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INVARIANT MEAN VALUE PROPERTY AND THE ASSOCIATED INTEGRAL EQUATIONS*

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Abstract

In this paper we consider a class of integral equations associated with the invariant mean value property for $\mathcal{M}$-harmonic functions. We have shown that nonconstant solutions of the integral equations are functions of unbounded variation and do not attain their supremum or infimum on $[0,1)$. We also discuss in detail the behavior of the kernel of the corresponding integral operator and obtained certain growth estimates of the integral operator.

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1 Introduction

Let $\mathbb{B}_n$ be the open unit ball of $\mathbb{C}^n$, $n \in \mathbb{N}$, with respect to the Euclidean metric. The group of all one-to-one holomorphic maps of $\mathbb{B}_n$ onto $\mathbb{B}_n$ (the

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automorphisms of $\mathbb{B}_n$) will be denoted by $Aut(\mathbb{B}_n)$. It is generated by the unitary operators on $\mathbb{C}^n$ and the involutions ϕ_a of the form

$$\phi_a(z) = \frac{a - \mathcal{P}z - (1-|a|^2)^{\frac{1}{2}}Qz}{1 - \langle z, a \rangle} \tag{1}$$

where $a \in \mathbb{B}_n$, $\mathcal{P}$ is the orthogonal projection onto the space spanned by a, $Qz = z - \mathcal{P}z$,

$$\langle z, a \rangle = \sum_{i=1}^{n} z_i \overline{a_i}, \text{ and } |a|^2 = \langle a, a \rangle .$$

Let $G_0 = \{\psi \in Aut(\mathbb{B}_n) : \psi(0) = 0\}$. It is well known [16] that G_0 is compact and that G_0 is a subgroup of the unitary group $\mathcal{U}_n$ of $\mathbb{C}^n$. Given $\psi \in Aut(\mathbb{B}_n)$, let $a = \psi^{-1}(0)$, then we have,

$$\psi \circ \phi_a(0) = \psi(a) = 0,$$

thus $\psi \circ \phi_a \in G_0$ and so there exists a unitary matrix U such that $\psi = U\phi_a$ where $U \in G_0$.
The invariant Laplacian $\widetilde{\triangle}$ is defined [13] for $f \in C^2(\mathbb{B}_n)$ by

$$(\widetilde{\triangle} f)(z) = \triangle(f \circ \phi_z)(0),$$

where $\triangle$ is the ordinary Laplacian. It commutes with every $\psi \in Aut(\mathbb{B}_n)$:

$$(\widetilde{\triangle} f) \circ \psi = \widetilde{\triangle}(f \circ \psi).$$

The $\mathcal{M}$-harmonic functions in $\mathbb{B}_n$ are those for which $\widetilde{\triangle} f = 0$. We recall that "$\mathcal{M}$-harmonic" is the same as "harmonic" when $n = 1$, but not when $n > 1$. For more details see [1],[3] and [2].
The letter ν denotes the Lebesgue measure on $\mathbb{C}^n$, normalized so that $\nu(\mathbb{B}_n) = 1$ and for $1 \le p \le \infty$, the space $L^p(\mathbb{B}_n)$ refers to the usual Lebesgue spaces and the integration is with respect to the measure ν. When $n = 1$, $d\nu = dA$, the normalized area measure on $\mathbb{D}$, the open unit disk in the complex plane $\mathbb{C}$. If $\widetilde{\triangle} f = 0$ then the mean value of f on spheres of radius $r < 1$ is $f(0)$. If f is also in $L^1(\mathbb{B}_n)$ it follows that

$$\int_{\mathbb{B}_n} (f \circ \psi) d\nu = f(\psi(0)) \tag{2}$$

for every $\psi \in Aut(\mathbb{B}_n)$. It happens as $\widetilde{\triangle} f = 0$ implies $\widetilde{\triangle}(f \circ \psi) = 0$ for all $\psi \in Aut(\mathbb{B}_n)$. The property described in equation (2) is called the invariant mean value property. It is invariant in the sense that $f \circ \psi$ has it for every $\psi \in Aut(\mathbb{B}_n)$ whenever f has it. One may now again ask about the converse:

(**Q**) If $f \in L^1(\mathbb{B}_n)$ satisfies (2) for every $\psi \in Aut(\mathbb{B}_n)$, does it follow that f is $\mathcal{M}$-harmonic?

The answer was found to be affirmative if $n \leq 11$, negative if $n \geq 12$. Rudin [13], [4], proved the result for $C(\overline{\mathbb{B}_n})$ instead of $L^1(\mathbb{B}_n)$. In 1991, Axler and Cuckovic [4] proved the result when $f \in L^1(\mathbb{D}, dA)$ and the radialization of $f \circ \phi$ is continuous on the closed unit disk for all $\phi \in Aut(\mathbb{D})$. Englis [6] proved the result for $L^\infty(\mathbb{D})$. In 1993, Ahern, Flores and Rudin [1] and Yi [15] proved the result for $L^1(\mathbb{B}_n), n \geq 1$. They proved that if $f \in L^1(\mathbb{B}_n)$ satisfies (2) for every $\psi \in Aut(\mathbb{B}_n)$, then f is $\mathcal{M}$-harmonic if $n \leq 11$ but this is not true if $n \geq 12$. Related work in this area can also be found in [5],[8] and [11].

It will be advantageous to recast the mean value property in terms of the linear operator T_0 defined by

$$(T_0 f)(z) = \int_{\mathbb{B}_n} (f \circ \phi_z) d\nu,$$

for $f \in L^1(\mathbb{B}_n), z \in \mathbb{B}_n$. Since every $\psi \in Aut(\mathbb{B}_n)$ has the form $\psi = \phi_z U$ for some $z \in \mathbb{B}_n$, and some unitary operator U, the rotation invariance of ν shows that an $f \in L^1(\mathbb{B}_n)$ satisfies (2) for every $\psi \in Aut(\mathbb{B}_n)$ if and only if $T_0 f = f$. For $f \in L^1(\mathbb{B}_n), z \in \mathbb{B}_n$,

$$(T_0 f)(z) = \int_{\mathbb{B}_n} f(\phi_z(w)) d\nu(w). \tag{3}$$

If we replace w by $\phi_z(w)$ in this integral, we obtain a second formula for $T_0 f$, namely

$$(T_0 f)(z) = \int_{\mathbb{B}_n} \frac{(1 - |z|^2)^{n+1}}{|1 - \langle z, w \rangle|^{2(n+1)}} f(w) d\nu(w). \tag{4}$$

The passage from (3) to (4) uses the identity

$$1 - |\phi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, w \rangle|^2}$$

and the fact that the (real) Jacobian of ϕ_z is

$$(J_{\mathbb{R}} \phi_z)(w) = \frac{(1 - |z|^2)^{n+1}}{|1 - \langle z, w \rangle|^{2n+2}}.$$

For details see [1] and [10].

Now suppose f is radial. That is, there is a function $g : [0, 1] \to \mathbb{C}$ such that $f(z) = g(|z|^2)$. The assumption $T_0 f = f$ leads then from (4) to the integral equation (the detail derivation is given in section 2)

$$g(x) = (1-x)^{n+1} \int_0^1 \frac{n+tx}{(1-tx)^{n+2}} g(t) t^{n-1} dt \tag{5}$$

and since the constants are the only radial $\mathcal{M}$-harmonic functions, our question (**Q**) has an affirmative answer if and only if the constants are the only solutions of (5) in $L^1[0, 1]$. From the results of Ahern, Flores and Rudin [1] it follows that this is so for all $n \leq 11$ but not for $n \geq 12$. In fact,(5) can be studied for all n, not necessarily integers, and then the question arises: At which point between 11 and 12 does the above mentioned integral equation (5) has non-constant solutions ? The equation (5) is also mentioned and studied by Axler, Cuckovic [4], Ahern, Flores and Rudin [1], Yi [15]. In 1995, Yi [15] gave a characterization of the functions fixed by the integral operator

$$Tg(x) = (1-x)^{n+1} \int_0^1 \frac{n+tx}{(1-tx)^{n+2}} g(t) t^{n-1} dt. \tag{6}$$

Yi [15] showed that equation (5) has only constant solutions if and only if the equation (2) has only $\mathcal{M}$-harmonic solutions in $L^1(\mathbb{B}_n, d\nu)$. In [15], a critical point $11 + \epsilon_0, 0 < \epsilon_0 < 1$, was found such that the target equation has only constant solutions if and only if $1 \leq n \leq 11 + \epsilon_0$.

Let $\Gamma(s)$ stands for the usual Gamma function, which is an analytic function of s in the whole complex plane except for simple poles at the points $\{0, -1, -2, \cdots\}$. In fact

$$\Gamma(z) = \frac{e^{-\beta z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}}$$

where β is the Euler's constant; its approximate value is 0.57722.

In this work we consider the integral equation (5) associated with the invariant volume mean value property. The spectral properties of the integral operator T associated with the integral equation (5) are discussed. It is proved that non-constant solutions of (5) are functions of unbounded variation for all $n \geq 1, \ n \in \mathbb{N}$; moreover, these non-constant solutions do not attain supremum and infimum in $[0, 1)$ but approach both at 1. Berezin transform and its connection with the integral operator T is established. We also study the behavior of the kernel of the integral operator and obtained some estimates on the growth rate of the integral operator T.

