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DIFFERENCE SEQUENCE SPACES OF $K-$FUNCTIONS *

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Abstract

In this paper we define certain difference sequence spaces via $n-$normed space and a sequence of Orlicz function without convexity. We also make an effort to investigate their structural and some topological properties. Finally, we broaden this idea to double sequences and establish a new matrix theoretic approach for construction of double sequence spaces over $n-$normed spaces.

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1 Introduction and preliminaries

In [8] Gähler introduced an attractive theory of 2-normed spaces. The notion was further generalized by Misiak [24] by introducing n-normed spaces. Also these spaces were studied by Gunawan ([9],[10]) in more detail. In [11] Gunawan and Mashadi gave a simple way to derive an $(n-1)$-norm from the n-norm. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of reals of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

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1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;

2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;

3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$, and

4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called as an n-normed space over the field $\mathbb{R}$.

Example 1.1. *Let $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $||x_1, x_2, \cdots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula*

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$.

Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in an n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in an n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
Throughout the paper we use the standard notation w, l_∞, c and c_0 to denote the set of all, bounded, convergent and null sequences of real numbers respectively. By $\mathbb{N}$ and $\mathbb{R}$ we denote the set of natural numbers and real numbers respectively.

In [27] Orlicz introduced functions called Orlicz functions and constructed the sequence space l_M. Krasnosel'skij and Rutickij further investigated the Orlicz space in [17]. Lindberg [18] initiated the theory of finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c_0 or $l^p(1 \leq p < \infty)$.
Subsequently, Lindenstrauss and Tzafriri ([19], [20], [21]) studied the Orlicz sequence spaces in more detail with an aim to solve many important and interesting structural problems in Banach spaces. For more detail about sequence spaces one may refer to ([26], [30], [31]) and references therein.
An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. see ([17], [27]).

Example 1.2. *Let us consider a function $M(x)$ defined as*

$$M(x) = \begin{cases} 0, & 0 \leq x < 1; \\ x - 1, & x \geq 1. \end{cases}$$

is an Orlicz function as this function is continuous and satisfies all the properties of an Orlicz function.

Lindenstrauss and Tzafriri [19] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \Big\{ x \in \omega : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0 \Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}.$$

Every Orlicz sequence space ℓ_M contains a subspace isomorphic to the classical sequence space $\ell_p (p \geq 1)$. The space $\ell_p, p \geq 1$ is itself an Orlicz sequence space for $M(x) = x^p$.
In ([20], [21]), Lindenstrauss and Tzafriri pointed out a possible generalization of the space ℓ_M to the case when M is an Orlicz function that does not satisfy the convexity condition. Later, Kalton [14] picked up the problem and succeeded in finding many interesting features distinguishing these two theories of sequence spaces. For more details, one can refer to Kamthan and Gupta [16]. A $K-$function is an Orlicz function M which is not convex.

Example 1.3. *For each $\nu \geq \frac{6+2\sqrt{5}}{4}$, the corresponding function $M_\nu : \mathbb{R} \to \mathbb{R}$, where*

$$M_\nu(x) = \begin{cases} |x|^\nu(|\log|x|| + 1), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

is a $K-$ function that satisfies the conditions of Orlicz function but for $1 < \nu < \frac{6+2\sqrt{5}}{4}$, M_ν is not convex on $[0, 1]$.

A K-function M is said to satisfy Δ_2-condition if for each $\alpha > 0$, we have

$$K_{M,\alpha} = \sup_{0<x<\infty} \frac{M(\alpha x)}{M(x)} < \infty.$$

The notion of difference sequence spaces was introduced by Kızmaz [15], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [33], who studied the spaces $l_\infty(\Delta_v)$, $c(\Delta_v)$ and $c_0(\Delta_v)$.
Let m, v be non-negative integers, then for Z a given sequence space Dutta [3] introduced

$$Z(\Delta^m_{(v)}) = \{x = (x_k) \in w : (\Delta^m_{(v)} x_k) \in Z\},$$

where $\Delta^m_{(v)} x_k = (\Delta^m_{(v)} x_k) = (\Delta^{m-1}_{(v)} x_k - \Delta^{m-1}_{(v)} x_{k-v})$ and $\Delta^0_{(v)} x_k = (x_k)$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^m_{(v)} x_k = \sum_{s=0}^{m} (-1)^s \binom{m}{s} x_{k-vs}.$$

