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BIFURCATION ANALYSIS OF A TIME-DELAYED TOURISM MODEL*

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Abstract

A stability and bifurcation analysis is undertaken in a neighborhood of the positive equilibrium of a tourism model with time delay. Choosing the time-delay as bifurcation parameter, a Hopf bifurcation analysis is undertaken, using center manifold reduction and normal form theory. As a result, the critical values of the delay are found which are responsible for the occurrence of oscillatory behavior in the system. Numerical simulations are presented to substantiate the theoretical results.

MSC: 34H20; 37L10; 91B55; 91B76.

keywords: tourism dynamics; asymptotic stability; oscillatory behavior; bifurcation; time delay.

1 Introduction

Nowadays the tourism industry has been expanded at global scale well beyond any prediction made in the past and became a well established industry alongside the traditional ones. It is an activity done by a person or a group of persons involving movement of people, goods and services from one place

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to another over geographical distributed areas [1]. The other side of the coin is linked to the negative impact over the the natural environment and resources. These must be kept under a close eye by all the factors involved in this industry. In order to study, analyze and predict the behavior of the factors describing this complex system, an efficient approach is provided by mathematical modeling.

Casagrandi and Rinaldi [2] introduced a minimal model containing the core features of several systems with three main elements: number of tourists, quality of the environment and tourist facilities. The findings show that sustainable and profitable tourism is a reachable goal as long as the economic agents expand carefully while observing an environmental friendly policy. Also, the link between sustainability and bifurcation theory is highlighted.

The minimal model of Casagrandi and Rinaldi was later investigated and exteded by Lacitignola et al. [3] and Wei et al. [4]. Lacitignola et al. analyzed its implementation for a real tourist destination taking into consideration the two main tourist categories (mass and eco-tourists). The results are presented in terms of bifurcation theory. Wei et al. presented a stability analysis, where various scenarios are analyzed having different investment parameters. More recently, Afsharnezhad et al. [1] studied the existence of transcritical, pitchfork and saddle-node bifurcations of a similar mathematical model with the coexistence of two main tourist classes.

In this paper, based on the existing minimal model of a given generic touristic site [2], we introduce a discrete time delay in the description of the number of tourists, with the aim of studying its effect in terms of bifurcation and normal form theory.

The structure of the paper is as follows. The mathematical model of a touristic site is presented in Section 2, where the discrete time delay is introduced. Also, the equilibrium points are determined. In Section 3, we present a bifurcation analysis, where the time delay is chosen as bifurcation parameter. The stability of the limit cycle is provided in Section 4. Numerical simulations are carried out in Section 5 and the conclusions are given in Section 6.

2 Mathematical model

The minimal model for a generic site has three variables as follows: $x_1(t)$ the number of tourists at time t, $x_2(t)$ stands for the quality of the natural environment and $x_3(t)$ is the capital flow of the tourist activities and should be dissociated from the flow of offered services for tourists. A two way positive influence between tourists ($x_1(t)$) and capital flow ($x_3(t)$) can be identified. At the same time, they influence in a negative manner the quality of the natural environment, but the upside of this is the increased number of tourists.

In [2], the rate of change of tourists is considered equal to the product between the attractiveness of the site and the number of tourists:

$$\dot{x_1}(t) = x_1(t)A\left(x_1(t), x_2(t), x_3(t)\right).$$

The attractiveness $A(x_1, x_2, x_3)$ is the algebraic difference between the absolute attractiveness and a reference value a [2]:

$$\dot{x_1}(t) = x_1(t)\left[g_1(x_2(t)) + g_2\left(\frac{x_3(t)}{x_1(t)+1}\right) - \alpha x_1(t) - a\right]$$

where $\alpha > 0$ is the congestion parameter and the functions g_1 and g_2 are given by:

$$g_i(x) = \mu_i \frac{x}{\varphi_i + x} \tag{1}$$

where $\mu_i, \varphi_i > 0$, $i = 1, 2$.

The rate of change of the environment is given by [2]:

$$\dot{x_2}(t) = rx_2(t)\left(1 - \frac{x_2(t)}{K}\right) - x_2(t)(\eta x_3(t) + \gamma x_1(t))$$

where the first term represents the quality of environment in the absence of tourists and capital and the second term is the flow of damages induced by tourism. The parameter $r > 0$ is the net growth rate, $K > 0$ is the quality of the environment in the presence of all civil and industrial activities (except tourism) of the generic site. The two parameters η, γ are positive. We assume that the quality of the environment at time t, $x_2(t)$, depends on the number of past tourists:

$$\dot{x_2}(t) = rx_2(t)\left(1 - \frac{x_2(t)}{K}\right) - x_2(t)(\eta x_3(t) + \gamma x_1(t-\tau)),$$

where the positive parameter τ is the time delay.

The rate of change of the capital flow is given by [2]:

$$\dot{x_3}(t) = \varepsilon x_1(t) - \delta x_3(t),$$

where the first term is the investment flow and the second one is the depreciation flow. The positive parameter ε is the investment rate and δ is related to the degradation of tourist structures thought to be very slow. We assume that the capital flow at time t, $x_3(t)$, depends on the number of past tourists:

$$\dot{x_3}(t) = \varepsilon x_1(t-\tau) - \delta x_3(t),$$

where the positive parameter τ is the time delay.