The organization of this paper is as follows: In section 2, we give a characterization of $\mathcal{M}$-harmonic functions u in terms of invariant mean value property and radialization of $u \circ \psi$, $\psi \in Aut(\mathbb{B}_n)$. This is an extension of the result of Axler and Cuckovic [4]. We then show that if f is radial and there is a function $g : [0,1] \to \mathbb{C}$ such that $f(z) = g(|z|^2)$ then f is a fixed point of the Berezin transform B if and only if

$$g(x) = (1-x)^{n+1} \int_0^1 \frac{n+tx}{(1-tx)^{n+2}} g(t) t^{n-1} dt = Tg(x). \tag{7}$$

Consequently we show that if the constant functions are the only functions $V \in C([0,1)) \cap L^1[0,1]$ such that

$$V(t) = (1-t)^{n+1} \int_0^1 \frac{n+ts}{(1-ts)^{n+2}} V(s) s^{n-1} ds$$

for every $t \in [0,1)$ then functions in $C(\mathbb{B}_n) \cap L^1(\mathbb{B}_n, d\nu)$ having the invariant volume mean value property is $\mathcal{M}$-harmonic. In section 3 we investigate whether constants are the only solutions of the integral equation (5) in $L^1[0,1]$ for all integers $n \geq 1$ and $n \leq 11$ and study the behavior of the solutions of the integral equations when $n \geq 12$. We show that the operator T leaves $L^\infty[0,1]$ invariant, is continuous on that space and has spectral radius 1, again on that space. Moreover we show that any fixed points of T other than the constants have unbounded variation and do not attain their supremum or infimum on $[0,1)$. In section 4 we study the behavior of the kernel of the integral operator and in section 5, we obtain certain estimates on the growth rate of the integral operator T.

2 Radial functions and the invariant MVP

Definition: Let f be a function in $L^1(\mathbb{D})$ and let $R(f) = rad(f)$ be the function defined by

$$R(f)(w) = rad(f)(w) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it} w) dt. \tag{8}$$

We say that $R(f)$ is the radialization of f and we say that f is radial if it is equal to its radialization. Thus a radial function is such that $f(z) = f(|z|)$. Let S_n be the boundary of the open unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$.

For a function u defined on $\mathbb{B}_n$, the radialization of u, denoted by Ru, is the function on $\mathbb{B}_n$ defined by

$$Ru(x) = \int_{S_n} u(|x|\xi) d\sigma(\xi). \tag{9}$$

Here $d\sigma$ be the normalized surface-area measure (Hausdorff measure) of S_n such that $\sigma(S_n) = 1$.

In the following theorem we give a characterization of $\mathcal{M}$-harmonic functions u in terms of invariant mean value property and radialization of $u \circ \psi$, $\psi \in Aut(\mathbb{B}_n)$. This is an extension of the result of Axler and Cuckovic [4].

Theorem 2.1. *Suppose that $u \in C(\mathbb{B}_n) \cap L^1(\mathbb{B}_n, d\nu)$. Then u is $\mathcal{M}$-harmonic on $\mathbb{B}_n$ if and only if*

$$\int_{\mathbb{B}_n} (u \circ \psi) d\nu = u(\psi(0)) \tag{10}$$

and

$$R(u \circ \psi) \in C(\overline{\mathbb{B}_n}) \tag{11}$$

for every $\psi \in Aut(\mathbb{B}_n)$.

Proof: Suppose that u is $\mathcal{M}$-harmonic on $\mathbb{B}_n$. Let $\psi \in Aut(\mathbb{B}_n)$. As we have already seen in section 1 that $u \circ \psi$ is $\mathcal{M}$-harmonic and so (10) holds. Hence

$$\int_{S_n} (u \circ \psi)(|x|\xi) d\sigma(\xi) = (u \circ \psi)(0). \tag{12}$$

Thus $R(u \circ \psi)(x) = (u \circ \psi)(0)$. That is, $R(u \circ \psi)$ is a constant function on $\mathbb{B}_n$, with value $u(\psi(0))$. So (11) also holds. Now suppose, (10) and (11) hold. Let $\psi \in Aut(\mathbb{B}_n)$ and let $v = R(u \circ \psi)$. From (11), we have $v \in C(\overline{\mathbb{B}_n})$. In what follows, we shall prove that v satisfies the invariant volume mean value property. Fix $\phi \in Aut(\mathbb{B}_n)$. Then

$$\int_{\mathbb{B}_n} (v \circ \phi) d\nu = \int_{\mathbb{B}_n} R(u \circ \psi)(\phi(w)) d\nu(w) \tag{13}$$

$$= \int_{\mathbb{B}_n} \int_{S_n} u(\psi(|\phi(w)|\xi)) d\sigma(\xi) d\nu(w). \tag{14}$$

For each $\xi \in S_n$, define $g_\xi \in Aut(\mathbb{B}_n)$ by $g_\xi(w) = \psi(|\phi(w)|\xi)$. Notice that $g_\xi(w) = (V \circ \phi_{z_\xi})(w)$ for some unitary V, and involution $\phi_{z_\xi}, z_\xi \in \mathbb{B}_n$. Thus,

$$\begin{aligned} & \int_{S_n} \int_{\mathbb{B}_n} |u(\psi(|\phi(w)|\xi))| d\nu(w) d\sigma(\xi) \\ = & \int_{S_n} \int_{\mathbb{B}_n} |(u \circ V)(w)| (J_{\mathbb{R}} \phi_{z_\xi})(w) d\nu(w) d\sigma(\xi) \\ = & \int_{S_n} \int_{\mathbb{B}_n} |(u \circ V)(w)| \tfrac{(1-|z_\xi|^2)^{n+1}}{|1-\langle z_\xi, w\rangle|^{2(n+1)}} d\nu(w) d\sigma(\xi) \\ \leq & \; K \int_{\mathbb{B}_n} |(u \circ V)(w)| d\nu(w) < \infty. \end{aligned}$$

Thus we can apply Fubini's theorem to obtain

$$\begin{aligned}
\int_{\mathbb{B}_n} (v \circ \phi) d\nu &= \int_{S_n} \int_{\mathbb{B}_n} u(\psi(|\phi(w)|\xi)) d\nu(w) d\sigma(\xi) \\
&= \int_{S_n} \int_{\mathbb{B}_n} (u \circ V \circ \phi_{z_\xi})(w) d\nu(w) d\sigma(\xi) \\
&= \int_{S_n} (u \circ V \circ \phi_{z_\xi})(0) d\sigma(\xi) \\
&= \int_{S_n} u(\psi(|\phi(0)|\xi)) d\sigma(\xi) \\
&= R(u \circ \psi)(\phi(0)) = v(\phi(0)).
\end{aligned}$$

Thus v is a continuous function on $\overline{\mathbb{B}_n}$ that has the invariant volume-mean-value property. Hence, v is $\mathcal{M}$-harmonic [13] on $\mathbb{B}_n$. Because v is also a radial function, the mean value property implies that v is a constant function on $\mathbb{B}_n$ with value $v(0)$. Recall that $v = R(u \circ \psi)$, so

$$\int_{S_n} (u \circ \psi)(|x|\xi) d\sigma(\xi) = u(\psi(0)) \tag{15}$$

for every $\psi \in Aut(\mathbb{B}_n), x \in \mathbb{B}_n$. In other words, u has the invariant mean value property (the usual version, not the volume mean value property). Thus, u is $\mathcal{M}$-harmonic on $\mathbb{B}_n$. $\square$

We now consider the Bergman space $L^2_a(\mathbb{B}_n)$ of holomorphic functions in $L^2(\mathbb{B}_n, d\nu)$. The reproducing kernel $K_{\mathbb{B}_n}(z, w)$ of $L^2_a(\mathbb{B}_n, d\nu)$ is holomorphic in z and antiholomorphic in w and

$$\int_{\mathbb{B}_n} |K_{\mathbb{B}_n}(z, w)|^2 d\nu(w) = K_{\mathbb{B}_n}(z, z) > 0 \tag{16}$$

for all $z \in \mathbb{B}_n$. Thus we define for each $\lambda \in \mathbb{B}_n$, a unit vector k_λ in $L^2_a(\mathbb{B}_n)$ by

$$k_\lambda(z) = \frac{K_{\mathbb{B}_n}(z, \lambda)}{\sqrt{K_{\mathbb{B}_n}(\lambda, \lambda)}}. \tag{17}$$

