}, \mathbb{R}) \in \mathcal{W}(\mathbb{Z})$. Then, from Theorem 6 we deduce the conclusion. $\square$

A criteria that extends the characterizations with ℓ^p-spaces can be formulated in terms of Orlicz sequence spaces. Indeed, we recall briefly the definitions introduced in [61] (see Example 2.3 in [61]).

Let $\varphi : \mathbb{R}_+ \to \bar{\mathbb{R}}_+$ be a nondecreasing left continuous function which is not identically 0 or ∞ on $(0, \infty)$. The mapping

$$Y_\varphi : \mathbb{R}_+ \to \bar{\mathbb{R}}_+, \quad Y_\varphi(t) = \int_0^t \varphi(\tau)\, d\tau$$

is called the *Young function* associated to φ. For each sequence $u \in \mathcal{S}(\mathbb{Z}, \mathbb{R})$, we consider

$$M_\varphi(u) := \sum_{k=-\infty}^{\infty} Y_\varphi(|u(k)|).$$

Consider $\ell_\varphi(\mathbb{Z}, \mathbb{R}) := \{u \in \mathcal{S}(\mathbb{Z}, \mathbb{R}) : \exists \lambda > 0$ such that $M_\varphi(\lambda u) < \infty\}$. This is a Banach sequence space with respect to the norm

$$|u|_\varphi := \inf\{\lambda > 0 : M_\varphi(\frac{1}{\lambda}\, u) \leq 1\}$$

called *the Orlicz sequence space* associated to φ. The Orlicz sequence spaces belong to $\mathcal{T}(\mathbb{Z})$ and generalize the ℓ^p-spaces.

Remark 12. If $\ell_\varphi(\mathbb{Z}, \mathbb{R})$ is an Orlicz sequence space, then either $\ell_\varphi(\mathbb{Z}, \mathbb{R}) \in \mathcal{W}(\mathbb{Z})$ or $\ell_\varphi(\mathbb{Z}, \mathbb{R}) = \ell^\infty(\mathbb{Z}, X)$ (see Lemma 2.5 in [61]).

Corollary 2. *Let $\ell_\varphi(\mathbb{Z}, \mathbb{R}), \ell_\psi(\mathbb{Z}, \mathbb{R})$ be Orlicz sequence spaces such that $(\ell_\varphi(\mathbb{Z}, \mathbb{R}),\ \ell_\psi(\mathbb{Z}, \mathbb{R})) \neq (\ell^\infty(\mathbb{Z}, \mathbb{R}), \ell^1(\mathbb{Z}, \mathbb{R}))$ and $\ell_\psi(\mathbb{Z}, \mathbb{R}) \subset \ell_\varphi(\mathbb{Z}, \mathbb{R})$. Then, the evolution family $\mathcal{U} = \{U(t,s)\}_{t \geq s}$ has a uniform exponential dichotomy if and only if the pair $(\ell_\varphi(\mathbb{Z}, \mathbb{R}), \ell_\psi(\mathbb{Z}, \mathbb{R}))$ is admissible for the system $(S_\mathcal{U})$.*

Proof. This follows from Theorem 6 and Remark 12. □

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