As a summary of the aforementioned considerations, the associated mathematical model of a generic touristic site is given by:

$$\begin{cases} \dot{x_1}(t) = x_1(t) A\left(x_1(t), x_2(t), x_3(t)\right) \\ \dot{x_2}(t) = r x_2(t)\left(1 - \dfrac{x_2(t)}{K}\right) - x_2(t)(\eta x_3(t) + \gamma x_1(t-\tau)) \\ \dot{x_3}(t) = \varepsilon x_1(t-\tau) - \delta x_3(t) \end{cases} \tag{2}$$

Besides the following trivial equilibrium states for system (2):

$$S_0 = (0,0,0), \quad S_1 = (0,K,0), \quad S_2 = (x^*, 0, \frac{\varepsilon}{\delta} x^*),$$

where $x^* = r\left(\eta \frac{\varepsilon}{\delta} + \gamma\right)^{-1}$, there exists at least one strictly positive equilibrium state S_+ if and only if the following equation has at least one strictly positive root x_{10} in the interval $\left(0, \frac{\delta r}{\eta\varepsilon + \gamma\delta}\right)$:

$$s_3 x^3 + s_2 x^2 + s_1 x + s_0 = 0, \tag{3}$$

where:

$$\begin{aligned} s_3 &= K a_1 a_2 \alpha, \\ s_2 &= -\alpha\delta(r a_2 a_3 - K a_1 \varphi_2) + K a_1 a_2 (a - \mu_1 K) - K a_1 \mu_2 \varepsilon, \\ s_1 &= -\alpha a_3 r \delta^2 \varphi_2 - a(a_3 a_2 r\delta - K a_1 \delta \varphi_2) + a_3 r \delta \varepsilon \mu_2 + \mu_1 K (r\delta a_2 - a_1 \delta \varphi_2), \\ s_0 &= (\mu_1 K - a a_3) r \delta^2 \varphi_2 \end{aligned}$$

and

$$a_1 = \eta\varepsilon + \gamma_1 \delta, \quad a_2 = \delta\varphi_2 + \varepsilon, \quad a_3 = \varphi_1 + K.$$

In this case, the coordinates of the positive equilibrium are
$S_+ = (x_{10}, x_{20}, x_{30})$, where

$$x_{20} = \frac{k(\delta r - (\eta\varepsilon + \gamma\delta)x_{10})}{\delta r}, \quad x_{30} = \frac{\varepsilon x_{10}}{\delta}. \tag{4}$$

3 Hopf bifurcation at the positive equilibrium

Considering the positive equilibrium $S_+ = (x_{10}, x_{20}, x_{30})$ and carrying out the translation $y_1(t) = x_1(t) - x_{10}$, $y_2(t) = x_2(t) - x_{20}$, $y_3(t) = x_3(t) - x_{30}$, we obtain:

$$\begin{cases} \dot{y}_1(t) = f_1(y_1(t), y_2(t), y_3(t)), \\ \dot{y}_2(t) = f_2(y_1(t-\tau), y_2(t), y_3(t)), \\ \dot{y}_3(t) = f_3(y_1(t-\tau), y_2(t), y_3(t)), \end{cases} \tag{5}$$

where

$$\begin{aligned} f_1(y_1, y_2, y_3) =& (y_1 + x_{10})\Big[\frac{\mu_1(y_2 + x_{20})}{y_2 + \varphi_2 + x_{20}} + \frac{\mu_2(y_3 + x_{30})}{y_3 + \varphi_2 y_1 + \varphi_2(x_{10} + 1) + x_{30}} \\ & - \alpha y_1 - \alpha x_{10} - a\Big] \\ f_2(y_1, y_2, y_3) =& -(y_2 + x_{20})\left(\frac{r}{K} y_2 + \eta y_3 + \gamma y_1\right) \\ f_3(y_1, y_2, y_3) =& \varepsilon y_1 - \delta y_3. \end{aligned}$$

Linearizing (5) at $(0,0,0)^T$, we obtain the system:

$$\dot{u}(t) = Au(t) + Bu(t-\tau), \tag{6}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 0 & 0 \\ b_{31} & 0 & 0 \end{pmatrix} \tag{7}$$

where

$$\begin{aligned} a_{11} &= -x_{10}\left[\frac{\mu_2 x_{30}}{(\varphi_2(x_{10}+1) + x_{30})^2} + \alpha\right], \\ a_{12} &= x_{10}\left(\frac{\mu_1}{\varphi_1 + x_{20}} - \frac{\mu_1 x_{20}}{(\varphi_1 + x_{20})^2}\right), \\ a_{13} &= x_{10}\left(\frac{\mu_2}{\varphi_2(x_{10}+1) + x_{30}} - \frac{\mu_2 x_{30}}{\varphi_2(x_{10}+1) + x_{30})^2}\right), \\ a_{22} &= -\frac{r}{K} x_{20}, \quad a_{23} = -\eta x_{20}, \quad a_{33} = -\delta, \\ b_{21} &= -\gamma x_{20}, \quad b_{31} = \varepsilon. \end{aligned}$$