For $z, \lambda \in \mathbb{B}_n$,

$$K_{\mathbb{B}_n}(z, \lambda) = \frac{n!}{(1 - z \cdot \overline{\lambda})^{n+1}} \tag{18}$$

where $z \cdot \overline{\lambda} = z_1 \overline{\lambda}_1 + z_2 \overline{\lambda}_2 + \ldots\ldots + z_n \overline{\lambda}_n$. For detail see [10]. If $f \in L^1(\mathbb{B}_n, d\nu)$, the Berezin transform of f is defined by

$$(Bf)(w) = \int_{\mathbb{B}_n} f(z) |k_w(z)|^2 d\nu(z) \tag{19}$$

where $k_w(z)$ is the normalized reproducing kernel at $w \in \mathbb{B}_n$. Notice that $k_w \in L^\infty(\mathbb{B}_n)$ for all $w \in \mathbb{B}_n$, so the definition makes sense. For $f \in L^1(\mathbb{B}_n, d\nu)$,

$$(Bf)(z) = \int_{\mathbb{B}_n} f(\phi_z(w))d\nu(w). \tag{20}$$

This can be verified as follows: for any $\psi \in Aut(\mathbb{B}_n)$, we denote by $J_\psi(z)$ the complex Jacobian determinant of the mapping $\psi : \mathbb{B}_n \to \mathbb{B}_n$. If $a \in \mathbb{B}_n$, then by a result of [16], there exists a unimodular constant $\theta(a)$ such that $J_{\phi_a}(z) = \theta(a)k_a(z)$ for all $z \in \mathbb{B}_n$ and $|J_{\phi_a}(z)| = |k_a(z)|^2$. Thus

$$(Bf)(z) = \int_{\mathbb{B}_n} f(w)|k_z(w)|^2 d\nu(w) = \int_{\mathbb{B}_n} (f \circ \phi_z)(w)d\nu(w). \tag{21}$$

If $f \in L^1(\mathbb{B}_n)$, we say f satisfies the invariant volume mean value property if

$$\int_{\mathbb{B}_n} (f \circ \psi)d\nu = f(\psi(0)) \tag{22}$$

for every $\psi \in Aut(\mathbb{B}_n)$. If

$$(T_0 f)(z) = \int_{\mathbb{B}_n} (f \circ \phi_z)d\nu \tag{23}$$

and $f \in L^1(\mathbb{B}_n)$, then we say f satisfies the invariant volume mean value property if and only if $T_0 f = f$. That is, if and only if $Bf = f$. Suppose f is radial and there is a function $g : [0, 1] \to \mathbb{C}$ such that $f(z) = g(|z|^2)$. In the following theorem, we shall prove that $Bf = f$ if and only if

$$g(x) = (1-x)^{n+1} \int_0^1 \frac{n+tx}{(1-tx)^{n+2}} g(t)t^{n-1}dt = Tg(x). \tag{24}$$

Theorem 2.2. *If f is radial and $f(z) = g(|z|^2)$ then $Bf = f$ if and only if*

$$g(x) = (1-x)^{n+1} \int_0^1 \frac{n+tx}{(1-tx)^{n+2}} g(t)t^{n-1}dt. \tag{25}$$

Proof: Recall that $(Bf)(z) = \displaystyle\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1}}{|1-\langle z, w\rangle|^{2(n+1)}} f(w)d\nu(w)$. If $f(w) = g(|w|^2) = g(r^2)$ then from [13] it follows that

$$(Bf)(z) = (1-|z|^2)^{n+1} 2n \int_0^1 I_{n+2}(rz) r^{2n-1} g(r^2)dr \tag{26}$$

and

$$I_{n+2}(rz) = \frac{\Gamma(n)}{\Gamma^2(n+1)} \sum_{k=0}^{\infty} \frac{\Gamma^2(k+n+1)}{\Gamma(k+1)\Gamma(k+n)} |rz|^{2k} \tag{27}$$

where we use polar coordinates $w = r\zeta, \zeta \in S_n$(the sphere that bounds the open unit ball $\mathbb{B}_n$). Since $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$, hence

$$I_{n+2}(rz) = \frac{(n-1)!}{n!n!} \sum_{k=0}^{\infty} \frac{(k+n)!(k+n)!}{k!(k+n-1)!} |rz|^{2k}. \tag{28}$$

If $n = 1$, then

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(k+1)(k+1)!}{k!} |rz|^{2k} &= \sum_{k=0}^{\infty} (k+1)^2 |rz|^{2k} \\
&= \frac{1}{2} \left[\sum_{k=0}^{\infty} 2(k+1)^2 |rz|^{2k} \right] \\
&= \frac{1}{2} \left[\sum_{k=1}^{\infty} (k+1)k |rz|^{2(k-1)} + \sum_{k=1}^{\infty} (k+1)k |rz|^{2k} \right].
\end{aligned}$$

Let $u = |rz|^2, s = |z|^2, t = r^2$. Thus $u = ts$. Then

$$\begin{aligned}
I_3(rz) &= \frac{1}{2} \left[\sum_{k=1}^{\infty} (k+1)k u^{k-1} + \sum_{k=1}^{\infty} (k+1)k u^k \right] \\
&= \frac{(1+u)}{2} \sum_{k=1}^{\infty} (k+1)k u^{k-1} \\
&= \frac{(1+u)}{2} \frac{d}{du} \left(\sum_{k=1}^{\infty} (k+1) u^k \right) \\
&= \frac{(1+u)}{2} \frac{d}{du} \left(\sum_{k=1}^{\infty} k u^{k-1} \right) \\
&= \frac{(1+u)}{2} \frac{d}{du} \left(\frac{1}{(1-u)^2} \right) = \frac{1+u}{(1-u)^3} \\
&= \frac{1+ts}{(1-ts)^3}.
\end{aligned}$$

We shall now show that $\sum_{k=0}^{\infty} \frac{(k+n)(k+n)!}{n!k!} |rz|^{2k} = \frac{n+ts}{(1-ts)^{n+2}}$ for all $n \in \mathbb{N}$. This identity is true for $n = 1$. So suppose $\sum_{k=0}^{\infty} \frac{(k+m)(k+m)!}{m!k!} |rz|^{2k} =$

$\dfrac{m+ts}{(1-ts)^{m+2}}$. We shall prove that

$$\sum_{k=0}^{\infty} \frac{(k+m+1)(k+m+1)!}{(m+1)!k!}|rz|^{2k} = \frac{(m+1)+ts}{(1-ts)^{m+3}}. \tag{29}$$

Since $\displaystyle\sum_{k=0}^{\infty} \frac{(k+m)(k+m)!}{m!k!}u^k = \frac{m+u}{(1-u)^{m+2}}$, by taking the derivatives of both the sides we get

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(k+m)(k+m)!}{m!k!}ku^{k-1} &= \frac{(1-u)^{m+2}-(m+2)(1-u)^{m+1}(-m-u)}{(1-u)^{2m+4}} \\ &= \frac{(1+m)^2+u(1+m)}{(1-u)^{m+3}}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(k+m)(k+m)!}{(k-1)!m!}u^{k-1} &= \sum_{k=1}^{\infty} \frac{k(k+m)(k+m)!}{m!k!}u^{k-1} \\ &= \frac{(1+m)(1+m+u)}{(1-u)^{m+3}}.\end{aligned}$$

That is,

$$\sum_{k=0}^{\infty} \frac{(k+m+1)(k+m+1)!}{(m+1)!k!}u^k = \frac{m+1+u}{(1-u)^{m+3}}. \tag{30}$$

Thus

$$\begin{aligned}(Bf)(z) &= (1-|z|^2)^{n+1}2n\int_0^1 \left(\frac{1}{n!n}\sum_{k=0}^{\infty} \frac{(k+n)(k+n)!}{k!}r^{2k}|z|^{2k}\right) r^{2n-1}g(r^2)dr \\ &= (1-|z|^2)^{n+1}\int_0^1 \left(\sum_{k=0}^{\infty} \frac{(k+n)(k+n)!}{n!k!}r^{2k}|z|^{2k}\right) r^{2n-2}g(r^2)2rdr.\end{aligned}$$

But

$$\sum_{k=0}^{\infty} \frac{(k+n)(k+n)!}{n!k!}r^{2k}|z|^{2k} = \frac{n+ts}{(1-ts)^{n+2}} \tag{31}$$

where $t=r^2, s=|z|^2$. Thus

$$(Bf)(z) = (1-s)^{n+1}\int_0^1 \frac{n+ts}{(1-ts)^{n+2}}g(t)t^{n-1}dt. \tag{32}$$

□

Theorem 2.3. *If the constant functions are the only functions $V \in C([0,1)) \cap L^1[0,1]$ such that*

$$V(t) = (1-t)^{n+1} \int_0^1 \frac{n+ts}{(1-ts)^{n+2}} V(s) s^{n-1} ds$$

for every $t \in [0,1)$ then functions in $C(\mathbb{B}_n) \cap L^1(\mathbb{B}_n, d\nu)$ having the invariant volume mean value property is $\mathcal{M}$-harmonic.