Taking $v = 1$, we get the spaces which were introduced and studied by Et and Çolak [7]. Taking $m = v = 1$, we get the spaces which were studied by Kızmaz [15].
We take here $x_{k-vs} = 0$ whenever $k - vs \leq 0$.
Recently, several authors combined the concepts of difference sequences and Orlicz functions to define new classes of sequences and investigated different relevant algebraic and topological properties (see for instance [2], [4], [5], [23]). Now we recall some basic definitions and results which will be useful for our paper. In this paper we consider only real vector spaces.
A vector space X equipped with a topology τ is called a topological vector space (TVS) if the operations $(x, y) \mapsto x + y$ from $X \times X \to X$ and $(\alpha, x) \mapsto \alpha x$ from $\mathbb{R} \times X \to X$ are continuous, where $X \times X$ and $\mathbb{R} \times X$ are equipped

with their usual product topologies and R with the usual metric topology. A topology τ on X such that (X, τ) becomes a TVS is referred to as a linear or vector topology on X. For more information about TVS see [32]. Recall that a subset U of a vector space X is absorbing if for each $x \in X$ there is $\lambda > 0$ such that $x \in \alpha U$ for all $\alpha \in \mathbb{R}$ with $|\alpha| > \lambda$ and U is called balanced if $\alpha U \subset U$ for each α with $|\alpha| \leq 1$.

Lemma 1.1. *[32] A vector space X equipped with a topology τ is a TVS if and only if there exists a local base β at the zero element 0 of X consisting of subsets of X such that*
(a) *Each U in β is absorbing and balanced;*
(b) *For each $U \in \beta$ there is a $V \in \beta$ with $V + V \subset U$.*

A TVS (X, τ) with $\tau \equiv \tau_q$, the topology generated by a norm q on X is called an F^*-space, and if in addition (X, τ_q) is complete, X is called an $F-$space.

Example 1.4. *The space $L^P([0,1])$ with $P < 1$ is not a locally convex. It is a $F-$ space.*

A sequence space X with a linear topology is called a $K-$space provided each of the maps $\pi_i : X \to \mathbb{R}$, $\pi_i(x) = x_i$ is continuous, $i \geq 1$. It is known that a sequence space X equipped with a linear topology is a $K-$space if and only if the identity map $I : X \to w$ is continuous, where w is endowed with the topology of pointwise convergence.
A $K-$space X is called a *Fréchet $K-$space* provided X is an $F-$space. For every absorbing and balanced set U of a vector space X, the function $p \equiv p_U : X \to \mathbb{R}^+$ defined by $p_U(x) = \inf\{\alpha : \alpha > 0, x \in \alpha U\}$ is called a Minkowski functional or the gauge associated with U. The function p_U associated with an absorbing and a balanced set U is also called a pseudonorm on X.

Lemma 1.2. *Every pseudonorm function p on X gives rise to a unique linear topology τ_p on X. Conversely, to every linear topology τ on X there corresponds a pseudonorm function p on X such that τ is equivalent to τ_p.*

2 Spaces of single $n-$normed difference sequences

In this section we define the space $l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$ and scrutinize its structural properties.

Let $\mathbf{M} = (M_k)$ be a sequence of $K-$functions, $p = (p_k)$ be a bounded sequence of positive real numbers, $u = (u_k)$ be a sequence of positive real numbers and m, v be a non-negative integers. By $w(X-n)$ denotes $X-$valued sequence spaces. Then we introduce the following difference sequence spaces:

$$l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|) =$$

$$\Big\{x \in w(X-n) : \sum_{k=1}^{\infty} M_k\Big(\|\frac{u_k \Delta^m_{(v)} x_k}{\rho}, z_1, \cdots, z_{n-1}\|\Big)^{p_k} < \infty, \text{ for some } \rho > 0$$

$$\text{and } z_1, \cdots, z_{n-1} \in X\Big\}.$$

By taking $m = 0$ and $M_k = M$ for all $k \geq 1$, we get the eminent space [14]. Again if $p = (p_k) = 1$, $u = (u_k) = 1$ and $n-$normed space is replaced by normed space then we get renowned space defined by Dutta and Kočinac [6].