The characteristic function for (6) is given by:

$$h(\lambda, \tau) = (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33}) - (m_{11}\lambda + m_{10})e^{-\lambda\tau}, \tag{8}$$

where

$$m_{11} = a_{12}b_{21} + a_{13}b_{31}, \quad m_{10} = a_{12}a_{23}b_{31} - a_{13}a_{22}b_{31} - a_{12}a_{33}b_{21}.$$

In what follows, we assume that the following hypothesis is fulfilled:

(H_1) : All roots of the polynomial $h(\lambda, 0)$ are in the left half-plane.

Necessary and sufficient conditions for the fulfillment of hypothesis (H_1) can be obtained using the Routh-Hurwitz criterion, in terms of inequalities involving the parameters a_{ii} and m_{1i}.

Considering the time delay τ as bifurcation parameter, the following statements have to be satisfied for the occurrence of a Hopf bifurcation at the equilibrium state S_+:

(S_1) : There exists a critical time delay denoted by τ_0 such that the function $h(\lambda, \tau_0)$ has a pair of pure imaginary roots $\lambda_{1,2}(\tau_0) = \pm i\omega$ and all the other roots have negative real part;

(S_2) : $\Re\left(\dfrac{d\lambda_{1,2}(\tau)}{d\tau}\Big|_{\tau=\tau_0}\right) \neq 0.$

To find the critical value τ_0 which insures the fulfillment of statement (S_1), we assume that there exists a pair of pure imaginary roots $\pm i\omega$ of the equation $h(\lambda, \tau) = 0$. This leads to the equation:

$$\begin{aligned}&(a_{11} + a_{22} + a_{33})\omega^2 - a_{11}a_{22}a_{33} - m_{10}\cos(\omega\tau) - m_{11}\sin(\omega\tau) - \\ &- i\Big[\omega^3 - \omega(a_{22}a_{33} + a_{11}a_{22} + a_{11}a_{33}) + m_{11}\cos(\omega\tau) - m_{10}\sin(\omega\tau)\Big] = 0.\end{aligned}$$

Separating the real and imaginary parts, we have:

$$\begin{cases} (a_{11} + a_{22} + a_{33})\omega^2 - a_{11}a_{22}a_{33} = m_{10}\cos(\omega\tau) + m_{11}\omega\sin(\omega\tau), \\ \omega^3 - (a_{22}a_{33} + a_{11}a_{22} + a_{11}a_{33})\omega = m_{10}\sin(\omega\tau) - m_{11}\omega\cos(\omega\tau). \end{cases} \tag{9}$$

Eliminating $\sin(\omega\tau)$ and $\cos(\omega\tau)$ from (9) we obtain:

$$\omega^6 + p_4\omega^4 + p_2\omega^2 + p_0 = 0, \tag{10}$$

where

$$\begin{aligned}p_4 &= (a_{11} + a_{22} + a_{33})^2 - 2(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}), \\ p_2 &= (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33})^2 - 2(a_{11} + a_{22} + a_{33})a_{11}a_{22}a_{33} - m_{11}^2, \\ p_0 &= a_{11}^2a_{22}^2a_{33}^2 - m_{10}^2.\end{aligned}$$

Let ω_0 be a positive root of (10). The critical value of the delay is:

$$\tau_0 = \frac{1}{\omega_0} \arccos\left(\frac{Q_1}{Q_2}\right) \tag{11}$$

where

$$\begin{aligned} Q_1 = & \left[\omega_0^2(a_{11}+a_{22}+a_{33}) - a_{11}a_{22}a_{33}\right] m_{10} + \\ & + \omega_0 \left[\omega_0(a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}) - \omega_0^3\right] m_{11} \\ Q_2 = & m_{10}^2 + m_{11}^2\omega_0^2 \end{aligned}$$

To insure that statement (S_2) holds, consider $\lambda = \lambda(\tau)$ the root of the equation $h(\lambda(\tau), \tau) = 0$ which satisfies $\lambda(\tau_0) = i\omega_0$. Differentiating with respect to τ, we have:

$$\frac{d\lambda(\tau)}{d\tau} = \frac{(m_{11}\lambda(\tau)+m_{10})e^{-\lambda(\tau)\tau}}{3\lambda(\tau)^2 - 2q_2\lambda(\tau) + q_1 - m_{11}e^{-\lambda(\tau)\tau} + (m_{11}\lambda(\tau)+m_{10})\tau e^{-\lambda(\tau)\tau}} \tag{12}$$

where

$$q_2 = a_{11}+a_{22}+a_{33}, \quad q_1 = a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}.$$