Proof: First suppose that f is a radial function in $C(\mathbb{B}_n) \cap L^1(\mathbb{B}_n, d\nu)$ having the invariant volume mean value property. We will show that f is constant on $\mathbb{B}_n$. Let $f(z) = g(|z|^2)$. Since $Bf = f$, by Theorem 2.2

$$g(t) = (1-t)^{n+1} \int_0^1 \frac{n+ts}{(1-ts)^{n+2}} g(s) s^{n-1} ds. \tag{33}$$

By the hypothesis, g is constant on [0,1), and thus so is f, as claimed. To complete the proof, now suppose that u is a function in $C(\mathbb{B}_n) \cap L^1(\mathbb{B}_n, d\nu)$ having the invariant volume mean value property. Let $\psi \in Aut(\mathbb{B}_n)$. Clearly $R(u \circ \psi)$ is a radial function on $\mathbb{B}_n$ and, as shown in the proof of Theorem 2.1, $v = R(u \circ \psi)$ has the invariant volume mean value property. That is,

$$\int_{\mathbb{B}_n} (v \circ \phi) d\nu = v(\phi(0)) \tag{34}$$

for all $\phi \in Aut(\mathbb{B}_n)$. By the first part of the proof, $R(u \circ \psi)$ is a constant function on $\mathbb{B}_n$. In particular, $R(u \circ \psi) \in C(\overline{\mathbb{B}_n})$, and so by Theorem 2.1,$u$ is $\mathcal{M}$-harmonic. □

3 Functions of bounded variation and the solutions of the integral equation

In this section we investigate whether constants are the only solutions of the integral equation (5) in $L^1[0,1]$ for all integers $n \geq 1$ and $n \leq 11$ and the behavior of the solutions when $n \geq 12$.
The following observations can be made about the solutions of the integral operator equation

$$V(t) = (1-t)^{n+1} \int_0^1 \frac{n+ts}{(1-ts)^{n+2}} V(s) s^{n-1} ds \tag{35}$$

for every $t \in (0,1)$. Casting it in a slightly different form, we want to show that the eigenspace corresponding to the eigenvalue 1 of the operator T is one dimensional, where

$$(Tu)(t) = (1-t)^{n+1} \int_0^1 \frac{n+ts}{(1-ts)^{n+2}} u(s) s^{n-1} ds \tag{36}$$

defined on $C([0,1)) \cap L^1([0,1]), n \leq 11, n \in \mathbb{N}$. The operator T is an integral operator with positive kernel which maps $L^1([0,1])$ to $C([0,1))$, so the domain may as well be considered as $L^1([0,1])$. But the operator T does not leave L^1 invariant because

$$G(s) = \int_0^1 (1-t)^{n+1} \frac{n+ts}{(1-ts)^{n+2}} dt$$

is an unbounded function of s and hence there exists $0 \leq f \in L^1$ such that

$$\int_0^1 f(s)G(s)ds = \infty.$$

We could regard T as an operator on $C([0,1))$, which it clearly leaves invariant. The difficulty here is that there is no natural norm on $C([0,1))$, although there is a metric which makes $C([0,1))$ into a Frechet space. The upshot of this is that it is difficult to find a convenient and natural invariant Banach space for T. Consider for a while the case $n=1$. We know in $\mathbb{D}$, the only measure left invariant by all Mobius transformations is the pseudo-hyperbolic measure $d\eta(z) = \dfrac{dA(z)}{(1-|z|^2)^2}$ where $dA(z) = \frac{1}{\pi} dxdy$. Therefore, the only harmonic function in $L^p(\mathbb{D}, d\eta)$ is constant zero. To see this e.g., for $L^2(\mathbb{D}, d\eta)$, denote

$$M(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt. \tag{37}$$

This is a nonnegative and nondecreasing function of r. Further,

$$\|f\|^2_{L^2(\mathbb{D},d\eta)} = \int_0^1 M(r) \frac{2r}{(1-r^2)^2} dr < \infty. \tag{38}$$

So $M(r)$ must tend to zero as $r \to 1$. Thus $M(r) \equiv 0$, whence $f=0$. Thus even though one can show that every space $L^p((0,1), \frac{dt}{(1-t)^2}), 1 \leq p \leq \infty$ is an invariant subspace [1], [6] of T but these spaces are no good in this context. This is because (except for L^∞) the corresponding spaces $L^p(\mathbb{D}, d\eta)$ do not contain nonzero harmonic functions, even no nonzero constants. Similar is the case for $\mathbb{B}_n, n \geq 1$. Let $d\eta'(z) = K_{\mathbb{B}_n}(z,z)d\nu(z)$. It is not so difficult to check that if $f \in L^2(\mathbb{B}_n, d\eta')$ is $\mathcal{M}$-harmonic then $f \equiv 0$. The argument is as follows: Denote the unit sphere, the boundary of the open unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ by S_n. Let $d\sigma$ be the normalized surface-area measure (Hausdorff measure) of S_n such that $\sigma(S_n) = 1$. Let $M(r) = \int_{S_n} |f(z)|^2 d\sigma(z)$. Then

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{B}_n, d\eta')} &= \int_{\mathbb{B}_n} |f(z)|^2 d\eta'(z) \\ &= \int_0^1 M(r) K_{\mathbb{B}_n}(z,z) 2nr^{2n-1} dr \\ &= n \int_0^1 M(r) n! \frac{t^{n-1}}{(1-t)^{n+1}} dt \qquad \text{where} \quad t = r^2. \end{aligned}$$

So $M(r)$ must tend to zero as $r \to 1$. Thus $M(r) \equiv 0$. Hence since f is $\mathcal{M}$-harmonic, by maximum principle $f \equiv 0$. Thus even though one can show that the space $L^2((0,1), \frac{t^{n-1}dt}{(1-t)^{n+1}})$ is an invariant subspace [1] of T but these spaces are no good in this context. This is because the space $L^2(\mathbb{B}_n, d\eta')$ do not contain nonzero harmonic functions, even no nonzero constants.

In the next theorem, we show that T leaves $L^\infty[0,1]$ invariant, is continuous on that space and has spectral radius 1, again on that space. Moreover we shall show that any fixed points of T other than the constants have unbounded variation and do not attain their supremum or infimum on $[0,1)$.

Theorem 3.1. *The operator T is a bounded operator from $L^\infty[0,1]$ into $L^\infty[0,1]$ and has spectral radius 1. Moreover, any fixed points of T other than the constants in $C[0,1) \cap L^1[0,1]$ have unbounded variation and do not attain their supremum or infimum on $[0,1)$.*

Proof: The points to note are:

(a) Constant functions are fixed points.

(b) If $u \geq 0$ on $[0,1), u \in C([0,1))$ and u is not identically zero then $Tu > 0$ on $[0,1)$.

These statements can be verified as follows: For $n = 1, 0 < r < 1, k = |\bar{y}r|$

and $R = |y|^2 < 1$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-\bar{y}re^{i\theta}|^4}d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-ke^{i\theta}|^4}d\theta\\ &= \frac{1}{2\pi i}\int_{\mathbb{T}}(1-kz)^{-2}\left(1-\frac{k}{z}\right)^{-2}\frac{dz}{z}\\ &= \frac{1+Rr^2}{(1-Rr^2)^3}.\end{aligned}$$

Here $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$. Thus

$$\begin{aligned}(1-t)^2\int_0^1\frac{1+ts}{(1-ts)^3}ds &= (1-t)^2\int_0^1\frac{1+r^2t}{(1-r^2t)^3}2rdr\\ &= (1-|y|^2)^2\int_0^1\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-\bar{y}re^{i\theta}|^4}d\theta\right)2rdr\\ &= \int_0^1(1-|y|^2)^2\left(\sum_{n=0}^{\infty}(n+1)^2r^{2n}|y|^{2n}\right)2rdr\\ &= (1-|y|^2)^2\sum_{n=0}^{\infty}(n+1)|y|^{2n} = 1.\end{aligned}$$

This can also be seen as follows. Notice that

$$\begin{aligned}\int_0^1\frac{1+ts}{(1-ts)^3}ds &= \int_0^1\frac{2}{(1-ts)^3}ds - \int_0^1\frac{1}{(1-ts)^2}ds\\ &= \frac{1}{(1-t)^2}.\end{aligned}$$

If $n \in \mathbb{Z}_+, n \geq 1$, let $v = 1-ts$. Then $ds = -\dfrac{dv}{t}$ and $n+ts = n+1-v$. Hence $s^{n-1} = \left(\dfrac{1-v}{t}\right)^{n-1}$. Thus

$$\begin{aligned}I = \int_0^1\frac{n+ts}{(1-ts)^{n+2}}s^{n-1}ds &= \int_1^{1-t}\left(\frac{n+1-v}{v^{n+2}}\right)\left(\frac{1-v}{t}\right)^{n-1}\left(-\frac{dv}{t}\right)\\ &= \frac{1}{t^n}\int_{1-t}^1\frac{(n+1-v)(1-v)^{n-1}}{v^{n+2}}dv.\end{aligned}$$