Theorem 2.1. *Let $\mathbf{M} = (M_k)$ be a sequence of $K-$functions, $p = (p_k)$ be a bounded sequence of positive real numbers, $u = (u_k)$ be a sequence of positive real numbers. Then the space $l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$ is a linear space.*

Proof. Let $x = (x_k)$ and $y = (y_k)$ be an arbitrary sequence in $l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$. Then for some $\rho_1, \rho_2 > 0$, we have

$$\sum_{k=1}^{\infty} M_k\Big(\|\frac{u_k \Delta^m_{(v)} x_k}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)^{p_k} < \infty$$

and

$$\sum_{k=1}^{\infty} M_k\Big(\|\frac{u_k \Delta^m_{(v)} y_k}{\rho_2}, z_1, \cdots, z_{n-1}\|\Big)^{p_k} < \infty.$$

Let $\rho = 2\max\{\rho_1, \rho_2\}$. One can suppose that there is a partition of $\mathbb{N}$ into two disjoint sets N_1 and N_2, at least one of which is infinite, such that $\|u_k \Delta^m_{(v)} x_k\| \leq \|u_k \Delta^m_{(v)} y_k\|$ for all $k \in N_1$ and $\|u_k \Delta^m_{(v)} x_k\| \leq \|u_k \Delta^m_{(v)} y_k\|$ for all $k \in N_2$. Since the operator $\Delta^m_{(v)}$ is linear and each M_k is non-decreasing, we have

$$\sum_{k\in N_1} M_k\Big(\big\|\frac{u_k\Delta^m_{(v)}(x_k+y_k)}{\rho}, z_1, \cdots, z_{n-1}\big\|\Big)^{p_k}$$
$$\leq \sum_{k\in N_1} M_k\Big(\big\|\frac{2u_k\Delta^m_{(v)}y_k}{\rho}, z_1, \cdots, z_{n-1}\big\|\Big)^{p_k}$$
$$\leq \sum_{k=1}^{\infty} M_k\Big(\big\|\frac{2u_k\Delta^m_{(v)}y_k}{\rho}, z_1, \cdots, z_{n-1}\big\|\Big)^{p_k}$$

and

$$\sum_{k\in N_2} M_k\Big(\big\|\frac{u_k\Delta^m_{(v)}(x_k+y_k)}{\rho}, z_1, \cdots, z_{n-1}\big\|\Big)^{p_k}$$
$$\leq \sum_{k\in N_2} M_k\Big(\big\|\frac{2u_k\Delta^m_{(v)}x_k}{\rho}, z_1, \cdots, z_{n-1}\big\|\Big)^{p_k}$$
$$\leq \sum_{k=1}^{\infty} M_k\Big(\big\|\frac{2u_k\Delta^m_{(v)}x_k}{\rho}, z_1, \cdots, z_{n-1}\big\|\Big)^{p_k}.$$

Therefore, we have

$$\sum_{k=1}^{\infty} M_k\Big(\big\|\frac{u_k\Delta^m_{(v)}(x_k+y_k)}{\rho}, z_1, \cdots, z_{n-1}\big\|\Big)^{p_k}$$
$$\leq \sum_{k=1}^{\infty} M_k\Big(\big\|\frac{2u_k\Delta^m_{(v)}x_k}{\rho}, z_1, \cdots, z_{n-1}\big\|\Big)^{p_k}$$
$$+ \sum_{k=1}^{\infty} M_k\Big(\big\|\frac{2u_k\Delta^m_{(v)}y_k}{\rho}, z_1, \cdots, z_{n-1}\big\|\Big)^{p_k}$$
$$< \infty$$

which gives $x+y \in l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$. Next, for any scalar α we can find $j \in \mathbb{N}$ so that $\frac{|\alpha|}{2^j} < \frac{1}{\rho_1}$. Since $M_k, k \in \mathbb{N}$ are non-decreasing functions, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} M_k\Big(|\alpha|\Big\|\frac{u_k\Delta^m_{(v)}(\alpha x_k)}{2^j}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k}\\
&= \sum_{k=1}^{\infty} M_k\Big(|\alpha|\Big\|\frac{u_k\Delta^m_{(v)}x_k}{2^j}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k}\\
&\leq \sum_{k=1}^{\infty} M_k\Big(|\alpha|\Big\|\frac{u_k\Delta^m_{(v)}y_k}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k}\\
&< \infty
\end{aligned}$$

which means that $\alpha x \in l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$. This completes the proof. □

Theorem 2.2. $l^{\mathbf{M}}(\Delta^i_{(v)}, p, u, \|\cdot, \cdots, \cdot\|) \subset l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$, $i = 1, 2, ..., m-1$.