Relation (12) can be written as:

$$\left.\frac{d\lambda(\tau)}{d\tau}\right|_{\tau=\tau_0} = \frac{A_1+iA_2}{B_1+iB_2} \tag{13}$$

where

$$\begin{aligned} A_1 = & -\omega_0(\omega_0 m_{11}\cos(\omega_0\tau_0) - m_{10}\sin(\omega_0\tau_0)), \\ A_2 = & \omega_0(m_{10}\cos(\omega_0\tau_0) + \omega_0 m_{11}\sin(\omega_0\tau_0)), \\ B_1 = & -3\omega_0^2 + q_1 + \tau_0(\omega_0 m_{11}\sin(\omega_0\tau_0) + m_{10}\cos(\omega_0\tau_0)), \\ B_2 = & -2q_2\omega_0 + \tau_0(\omega_0 m_{11}\cos(\omega_0\tau_0) - m_{10}\sin(\omega_0\tau_0)). \end{aligned} \tag{14}$$

The following notations will be used:

$$M = \Re\left(\frac{d\lambda(\tau)}{d\tau}\right)\Big|_{\tau=\tau_0} = \frac{A_1B_1+A_2B_2}{B_1^2+B_2^2}, \quad N = \Im\left(\frac{d\lambda(\tau)}{d\tau}\right)\Big|_{\tau=\tau_0} = \frac{A_2B_1-A_1B_2}{B_1^2+B_2^2}. \tag{15}$$

Hence, the following result is obtained

Proposition 1. *If ω_0 is a positive root of (10) and $M \neq 0$ then at $\tau = \tau_0$ given by* (11) *a Hopf bifurcation occurs for system (2) in a neighborhood of the positive equilibrium S_+.*

4 Stability of the limit cycle

In what follows, the first Lyapunov coefficient will be computed, to obtain information about the stability of the limit cycle that appears due to the Hopf bifurcation. We follows the guidelines given in [5, 6, 7, 8]. First we transform system (5) with $\tau = \tau_0 + \mu,\ \mu > 0$, where τ_0 is the critical value of the bifurcation parameter, into an equation of the form

$$\dot{y}_t = \mathcal{A}(\mu)y_t + \mathcal{R}(\mu, y_t) \tag{16}$$

where $y = (y_1, y_2, y_3)^\top,\ y_t = y(t+\theta),\ \theta \in [-\tau, 0]$, and the operators $\mathcal{A}$ and $\mathcal{R}$ are defined for $\phi \in C^1([-\tau_0, 0], \mathbb{C}^3)$ as follows:

$$\mathcal{A}(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta} & ,\ \theta \in [-\tau, 0) \\ A\phi(0) + B\phi(-\tau) & ,\ \theta = 0 \end{cases} \tag{17}$$

with $A,\ B$ are given by (7) and

$$\mathcal{R}(\mu, \phi(\theta)) = \begin{cases} (0,0,0)^\top & ,\ \theta \in [-\tau, 0) \\ (F_1(\mu,\theta), F_2(\mu,\theta), F_3(\mu,\theta))^\top & ,\ \theta = 0 \end{cases} \tag{18}$$

with

$$\begin{aligned} F_1(\mu,\theta) =& a_{200}m_1^2 + 2a_{110}m_1m_2 + 2a_{101}m_1m_3 + a_{020}m_2^2 + a_{002}m_3^2 + \\ &+ 3a_{201}m_1^2m_3 + a_{300}m_1^3 + a_{030}m_2^3 + a_{003}m_3^3 + 3a_{120}m_1m_2^2 + \\ &+ 3a_{102}m_1m_3^2 \\ F_2(\mu,\theta) =& b_{020}m_2^2 + 2b_{011}m_2m_3 + 2b_{110}m_4m_2 \\ F_3(\mu,\theta) =& 0 \end{aligned}$$

where

$$m_1 = \phi_1(0), m_2 = \phi_2(0), m_3 = \phi_3(0), m_4 = \phi_1(-\tau)$$

and

$$\begin{aligned} a_{200} &= -\frac{2\mu_2 x_{30}\varphi_2}{((x_{10}+1)\varphi_2 + x_{30})^2} - 2\alpha + \frac{2\mu_2 x_{10}x_{30}\varphi_2^2}{((x_{10}+1)\varphi_2 + x_{30})^3} \\ a_{020} &= -\frac{2\mu_1 x_{10}}{(x_{20}+\varphi_1)^2} + \frac{2\mu_1 x_{10}x_{20}}{(\varphi_1 + x_{20})^3} \\ a_{110} &= \frac{\mu_1}{x_{20}+\varphi_1} - \frac{\mu_1 x_{20}}{(\varphi_1 + x_{20})^2} \end{aligned}$$