But

$$(n+1-v)(1-v)^{n-1} = (n+1)\left[\binom{n-1}{0} - \binom{n-1}{1}v + \binom{n-1}{2}v^2 - \cdots \right.$$
$$\left. + (-1)^{n-2}\binom{n-1}{n-2}v^{n-2} + (-1)^{n-1}\binom{n-1}{n-1}v^{n-1}\right]$$
$$+ \left[-\binom{n-1}{0}v + \binom{n-1}{1}v^2 - \binom{n-1}{2}v^3 + \cdots \right.$$
$$\left. + (-1)^{n-1}\binom{n-1}{n-2}v^{n-1} + (-1)^{n}\binom{n-1}{n-1}v^{n}\right]$$

and

$$\frac{(n+1-v)(1-v)^{n-1}}{v^{n+2}} = (n+1)\binom{n-1}{0}v^{-n-2}$$
$$- \left[(n+1)\binom{n-1}{1} + \binom{n-1}{0}\right]v^{-n-1} + \left[(n+1)\binom{n-1}{2} + \binom{n-1}{1}\right]v^{-n}$$
$$+ \cdots + (-1)^{n-2}\left[(n+1)\binom{n-1}{n-2} + \binom{n-1}{n-3}\right]v^{-4}$$
$$+ (-1)^{n-1}\left[(n+1)\binom{n-1}{n-1} + \binom{n-1}{n-2}\right]v^{-3} + (-1)^{n}\binom{n-1}{n-1}v^{-2}$$
$$= (n+1)\binom{n}{0}v^{-n-2} - n\binom{n}{1}v^{-n-1} + (n-1)\binom{n}{2}v^{-n}$$
$$- (n-2)\binom{n}{3}v^{-n+1} + \cdots$$
$$+ (-1)^{n-2}3\binom{n}{n-2}v^{-4} + (-1)^{n-1}2\binom{n}{n-1}v^{-3} + (-1)^{n}\binom{n}{n}v^{-2}$$
$$= \sum_{k=1}^{n+1}(-1)^{n+1-k}k\binom{n}{n+1-k}v^{-k-1}. \tag{39}$$

Substituting (39) in the integral expression of I, we get

$$I = \frac{1}{t^n}\sum_{k=1}^{n+1}(-1)^{n+1-k}k\binom{n}{n+1-k}\int_{1-t}^{1} v^{-k-1}dv$$
$$= \frac{1}{t^n}\sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n}{n+1-k}\left[\frac{1}{(1-t)^k} - 1\right]$$
$$= \frac{1}{t^n(1-t)^{n+1}}\left\{\sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n}{n+1-k}\left[(1-t)^{n+1-k} - (1-t)^{n+1}\right]\right\}$$
$$= \frac{1}{t^n(1-t)^{n+1}}t^n = \frac{1}{(1-t)^{n+1}}$$

as

$$\sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n}{n+1-k}\left[(1-t)^{n+1-k}-(1-t)^{n+1}\right]=t^n \qquad (40)$$

for all $n \in \mathbb{N}$. Thus it follows that if $u \in L^\infty$ then $Tu \in L^\infty$ and $\|Tu\|_\infty \leq \|u\|_\infty$. Therefore $\|T\| \leq 1$. When u is a constant function, we have $\|Tu\|_\infty = \|u\|_\infty$. Hence $\|T\| = 1$ and the spectral radius of T is 1. From these, it also follows that if u is a nonconstant fixed point in $C([0,1))$ then

(i) $\inf\limits_{t\in[0,1)} u < u(t)$ for all $t \in [0,1)$. (ii) $\sup\limits_{t\in[0,1)} u > u(t)$ for all $t \in [0,1)$.

(iii) $\liminf\limits_{t\to 1} u(t) = \inf\limits_{t\in[0,1)} u$. (iv) $\limsup\limits_{t\to 1} u(t) = \sup\limits_{t\in[0,1)} u$.

If u is unbounded below, then (i) and (iii) are trivial. If u is bounded below, let $\alpha = \inf\limits_{t\in[0,1)} u$. Now, $u(t) - \alpha \geq 0$ and is not identically zero since u is not constant. By (a) and (b), $u - \alpha = T(u - \alpha) > 0$ on $[0,1)$, proving (i). Again (iii) is now immediate by continuity. If u is unbounded above, then (ii) and (iv) are trivial. If u is bounded above, the same argument as above applied to $\sup\ u - u$ shows (ii) and (iv). What (i)-(iv) show is that nonconstant $C([0,1))$ fixed points of T look something like the plot in Figure 1

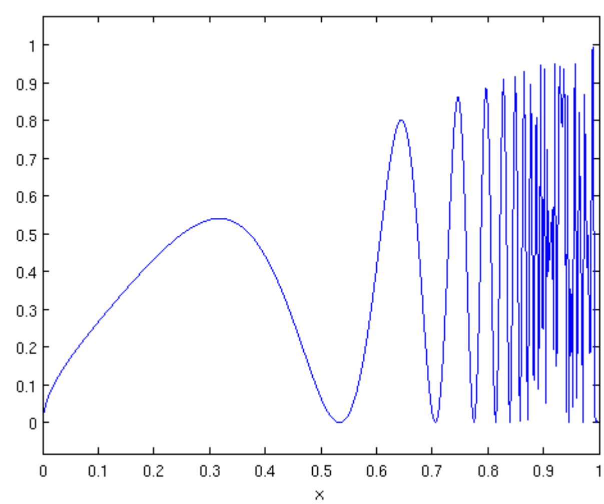

Figure 1: Unbounded variation of the fixed points of T

where either the infimum or supremum may be infinite. They oscillate infinitely many times, thus having unbounded variation on $[0, 1)$, not attaining their supremum or infimum anywhere, but approaching both at 1. □

4 On the kernel of the integral operator

In this section we study the behavior of the kernel of the integral operator T defined in (24). First we make the following observation.

Lemma 4.1. *If $r > 0$ and $s > 0$ then the following hold:*

1. $\frac{1}{2\pi}\displaystyle\int_0^{2\pi} \frac{1}{|1-rse^{i\theta}|^2}d\theta = \frac{1}{1-r^2s^2}$;
2. $\frac{1}{2\pi}\displaystyle\int_0^{2\pi} \frac{1}{|1-rse^{i\theta}|^4}d\theta = \frac{1+r^2s^2}{(1-r^2s^2)^3}$;
3. $\frac{1}{2\pi}\displaystyle\int_0^{2\pi} \frac{\Re e^{i\theta}}{|1-rse^{i\theta}|^2}d\theta = \frac{rs}{1-r^2s^2}$;
4. $\frac{1}{2\pi}\displaystyle\int_0^{2\pi} \frac{\Re e^{i\theta}}{|1-rse^{i\theta}|^4}d\theta = \frac{2rs}{(1-r^2s^2)^3}$.

Proof: These equalities are quite simple to establish, using the Taylor series for $\frac{1}{1-x}$ and the orthonormality of the exponentials in $L^2[0, 2\pi]$. It is also possible to verify the Lemma directly using the residue theorem though the calculations do become tedious. □

One interesting property about the kernel is that the function $\dfrac{n+ts}{(1-ts)^{n+2}}$ is an increasing function of t for fixed s. This implies that if $u \geq 0$ then $\displaystyle\int_0^1 \frac{n+ts}{(1-ts)^{n+2}}u(s)ds$ is an increasing function of t, so if u is non-negative then $(Tu)(t) = (1-t)^{n+1}f(t)$ where f is increasing. In particular, any nonnegative fixed point of T has this property. One may think that this property might contradict the oscillatory property, but it seems it does not. In the one-dimensional case (when $n = 1$) the function given by

$$f(t) = \frac{1}{(1-t)^2}\{2 + \sin[\log(1-t)]\} \tag{41}$$

is monotone, but $(1-t)^2 f(t)$ oscillates infinitely many times on $[0, 1)$. It does, however, attain its supremum and infimium, so it is not quite of the

right form. If f is bounded, this does contradict the oscillatory property. May be one could show that the operator does not map any functions at all to functions with this property.
Thus we have shown that any fixed point of T (other than constants) oscillates infinitely many times, so all its derivatives also oscillate. Consider now the case $n = 1$. Let $k(s,t) = \frac{1+ts}{(1-ts)^3}$. Then

$$\frac{d^m}{ds^m}\int_0^1 k(s,t)u(t)dt = \int_0^1 [\frac{\partial^m}{\partial s^m}k(s,t)]u(t)dt. \tag{42}$$

Now if $u \geq 0$ and $\frac{\partial^m}{\partial s^m}k(s,t) \geq 0$ then this is nonnegative. Thus, if $\frac{\partial^m}{\partial s^m}k(s,t) \geq 0$ then T has no nonnegative, non-oscillatory fixed points, hence no bounded above or bounded below fixed points at all. Further it may be observed that $\frac{\partial^m}{\partial s^m}k(s,t)$ is not single-signed in a neighborhood of the singularity at least for any m less than about 200. Finally, if k did satisfy the condition that $\frac{\partial^m}{\partial s^m}k(s,t) \geq 0$ for some integer $m \geq 1$ then if u is bounded,

$$\frac{d^m}{ds^m}T(u - \inf u) \geq 0 \text{ which implies } \frac{d^m}{ds^m}Tu \geq 0 \tag{43}$$

and

$$\frac{d^m}{ds^m}T(\sup\ u - u) \geq 0 \text{ which implies } \frac{d^m}{ds^m}Tu \leq 0. \tag{44}$$

So if u is bounded, then Tu is a polynomial of degree $\leq n-1$. That is, the images of bounded functions are equal to polynomials in a neighborhood of 1. In Theorem 3.1, we have verified that there exists no $f \in L^1(\mathbb{D}, dA)$ that is radial, real analytic, nonconstant such that $Bf = f$ and f is a function of bounded variation. But the following is true.

Remark 4.1. *There exist functions f that are radial, real analytic and of unbounded variation but which are not in $L^1(\mathbb{D}, dA)$ and for which $Bf = f$ (in a sense obtained by extending the definition of B). Notice that Bf cannot be defined by the usual integral if f is not in $L^1(\mathbb{D}, dA)$. For instance, the spherical functions*

$$f_\lambda(z) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1-|z|^2}{|z-e^{i\theta}|^2}\right)d\theta \tag{45}$$

satisfy $Bf_\lambda = \frac{\pi\lambda(1-\lambda)}{sin\pi\lambda} f_\lambda$ *for any* $\lambda \in \mathbb{C}$, *but the integral defining* Bf_λ *converges only for* λ *in the strip* $-1 < Re\ \lambda < 2$. *For other* λ, *one has to extend the definition of* B *somehow, e.g., by defining* $Bf = g$ *if*

$$\int_{\mathbb{D}} g(z)\phi(z)\frac{dA(z)}{(1-|z|^2)^2} = \int_{\mathbb{D}} Bf(z)\phi(z)\frac{dA(z)}{(1-|z|^2)^2} \tag{46}$$

for all smooth functions ϕ *with compact support. Now the function* $m(\lambda) = \frac{\pi\lambda(1-\lambda)}{sin\pi\lambda}$ *satisfies* $m(\lambda) = m(1-\lambda)$, *hence* $m(\frac{1}{2}+z) = M(z^2)$ *for some entire function* M; *since* m *has order* 1, M *has order one half, and therefore* $M(z) - a$ *has infinitely many zeroes in* $\mathbb{C}$, *for any* a; *taking* $a = 1$ *it follows that* $M-1$, *hence also* $m-1$, *has infinitely many zeroes. Consequently, there exist infinitely many* λ *with* $m(\lambda) = 1$, *hence infinitely many* λ *for which* $Bf_\lambda = f_\lambda$. *All these functions* f_λ *are radial, real analytic, of unbounded variation, only none of them belongs to* $L^1(\mathbb{D}, dA)$ *except* $f_0 = f_1 = 1$.

Let $d\eta(z) = \dfrac{dA(z)}{(1-|z|^2)^2}$ and $f \in L^1(\mathbb{D}, d\eta)$. Then it is shown in [6] that $Bf = f * h$ where $h(z)$ is the function on $\mathbb{D}$ given by $h(z) = (1-|z|^2)^2$ and $*$ denotes convolution.
For a radial function f in $L^1(\mathbb{D}, d\eta)$, define its Fourier transform by

$$\check{\mathbf{f}}(s) = \int_{\mathbb{D}} f(z)\left(\frac{1-|z|^2}{|1-z|^2}\right)^s d\eta(z), s \in \mathbb{C}. \tag{47}$$

This integral exists if s lies in the vertical strip $\Omega = \{s \in \mathbb{C} : 0 < \text{Re } s < 1\}$.

Further, $\check{\mathbf{f}}$ is holomorphic in this strip and satisfies $\check{\mathbf{f}}(s) = \check{\mathbf{f}}(1-s), s \in \Omega$ and $(\mathbf{f} \check{*} \mathbf{g}) = \check{\mathbf{f}} \cdot \check{\mathbf{g}}$ for all f, g radial in $L^1(\mathbb{D}, d\eta)$. It is shown in [6] that $\check{\mathbf{h}} = \phi$ where $\phi(s) := \dfrac{\pi s(1-s)}{\sin \pi s}$.

Theorem 4.1. *Suppose* $f \in L^1(\mathbb{D}, d\eta)$ *is radial and is a function of unbounded variation. Then* $Bf \neq f$.

Proof: If $Bf = f$ then f is real analytic [1] and from the above discussion it follows that $\check{\mathbf{f}} = \check{\mathbf{f}} \cdot \check{\mathbf{h}} = \check{\mathbf{f}} \cdot \phi$. That is, $\check{\mathbf{f}}(1-\phi) = 0$. The function ϕ is holomorphic in the vertical strip (see [6]) $\Omega_1 = \{s \in \mathbb{C} : -1 < \text{Re}s < 2\}$, $\phi(0) = \phi(1) = 1$ and $1-\phi$ has only two zeroes 0 and 1 in Ω_1. Thus it follows that $\check{\mathbf{f}} = 0$ almost everywhere in Ω. For $s \in \mathbb{C}$,

$$\begin{aligned} \check{\mathbf{f}}(s) &= \int_{\mathbb{D}} f(z)\left(\frac{1-|z|^2}{|1-z|^2}\right)^s d\eta(z) \\ &= \int_0^1 f(r^2)(1-r^2)^s\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|^{2s}}\right)2rdr. \end{aligned} \tag{48}$$

By the binomial formula,

$$(1-z)^{-s} = \sum_{k=0}^{\infty} \frac{\Gamma(k+s)}{\Gamma(s)} \frac{z^k}{k!}, z \in \mathbb{D}, s \in \mathbb{C}, \tag{49}$$

with convergence uniform on compact subsets of $\mathbb{D}$. (Here Γ is the Euler's Gamma function: $\Gamma(n+1) = n!$). Consequently, for $0 \leq r < 1$,

$$|1-re^{i\theta}|^{-2s} = (1-re^{i\theta})^{-s}(1-re^{-i\theta})^{-s} = \sum_{m,k=0}^{\infty} \frac{\Gamma(k+s)\Gamma(m+s)}{k!m!\Gamma(s)^2} r^{m+k} e^{i(m-k)\theta}, \tag{50}$$

with convergence uniform in θ. Hence,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^{2s}} = \sum_{m,k=0}^{\infty} \frac{\Gamma(k+s)\Gamma(m+s)}{k!m!\Gamma(s)^2} r^{m+k} \int_0^{2\pi} e^{i(m-k)\theta} \frac{d\theta}{2\pi} \tag{51}$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(m+s)^2}{\Gamma(s)^2} \frac{r^{2m}}{m!^2},$$

so the right-hand side of (48) equals

$$\int_0^1 f(r^2)(1-r^2)^s \sum_{m=0}^{\infty} \frac{\Gamma(m+s)^2}{\Gamma(s)^2} \frac{r^{2m}}{m!^2} 2rdr$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(m+s)^2}{m!^2\Gamma(s)^2} \int_0^1 f(r^2)(1-r^2)^s r^{2m} 2rdr$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(m+s)^2}{m!^2\Gamma(s)^2} \int_0^1 f(t)(1-t)^s t^m dt.$$

Thus $\check{\mathbf{f}}(s) = 0$ implies $\sum_{m=0}^{\infty} \frac{\Gamma(m+s)^2}{m!^2\Gamma(s)^2} \int_0^1 f(t)(1-t)^s t^m dt = 0.$

Suppose now f is a function of unbounded variation and $Bf = f$. Then f is real analytic and

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+s)^2}{m!^2\Gamma(s)^2} \int_0^1 f(t)(1-t)^s t^m dt = 0. \tag{52}$$

Let $f(t) = \sum_{n=0}^{\infty} a_n t^n$ be the Taylor series expansion of f around origin. Since

$$(k+1)\int_0^1 x^n(1-x)^k dx = \frac{\Gamma(n+1)\Gamma(k+2)}{\Gamma(n+k+2)} \asymp (n+1)^{-k-1}$$

for nonnegative real k, n, it follows from (52) that

$$\sum_{m,n=0}^{\infty} \frac{a_n}{s+1} \frac{\Gamma(m+s)^2}{m!^2\Gamma(s)^2} \frac{1}{(m+n+1)^{s+1}} = 0 \tag{53}$$

for all $s \in \mathbb{R}, 0 < s < 1$. This is possible only when $a_n = 0$ for all n. But that implies $f \equiv 0$. This is a contradiction as f is a function of unbounded variation. Hence $Bf \neq f$. $\square$

5 Growth estimates of the integral operator

In this section we obtain certain growth estimates of the integral operator T when $n = 1$. Thus for this section

$$Tg(x) = (1-x)^2 \int_0^1 \frac{1+tx}{(1-tx)^3} g(t)dt.$$

Theorem 5.1. *Let $a = 2 - 2\epsilon, 0 < \epsilon < \frac{3}{2}$. Then there exists constant C_ϵ depending only on ϵ such that*

$$\int_0^1 (1-t)^a \frac{1+ts}{(1-ts)^3} dt \leq C_\epsilon (1-s)^{-2\epsilon}.$$

Proof Notice that

$$\int_0^1 (1-t)^{2-2\epsilon} \frac{1+ts}{(1-ts)^3} dt \leq 2 \int_0^1 \frac{(1-t)^{2-2\epsilon}}{(1-ts)^3} dt. \tag{54}$$

Let $\alpha = -\frac{\epsilon}{2}$ and $\beta = 2 - \frac{3\epsilon}{2}$. Then $\alpha + \beta = 2 - 2\epsilon$. Now let $k = -\frac{\epsilon}{2}$.