$$a_{002} = -\frac{2\mu_2 x_{10}}{((x_{10}+1)\varphi_1 + x_{30})^2} + \frac{2\mu_2 x_{10} x_{30}}{((\varphi_2(x_{10}+1)+x_{30})^3}$$
$$a_{101} = \frac{\mu_2}{(x_{10}+1)\varphi_2 + x_{30}} - \frac{\mu_2 x_{30}}{(\varphi_2(x_{10}+1)+x_{30})^2} - \frac{\mu_2 x_{10}\varphi_2}{(\varphi_2(x_{10}+1)+x_{30})^2} + \\ + \frac{2x_{10}\mu_2 x_{30}\varphi_2}{(\varphi_2(x_{10}+1)+x_{30})^3}$$
$$a_{120} = -\frac{2\mu_1}{(x_{20}+\varphi_1)^2} + \frac{2\mu_1 x_{20}}{(\varphi_1 + x_{20})^3}$$
$$a_{102} = \frac{2\mu_2}{((x_{10}+1)\varphi_2 + x_{30})^2} + \frac{2\mu_2 x_{30}}{(\varphi_2(x_{10}+1)+x_{30})^3} + \frac{4\mu_2 x_{10}\varphi_2}{(\varphi_2(x_{10}+1)+x_{30})^2} - \\ - \frac{6x_{10}\mu_2 x_{30}\varphi_2}{(\varphi_2(x_{10}+1)+x_{30})^4}$$
$$a_{003} = \frac{6\mu_2 x_{10}}{((x_{10}+1)\varphi_2 + x_{30})^3} - \frac{6\mu_2 x_{10} x_{30}}{(\varphi_2(x_{10}+1)+x_{30})^4}$$
$$a_{300} = \frac{6\mu_2 x_{30}\varphi_2^2}{((x_{10}+1)\varphi_2 + x_{30})^3} - \frac{6\mu_2 x_{10} x_{30}\varphi_2^3}{((x_{10}+1)\varphi_2 + x_{30})^4}$$
$$a_{201} = -\frac{2\mu_2\varphi_2}{((x_{10}+1)\varphi_2 + x_{30})^2} + \frac{4\mu_2 x_{30}\varphi_2 + 2x_{10}\mu_2\varphi_2^2}{(\varphi_2(x_{10}+1)+x_{30})^3} - \frac{6\mu_2 x_{10} x_{30}\varphi_2^2}{(\varphi_2(x_{10}+1)+x_{30})^4}$$
$$b_{020} = -\frac{2r}{k}, \quad b_{011} = -\eta, \quad b_{110} = -\gamma$$

The adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ is defined for $\psi \in C^1([0,\tau],\mathbb{C}^3)$ as follows:

$$\mathcal{A}^*(\mu)(\psi(s)) = \begin{cases} -\frac{d\psi(s)}{ds} & , \ s \in [0,\tau) \\ \psi^\top(0)A + \psi^\top(\tau)B & , \ s = \tau \end{cases} \tag{19}$$

For $\phi \in C^1([-\tau,0],\mathbb{C}^3)$ and $\psi \in C^1([0,\tau],\mathbb{C}^3)$ we define the bilinear form:

$$<\psi,\phi> = \bar{\psi}(0)^\top\phi(0) - \int_{\theta=-\tau}^{0}\int_{s=0}^{\theta}\bar{\psi}^\top(s-\theta)d\eta(\theta)\phi(s)ds \tag{20}$$

where $\eta(\theta) = B\delta(\theta+\tau)$ for $\theta \in [-\tau,0)$ and δ is the Dirac distribution.

Using (17) and (19) we obtain:

Proposition 2.

1. *An eigenvector ϕ of $\mathcal{A}$ associated to the eigenvalue $i\omega_0$ is*

$$\phi(\theta) = me^{i\omega_0\theta}, \ \theta \in [-\tau,0] \tag{21}$$

where $m = (m_1, m_2, m_3)^\top$ *is given by:*

$$\begin{aligned} m_1 &= -a_{12}(i\omega_0 - a_{33}),\\ m_2 &= b_{31}a_{13}e^{-i\omega_0\tau_0} - (i\omega_0 - a_{11})(i\omega_0 - a_{33}),\\ m_3 &= -a_{12}b_{31}e^{-i\omega_0\tau_0}. \end{aligned} \tag{22}$$

2. *An eigenvector* ψ *of* $\mathcal{A}^*$ *associated to the eigenvalue* $-i\omega_0$ *is:*

$$\psi(s) = le^{-i\omega_0 s},\ s \in [0, \tau], \tag{23}$$

where $l = (l_1, l_2, l_3)^\top$ *is given by:*

$$\begin{aligned} l_1 &= (i\omega_0 - a_{22})(i\omega_0 - a_{33}),\\ l_2 &= a_{12}(i\omega_0 - a_{33}),\\ l_3 &= a_{12}a_{23} + a_{13}(i\omega_0 - a_{22}). \end{aligned}$$

3. *With respect to (20) we have*

$$\begin{aligned} &< \psi(s), \phi(\theta) >= e_{11} && < \psi(s), \bar{\phi}(s) >= e_{12}\\ &< \bar{\psi}(s), \phi(\theta) >= e_{21} && < \bar{\psi}(s), \bar{\phi}(\theta) >= e_{22} \end{aligned}$$

where

$$\begin{aligned} e_{11} = \bar{e}_{22} &= \bar{l}_1 m_1 + \bar{l}_2 m_2 + \bar{l}_3 m_3 + e^{-i\omega_0\tau_0} m_1(b_{21}l_2 + b_{31}l_1),\\ e_{12} = \bar{e}_{21} &= (i\omega_0 + a_{22})(i\omega_0 + a_{33})l_1 - a_{12}(i\omega_0 + a_{33})l_2 -\\ &\quad - a_{12}b_{31}e^{i\omega_0\tau_0} l_3 - \tau_0 e^{-i\omega_0\tau_0}(b_2 l_2 l_1 + b_{31} l_1 l_3). \end{aligned}$$

We consider

$$P = \begin{pmatrix} e_{11} & e_{12}\\ e_{21} & e_{22} \end{pmatrix}^{-1} = \begin{pmatrix} f_{11} & f_{12}\\ f_{21} & f_{22} \end{pmatrix}$$

and

$$\begin{aligned} n_1 &= f_{11}l_1 + f_{12}\bar{l}_1,\\ n_2 &= f_{11}l_2 + f_{12}\bar{l}_2,\\ n_3 &= f_{11}l_3 + f_{13}\bar{l}_3 \end{aligned} \tag{24}$$

and

$$\psi^*(s) = f_{11}\psi(s) + f_{12}\bar{\psi}(s), \quad \bar{\psi}^*(s) = f_{21}\psi(s) + f_{22}\bar{\psi}(s). \tag{25}$$

Using (21) and (23) we have:

$$< \psi^*(s), \phi(\theta) >= 1 \qquad\qquad < \psi^*(s), \bar{\phi}(s) >= 0$$

$$<\bar{\psi}^*(s),\phi(\theta)>=0 \qquad\qquad <\bar{\psi}^*(s),\bar{\phi}(s)>=1$$

In the following, the coordinates of the center manifold Ω_0 at $\mu = 0$ will be defined. We consider

$$z(t) =<\psi, y_t>,\quad w(t,\theta) = y_t - 2\Re\{z(t)\phi(\theta)\} \tag{26}$$

On the center manifold Ω_0, $w(t,\theta) = w(z(t),\bar{z}(t),\phi)$ where

$$w(z,\bar{z},\theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + ... \tag{27}$$

and $z,\ \bar{z}$ are the local coordinates of the center manifold Ω_0 in the direction of ϕ and ψ, respectively.

For $\mu = 0$, the equation on the center manifold is reduced to:

$$\dot{z}(t) = \lambda_1 z(t) + <\psi, \mathcal{R}(w(t,\theta) + 2Re\{z(t)\phi(\theta)\})>$$

that can be rewrite as:

$$\dot{z}(t) = \lambda_1 z(t) + g(z(t),\bar{z}(t)) \tag{28}$$

with

$$g(z,\bar{z}) = \bar{\psi}(0)^\top \mathcal{R}(w(z,\bar{z},\theta) + 2Re(z\phi(\theta))).$$

We expand the function $g(z,\bar{z})$ on the center manifold Ω_0 and obtain:

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2}. \tag{29}$$

Therefore, the following result is obtained:

Proposition 3. *For system (5), we have:*

$$\begin{aligned} g_{20} &= \bar{n}_1 F_{120} + \bar{n}_2 F_{220} + \bar{n}_3 F_{320},\\ g_{11} &= \bar{n}_1 F_{111} + \bar{n}_2 F_{211} + \bar{n}_3 F_{311},\\ g_{02} &= \bar{n}_1 F_{102} + \bar{n}_2 F_{202} + \bar{n}_3 F_{302},\\ g_{21} &= \bar{n}_1 F_{121} + \bar{n}_2 F_{221} + \bar{n}_3 F_{321}, \end{aligned} \tag{30}$$

where $\bar{n}_1,\ \bar{n}_2, \bar{n}_3$ *are given by (24) and*

$$F_{120} = \frac{1}{2}a_{200}m_1^2 + a_{110}m_1m_2 + a_{101}m_1m_3 + \frac{1}{2}a_{020}m_2^2 + \frac{1}{2}a_{002}m_3^2,$$
$$F_{111} = a_{200}m_1\bar{m}_1 + a_{110}(m_1\bar{m}_2 + m_2\bar{m}_1) + a_{101}(m_1\bar{m}_3 + \bar{m}_1m_3) + \\ + a_{020}m_2\bar{m}_2 + a_{002}m_3\bar{m}_3,$$

$$F_{102} = \frac{1}{2}a_{200}\bar{m}_1^2 + a_{110}\bar{m}_1\bar{m}_2 + a_{101}\bar{m}_1\bar{m}_3 + \frac{1}{2}a_{020}\bar{m}_2^2 + \frac{1}{2}a_{002}\bar{m}_3^2,$$
$$F_{220} = \frac{1}{2}b_{020}m_2^2 + b_{011}m_2m_3 + b_{110}m_1m_2e^{-i\omega_0\tau_0},$$
$$F_{211} = b_{110}(m_1\bar{m}_2e^{-i\omega_0\tau_0} + m_2\bar{m}_1e^{i\omega_0\tau_0}) + b_{020}m_2\bar{m}_2 + b_{011}(m_2\bar{m}_3 + \bar{m}_2m_3),$$
$$F_{202} = \frac{1}{2}b_{020}\bar{m}_2^2 + b_{011}\bar{m}_2\bar{m}_3 + d_{110}\bar{m}_1\bar{m}_2e^{i\omega_0\tau_0},$$
$$F_{320} = 0, F_{311} = 0, F_{302} = 0.$$