$$\begin{aligned}
\int_0^1 \frac{(1-t)^k}{1-ts} dt &= \int_0^1 \frac{r^k}{(1-s)+sr} dr \ \ (\textit{letting } r = 1-t, dt = -dr) \\
&= -\frac{1}{k} \int_1^\infty \frac{R^{\frac{1}{k}}}{(1-s)+sR^{\frac{1}{k}}} dR \ \left(\textit{by letting } R = r^k, dr = \frac{1}{k} R^{\frac{1}{k}-1} dR\right) \\
&= l \int_1^\infty \frac{dR}{s+(1-s)R^l} \ \left(\textit{letting } V = (1-s)^{\frac{1}{l}} R, dR = (1-s)^{-\frac{1}{l}} dV\right) \\
&= l(1-s)^{-\frac{1}{l}} \int_{(1-s)^{\frac{1}{l}}}^\infty \frac{dV}{s+V^l} \ \left(\textit{where } l = -\frac{1}{k} \in (1,\infty)\right).
\end{aligned}$$

But

$$\int_1^\infty \frac{dV}{s+V^l} \leq \int_1^\infty \frac{dV}{V^l} = \frac{1}{l-1},$$

$$\int_{(1-s)^{\frac{1}{l}}}^{1} \frac{dV}{s+V^l} \leq \int_{(1-s)^{\frac{1}{l}}}^{1} dV \leq 1,$$

so, indeed,

$$\begin{aligned}\int_0^1 \frac{(1-t)^{-\frac{\epsilon}{2}}}{(1-ts)} dt &\leq l(1-s)^{-\frac{1}{l}} \frac{l}{l-1} \\ &= \frac{1}{\frac{\epsilon}{2}(1-\frac{\epsilon}{2})}(1-s)^{-\frac{\epsilon}{2}}.\end{aligned}$$

Now

$$\begin{aligned}\frac{(1-t)^{2-2\epsilon}}{(1-ts)^3} &= (1-s)^{-\frac{3\epsilon}{2}} \left(\left(\frac{1-s}{1-ts} \right)^{\frac{3\epsilon}{2}} \right) \left(\frac{1-t}{1-ts} \right)^{2-\frac{3\epsilon}{2}} \frac{(1-t)^{-\frac{\epsilon}{2}}}{(1-ts)} \\ &\leq (1-s)^{-\frac{3\epsilon}{2}} \frac{(1-t)^{-\frac{\epsilon}{2}}}{(1-ts)}, as\ \frac{1-t}{1-ts}, \frac{1-s}{1-ts} \in (0,1).\end{aligned}$$

Thus

$$\begin{aligned}\int_0^1 \frac{(1-t)^{2-2\epsilon}(1+ts)}{(1-ts)^3} dt &\leq 2\int_0^1 \frac{(1-t)^{2-2\epsilon}}{(1-ts)^3} dt \\ &\leq 2(1-s)^{-\frac{3\epsilon}{2}} \frac{(1-s)^{-\frac{\epsilon}{2}}}{\frac{\epsilon}{2}(1-\frac{\epsilon}{2})} \\ &= C_\epsilon (1-s)^{-2\epsilon}\end{aligned}$$

where C_ϵ is a constant depending on ϵ. □

Let

$$Tg(x) = (1-x)^2 \int_0^1 \frac{1+tx}{(1-tx)^3} g(t) dt.$$

The operator T is bounded [7] [14] on $L^2[0,1]$ and

$$\begin{aligned}\langle g, T^* f \rangle &= \langle Tg, f \rangle \\ &= \int_0^1 Tg(x)\overline{f(x)} dx \\ &= \int_0^1 (1-x)^2 \int_0^1 \frac{1+tx}{(1-tx)^3} g(t)\overline{f(x)} dt dx \\ &= \int_0^1 g(t) \left(\int_0^1 (1-x)^2 \frac{1+tx}{(1-tx)^3} \overline{f(x)} dx \right) dt.\end{aligned}$$

Thus

$$\begin{aligned}(T^*f)(t) &= \overline{\int_0^1 (1-x)^2 \frac{1+tx}{(1-tx)^3} \overline{f(x)} dx} \\ &= \int_0^1 (1-x)^2 \frac{1+tx}{(1-tx)^3} f(x) dx.\end{aligned}$$

Theorem 5.2. *T is not a bounded operator on $L^1[0,1]$.*

Proof: The operator T is defined as

$$(Tg)(x) = (1-x)^2 \int_0^1 \frac{1+tx}{(1-tx)^3} g(t) dt.$$

Notice that

$$\frac{1+tx}{(1-tx)^3} = \sum_{n=0}^{\infty} (n+1)^2 t^n x^n,$$

$0 < x < 1, 0 < t < 1$. If T were bounded on $L^1[0,1]$, its adjoint $T^* = S$, where

$$(Sf)(t) = \int_0^1 (1-x)^2 \frac{1+tx}{(1-tx)^3} f(x) dx$$

would be a bounded operator on $L^\infty[0,1]$.
Now

$$\begin{aligned}(S1)(t) &= \int_0^1 (1-x)^2 \frac{1+tx}{(1-tx)^3} dx \\ &= \int_0^1 (1-x)^2 \sum_{n=0}^{\infty} (n+1)^2 t^n x^n dx \\ &= \sum_{n=0}^{\infty} (n+1)^2 t^n \int_0^1 (1-x)^2 x^n dx \\ &= \sum_{n=0}^{\infty} \frac{2(n+1)}{(n+2)(n+3)} t^n.\end{aligned}$$

As $t \to 1$, this expression behaves asymptotically like $log \frac{1}{1-|z|^2}$, hence $S1 \notin L^\infty[0,1]$, so $S = T^*$ cannot be a bounded operator on $L^\infty[0,1]$. $\square$
Let $L(H)$ be the set of all bounded linear operators from the Hilbert space H into itself. Let $H^2(\mathbb{D})$ be the Hardy space of the disk and $L_a^2(\mathbb{D})$ be the Bergman space of the open unit disk $\mathbb{D}$. Let $W : \mathbb{H}^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ be a map

that sends z^k into $e_k = \sqrt{k+1}z^k$. Thus W is a Hilbert space isomorphism of $\mathbb{H}^2(\mathbb{D})$ onto $L^2_a(\mathbb{D})$. Thus an operator $T \in L(L^2_a)$ can be transferred to $W^*TW \in L(\mathbb{H}^2)$. Let $g_\lambda(z) = \frac{1}{1-\bar{\lambda}z}, \lambda, z \in \mathbb{D}$, the reproducing kernel of the Hardy space $\mathbb{H}^2(\mathbb{D})$. Let $h_\lambda(z) = \frac{g_\lambda}{\|g_\lambda\|}, \lambda \in \mathbb{D}$ be the normalized reproducing kernel.

$$H_\lambda(z) = Wh_\lambda(z) = (1-|\lambda|^2)^{1/2}\sum_{k=0}^{\infty}\sqrt{(k+1)}\bar{\lambda}^k z^k. \tag{55}$$

Let ϕ be a radial function in $L^\infty(\mathbb{D})$ and T_ϕ be the Toeplitz operator on $L^2_a(\mathbb{D})$ with symbol ϕ defined by $T_\phi f = P(\phi f)$ where P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$. Let $\widetilde{\phi}(\lambda) = \langle T_\phi k_\lambda, k_\lambda\rangle$, the Berezin transform of ϕ and $\mathring{\phi}(\lambda) = \langle T_\phi H_\lambda, H_\lambda\rangle$. Since ϕ is radial,

$$\begin{aligned}
\langle T_\phi e_j, e_k\rangle &= \int_{\mathbb{D}} \phi(|z|)\sqrt{j+1}\sqrt{k+1}z^j\bar{z}^k dA(z)\\
&= \sqrt{j+1}\sqrt{k+1}\int_0^1 \phi(r)r^{j+k}\int_0^{2\pi} e^{i(j-k)t}\frac{dt}{\pi}rdr\\
&= \begin{cases} 0, \text{ when } j \neq k;\\ \text{some number } c_k, |c_k| \leq ||T_\phi|| \text{ for some } k \text{ , when } j = k.\end{cases}
\end{aligned}$$

In other words, T_ϕ is a diagonal operator with respect to the basis $\{e_j\}$. Now if $t = |\lambda|^2$, then

$$\begin{aligned}
\mathring{\phi}(\lambda) &= \langle T_\phi H_\lambda, H_\lambda\rangle\\
&= (1-|\lambda|^2)\sum_{n=0}^{\infty} c_n|\lambda|^{2n}\\
&= (1-t)\sum_{n=0}^{\infty} c_n t^n.
\end{aligned}$$

Further,

$$\begin{aligned}
\widetilde{\phi}(\lambda) &= \langle T_\phi k_\lambda, k_\lambda\rangle\\
&= (1-|\lambda|^2)^2\sum_{n=0}^{\infty}(n+1)c_n|\lambda|^{2n}\\
&= (1-t)^2\sum_{n=0}^{\infty}(n+1)c_n t^n.
\end{aligned}$$

Thus both $\mathring{\phi}$ and $\widetilde{\phi}$ are radial functions and we shall show below that if $\widetilde{\phi}$ has a limit as $t \to 1$, then $\mathring{\phi}$ has a limit as $t \to 1$ and both limits are equal.