The components of the vectors:

$$w_{20} = (w_{120}, w_{220}, w_{320})^\top, w_{11} = (w_{111}, w_{211}, w_{311})^\top, w_{02} = (w_{102}, w_{202}, w_{302})^\top$$

are given by:

$$\begin{aligned}
w_{120} &= -\frac{g_{20}}{i\,\omega_0}m_1 - \frac{\bar{g}_{02}}{3i\,\omega_0}\bar{m}_1 + E_{21}\\
w_{220} &= -\frac{g_{20}}{i\,\omega_0}m_2 - \frac{\bar{g}_{02}}{3i\,\omega_0}\bar{m}_2 + E_{22}\\
w_{320} &= -\frac{g_{20}}{i\,\omega_0}m_3 - \frac{\bar{g}_{02}}{3i\,\omega_0}\bar{m}_3 + E_{23}\\
w_{111} &= \frac{g_{11}}{i\,\omega_0}m_1 - \frac{\bar{g}_{11}}{i\,\omega_0}\bar{m}_1 + E_{11}\\
w_{211} &= \frac{g_{11}}{i\,\omega_0}m_2 - \frac{\bar{g}_{11}}{i\,\omega_0}\bar{m}_2 + E_{12}\\
w_{311} &= \frac{g_{11}}{i\,\omega_0}m_3 - \frac{\bar{g}_{11}}{i\,\omega_0}\bar{m}_3 + E_{13}\\
w_{102} &= \bar{w}_{120}, \quad w_{202} = \bar{w}_{220}, \quad w_{302} = \bar{w}_{302},
\end{aligned}$$

where $E_1 = (E_{11}, E_{12}, E_{13})^\top,\ E_2 = (E_{21}, E_{22}, E_{23})^\top$ *and*

$$\begin{aligned}
E_1 &= Q_1(F_{120}, F_{220}, F_{320})^T,\\
E_2 &= Q_2(F_{111}, F_{211}, F_{311})^T,\\
Q_1 &= -(A + e^{-2i\omega_0\tau_0}B - 2i\omega_0E)^{-1},\\
Q_2 &= -(A + B)^{-1}.
\end{aligned}$$

Proof. The projection of the solution $(y_1(t), y_2(t), y_3(t))$ of the system (5) on Ω_0 is given by:

$$y_1(t) = m_1z(t) + \bar{m}_1\bar{z}(t) + \frac{1}{2}w_{120}z(t)^2 + w_{111}z(t)\bar{z}(t) + \frac{1}{2}w_{102}\bar{z}(t)^2$$

$$y_2(t) = m_2 z(t) + \bar{m}_2 \bar{z}(t) + \frac{1}{2} w_{220} z(t)^2 + w_{211} z(t)\bar{z}(t) + \frac{1}{2} w_{202} \bar{z}(t)^2$$
$$y_3(t) = m_3 z(t) + \bar{m}_3 \bar{z}(t) + \frac{1}{2} w_{320} z(t)^2 + w_{311} z(t)\bar{z}(t) + \frac{1}{2} w_{302} \bar{z}(t)^2$$

with m_1, m_2, m_3 are given by (22).

The nonlinear part (5) can be written as:

$$\frac{1}{2} F_{20} z(t)^2 + F_{11} z(t)\bar{z}(t) + \frac{1}{2} F_{02} \bar{z}^2(t) + \frac{1}{2} F_{12} z^2(t)\bar{z(t)}$$

where

$$F_{20} = (F_{120}, F_{220}, F_{320})^\top, \quad F_{11} = (F_{111}, F_{211}, F_{311})^\top$$
$$F_{02} = (F_{102}, F_{202}, F_{302})^\top, \quad F_{12} = (F_{112}, F_{212}, F_{312})^\top$$

and the components of F_{20} and F_{11} are given above. The components of F_{21} are:

$$\begin{aligned} F_{121} =& a_{120}(\bar{m}_1 m_2^2 + 2m_1 m_2 \bar{m}_2) + a_{102}(\bar{m}_1 m_3^2 + 2m_1 m_3 \bar{m}_3) + \\ &+ \frac{1}{2} a_{003} m_3^2 \bar{m}_3 + \frac{1}{2} a_{200} m_1^2 m_1 + a_{201}(m_1^2 m_3 + 2m_1 m_3 \bar{m}_1), \\ F_{221} =& 0, F_{321} = 0 \end{aligned}$$

and a_{120}, a_{102}, a_{003}, a_{300}, a_{201} are given above.

The relations (30) are obtained using:

$$g_{20} = \bar{\psi}^\top(0) F_{20}, \quad g_{11} = \bar{\psi}^\top(0) F_{11}, \quad g_{02} = \bar{\psi}^\top(0) F_{02}, \quad g_{21} = \bar{\psi}^\top(0) F_{21}$$

□

Therefore, we can compute the following parameters [8]:

$$\begin{aligned} C(0) &= \frac{i}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2} \\ \mu_2 &= -\frac{Re(C(0))}{M}, \\ T_2 &= -\frac{Im(C(0)) + \mu_2 N}{\omega_0}, \\ \beta_2 &= 2Re(C(0)) \end{aligned}$$

where M and N are given by (15).

The parameter μ_2 defined above determines the direction of the Hopf bifurcation; β_2 determines the stability of the bifurcating periodic orbit and T_2 determines the period of the bifurcating periodic orbit.

Proposition 4. *(see [8])*

1. *If* $\mu_2 > 0(< 0)$ *the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exists for* $\tau > \tau_0(< \tau_0)$.

2. *If* $\beta_2 < 0(> 0)$ *the solutions are orbitally stable (unstable).*

3. *If* $T_2 > 0(< 0)$ *the period increases (decreases).*

5 Numerical results

In this section, we consider the following values for the system parameters [2]: $r = \alpha = \eta = \gamma = \varphi_1 = K = 1$; $\varepsilon = 0.1$; $\delta = 0.1$; $\varphi_2 = 0.5$; $a = 5$, $\mu_1 = \mu_2 = 10$.

The coordinates of the positive equilibrium are

$$S_+ = (0.381387, 0.237227, 0.381387).$$

In the absence of time delay, the equilibrium S_+ is asymptotically stable. If a discrete time delay is included in the mathematical model, the critical value of delay responsible for the occurrence of a Hopf bifurcation with the loss of asymptotic stability of S_+ is $\tau_0 = 1.9299$. We obtain that S_+ is asymptotically stable for $\tau \in (0, \tau_0)$ and unstable for $\tau > \tau_0$.

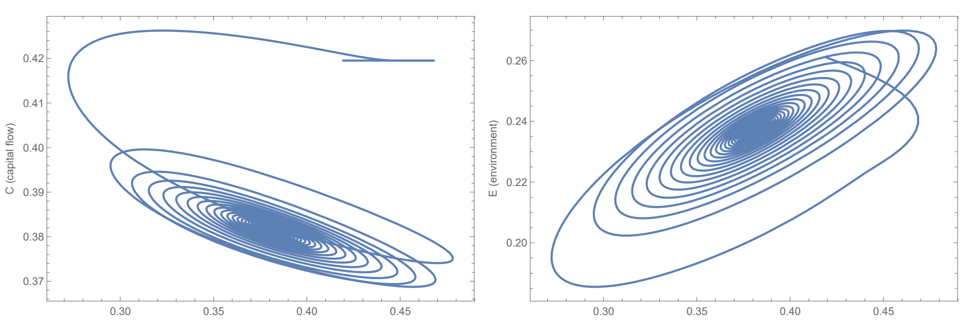

Figure 1: Trajectories in the phase planes (x_1, x_3) and (x_1, x_2) respectively, in the case of a discrete time delay $\tau = 1.8$, choosing an initial condition in a neighborhood of the positive equilibrium S_+, which is asymptotically stable.

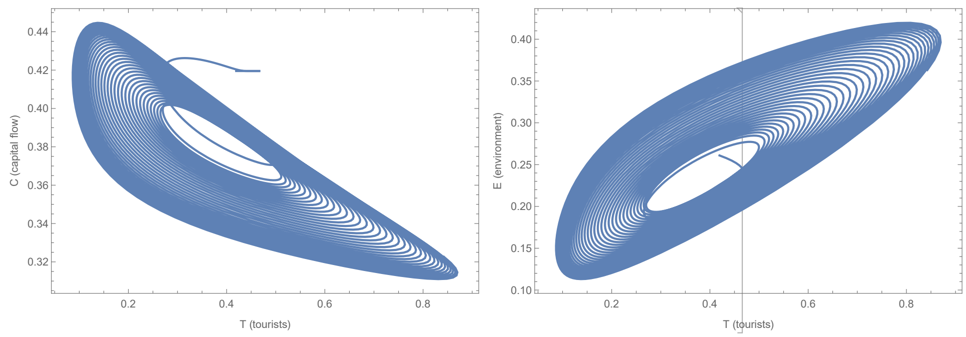

Figure 2: Trajectories in the phase planes (x_1, x_3) and (x_1, x_2) respectively, in the case of a discrete time delay $\tau = 2$, choosing an initial condition in a neighborhood of the positive equilibrium S_+, which is asymptotically stable. An asymptotically stable limit cycle is present.

6 Conclusions

In the current paper, the sustainable tourism for a generic site was studied using the bifurcation and normal form theory. We started from an existing minimal model with three variables: the number of tourists, the quality of the environment and the capital flow as the framework for the tourists activities. We assumed that the past tourists have an effect on the number of present environment and capital flow and therefore the time delay is introduced.

In the numerical simulations that were done, we notice the presence of oscillations corresponding to critical values of the bifurcation parameter.

It is worth mentioning that a various approaches of the minimal model, like including environmental perturbations, can be taken into account. Also, the tourists memory is one aspect that should be factored in by introducing the fractional derivative.

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