Theorem 5.3. *If* $\lim_{t\to 1}(1-t)^2\sum_{n=0}^{\infty}(n+1)c_n t^n$ *exists, then so does* $\lim_{t\to 1}(1-t)\sum_{n=0}^{\infty}c_n t^n$ *and the two are equal.*

Proof: We have seen that if $|\lambda|^2 = t$, then $(1-t)\sum_{n=0}^{\infty}c_n t^n = \mathring{\phi}(\lambda)$ and $\widetilde{\phi}(\lambda) = (1-t)^2\sum_{n=0}^{\infty}(n+1)c_n t^n$ where $c_n = \langle T_\phi e_n, e_n\rangle$. Since $t \in (0,1)$ is real, splitting the coefficients into real and imaginary parts will produce the same splitting of both $\widetilde{\phi}$ and $\mathring{\phi}$; thus there is no loss of generality in assuming that all the c_n's are real.
Let $C = \| T_\phi \| + 1$, then C is finite, and since

$$\lim_{t\to 1}(1-t)\sum_{n=0}^{\infty}Ct^n = \lim_{t\to 1}(1-t)^2\sum_{n=0}^{\infty}(n+1)Ct^n = C,$$

we may replace c_n by $c_n + C$ throughout. Thus, it suffices to consider the case when all $c_n \geq 1$. Then clearly

$$\lim_{t\to 1}\sum_{n=0}^{\infty}c_n t^n = +\infty.$$

Observe that

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty}c_n t^{n+1}\right) = \sum_{n=0}^{\infty}(n+1)c_n t^n \tag{56}$$

and

$$\frac{d}{dt}\left(\frac{t}{1-t}\right) = \frac{1}{(1-t)^2}.$$

Therefore using L'Hospital's rule (56) yields that if

$$\lim_{t\to 1}(1-t)^2\sum_{n=0}^{\infty}(n+1)c_n t^n$$

exists, then so does

$$\lim_{t\to 1}(1-t)\sum_{n=0}^{\infty} c_n t^n$$

and the two are equal. □

Remark 5.1. *Notice that the fixed point $V(t)$ of the operator T satisfies the integral equation*

$$V(t) = (1-t)^2 \int_0^1 \frac{1+ts}{(1-ts)^3} V(s)ds. \tag{57}$$

If $m \in \mathbb{Z}_+$,

$$V\left(\frac{1}{m}\right) = (m-1)^2 \int_0^1 \frac{m+s}{(m-s)^3} V(s)ds.$$

Thus $V(1) = 0, V(0) = \displaystyle\int_0^1 V(s)ds$ and $V(\frac{1}{2}) = \displaystyle\int_0^1 \frac{2+s}{(2-s)^3} V(s)ds$.

If constants are the only solutions of the integral equation (57) then $V'(t) = 0$ for all t and for all $V \in C[0,1) \cap L^1[0,1]$ satisfying the integral equation

$$V(t) = (1-t)^2 \int_0^1 \frac{1+ts}{(1-ts)^3} V(s)ds.$$

Notice that

$$\frac{1+ts}{(1-ts)^3} = \frac{2}{(1-ts)^3} - \frac{1}{(1-ts)^2}.$$

Hence

$$\begin{aligned} \frac{d}{dt}\left[\frac{1+ts}{(1-ts)^3}\right] &= 2\left[\frac{-3(1-ts)^2(-s)}{(1-ts)^6}\right] - \frac{(-2)(1-ts)(-s)}{(1-ts)^4} \\ &= \frac{6s}{(1-ts)^4} - \frac{2s}{(1-ts)^3}. \end{aligned}$$

Thus

$$
\begin{aligned}
V'(t) &= -2(1-t)\int_0^1 \frac{1+ts}{(1-ts)^3}V(s)ds + (1-t)^2\int_0^1 \left[\frac{6s}{(1-ts)^4} - \frac{2s}{(1-ts)^3}\right]V(s)ds\\
&= 2(1-t)\int_0^1 \left[-\left(\frac{1+ts}{(1-ts)^3}\right) + (1-t)s\left(\frac{3}{(1-ts)^4} - \frac{1}{(1-ts)^3}\right)\right]V(s)ds\\
&= 2(1-t)\int_0^1 \left[-\left(\frac{1+ts}{(1-ts)^3}\right) + (1-t)s\left(\frac{3-1+ts}{(1-ts)^4}\right)\right]V(s)ds\\
&= 2(1-t)\int_0^1 \left[-\left(\frac{1+ts}{(1-ts)^3}\right) + (1-t)s\left(\frac{2+ts}{(1-ts)^4}\right)\right]V(s)ds\\
&= 2(1-t)\int_0^1 \left[\frac{-1+t^2s^2+2s+ts^2-2ts-t^2s^2}{(1-ts)^4}\right]V(s)ds\\
&= 2(1-t)\int_0^1 \left(\frac{-1+2s-2ts+ts^2}{(1-ts)^4}\right)V(s)ds\\
&= 2(1-t)\int_0^1 \frac{2(1-ts)-s(1-ts)+3(s-1)}{(1-ts)^4}V(s)ds\\
&= 2(1-t)\int_0^1 \frac{2}{(1-ts)^3}V(s)ds - 2(1-t)\int_0^1 \frac{s}{(1-ts)^3}V(s)ds\\
&+ 2(1-t)\int_0^1 \frac{3s-3}{(1-ts)^4}V(s)ds.
\end{aligned}
$$

Thus $V'(t)=0$ *if and only if*

$$\int_0^1 \frac{2-s}{(1-ts)^3}V(s)ds = \int_0^1 \frac{3(1-s)}{(1-ts)^4}V(s)ds.$$

If $0<t<1, 0<s<1$, *then it is easy to check that*

$$\frac{1+ts}{(1-ts)^3} = \sum_{m=0}^{\infty}(m+1)^2t^ms^m.$$

Thus

$$
\begin{aligned}
(1-x)^2\int_0^1 \frac{1+tx}{(1-tx)^3}dt &= (1-x)^2\int_0^1\left(\sum_{m=0}^{\infty}(m+1)^2t^mx^m\right)dt\\
&= (1-x)^2\sum_{m=0}^{\infty}\left[(m+1)^2x^m\frac{t^{m+1}}{m+1}\right]_0^1\\
&= (1-x)^2\sum_{m=0}^{\infty}(m+1)x^m.
\end{aligned}
$$

Suppose

$$Tg(x) = (1-x)^2 \int_0^1 \frac{1+tx}{(1-tx)^3} g(t)dt = (1-x)^2 \sum_{m=0}^{\infty} (m+1)a_m x^m.$$

In the following theorem, we shall show that if

$$Tg(x) = (1-x)^2 \sum_{m=0}^{\infty} (m+1)a_m x^m$$

and if the sequence $\{m(a_m - a_{m-1})\}$ *is bounded then* $\lim_{x\to 1^-} Tg(x) = 0$ *implies* $\lim_{m\to\infty} a_m = 0.$

Theorem 5.4. *Let* $Tg(x) = (1-x)^2 \sum_{m=0}^{\infty} (m+1)a_m x^m$*. Suppose the sequence* $\{m(a_m - a_{m-1})\}$ *is bounded and* $\lim_{x\to 1^-} Tg(x) = 0$*. Then* $\lim_{m\to\infty} a_m = 0.$

Proof: Let $b_0 = a_0$, let $b_m = (m+1)a_m - ma_{m-1}$, for $m \geq 0$. Now

$$\begin{aligned} 0 = \lim_{x\to 1^-} Tg(x) &= \lim_{x\to 1^-} (1-x)^2 \sum_{m=0}^{\infty} (m+1)a_m x^m \\ &= \lim_{x\to 1^-} (1-x) \left[a_0 + \sum_{m=1}^{\infty} ((m+1)a_m - ma_{m-1})x^m \right] \\ &= \lim_{x\to 1^-} (1-x) \left[a_0 + \sum_{m=1}^{\infty} b_m x^m \right]. \end{aligned}$$

Thus,

$$\lim_{x\to 1^-} (1-x) \sum_{m=0}^{\infty} b_m x^m = 0.$$

It is given that $\{b_m\}$ is bounded since $b_m = m(a_m - a_{m-1}) + a_m$. A result from [12] states that for a sequence $\{c_m\}$ such that

$$\lim_{x\to 1^-} (1-x) \sum_{m=0}^{\infty} c_m x^m = 0$$

and such that $\{c_m\}$ is bounded, we have that

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} c_k = 0.$$

Hence we obtain,

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} b_k = 0.$$

But

$$\begin{aligned} \frac{1}{m+1} \sum_{k=0}^{m} b_k &= \frac{1}{m+1} \left(a_0 + \sum_{k=1}^{m} [(k+1)a_k - k a_{k-1}] \right) \\ &= \frac{1}{m+1} [(m+1)a_m] = a_m \end{aligned}$$

and we have that $\lim_{m\to\infty} a_m = 0$. □

Notice that the condition that $m(a_m - a_{m-1})$ is bounded is a criterion on the oscillation of the sequence $\{a_m\}$, it requires that the rate of oscillation cannot be slower than $\frac{1}{m}$. Notice that as long as the oscillation of a sequence tends to zero, i.e., as long as $a_m - a_{m-1} \longrightarrow 0$, as $m \longrightarrow \infty$, we have that either the sequence converges, or that the sequence has an uncountable set of accumulation points.

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