Proof. The proof is easy so we omit it. □

3 Some Topological Properties of Space $l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$

In this section we intend to define a linear topology on $l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$. Before defining it we prove some other results which are useful to introduce this topology.

$$B_{\mathbf{M}}(\epsilon)(p, u, \|\cdot, \cdots, \cdot\|) =$$

$$\Big\{x \in w(X{-}n) : \sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{u_k\Delta^m_{(v)}x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k} < \infty, \text{ for some } \rho > 0 \text{ and } z_1, \cdots, z_{n-1} \in X\Big\}.$$

and

$$\beta_{\mathbf{M}}(p, u, \|\cdot, \cdots, \cdot\|) = \{\rho B_{\mathbf{M}}(\epsilon)(p, u, \|\cdot, \cdots, \cdot\|) : \rho, \epsilon > 0\}.$$

Clearly each element in $\beta_{\mathbf{M}}(p, u, \|\cdot, \cdots, \cdot\|)$ contains the zero sequence $0-$ the origin of $l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$.

Theorem 3.1. *The family* $\beta_{\mathbf{M}}(p,u,\|\cdot,\cdots,\cdot\|)$ *satisfies the following properties:* (i) *If* $x\in l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$*, then for each member* $\rho B_{\mathbf{M}}(\epsilon)(p,u,\|\cdot,\cdots,\cdot\|)$ *of* $\beta_{\mathbf{M}}(p,u,\|\cdot,\cdots,\cdot\|)$ *we have*
$x\in \lambda_0\rho B_{\mathbf{M}}(\epsilon)(p,u,\|\cdot,\cdots,\cdot\|)$*, for some* $\lambda_0>0$ *and thus for all* $\lambda\in\mathbb{R}$ *with* $\lambda\geq\lambda_0$.
(ii) *For each element* $U=\rho B_{\mathbf{M}}(\epsilon)(p,u,\|\cdot,\cdots,\cdot\|)$ *in* $\beta_{\mathbf{M}}(p,u,\|\cdot,\cdots,\cdot\|)$ *and each* $\lambda\in(0,1], \lambda U\subset U$,
(iii) $\frac{\rho}{2}B_{\mathbf{M}}(\frac{\epsilon}{2})(p,u,\|\cdot,\cdots,\cdot\|)+\frac{\rho}{2}B_{\mathbf{M}}(\frac{\epsilon}{2})(p,u,\|\cdot,\cdots,\cdot\|)$
$\subset\rho B_{\mathbf{M}}(\epsilon)(p,u,\|\cdot,\cdots,\cdot\|)$,
(iv) $\cap\{U:U\in\beta_{\mathbf{M}}(p,u,\|\cdot,\cdots,\cdot\|)\}=\{0\}$.

Proof. (i) Let $x\in l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$. Then we can find $\gamma>0$ with

$$\sum_{k=1}^{\infty}M_k\Big(\|\frac{u_k\Delta^m_{(v)}x_k}{\gamma\rho},z_1,\cdots,z_{n-1}\|\Big)^{p_k}<\infty.$$

Hence, there is $j\in\mathbb{N}$ such that

$$\sum_{k=j+1}^{\infty}M_k\Big(\|\frac{u_k\Delta^m_{(v)}x_k}{\gamma\rho},z_1,\cdots,z_{n-1}\|\Big)^{p_k}<\frac{\epsilon}{2}.$$

There are also positive numbers $\gamma_1,\gamma_2,....,\gamma_j$ such that

$$M_1\Big(\|\frac{u_k\Delta^m_{(v)}x_1}{\gamma_1\gamma\rho},z_1,\cdots,z_{n-1}\|\Big)^{p_k}<\frac{\epsilon}{2^2},$$

$$M_2\Big(\|\frac{u_k\Delta^m_{(v)}x_2}{\gamma_2\gamma\rho},z_1,\cdots,z_{n-1}\|\Big)^{p_k}<\frac{\epsilon}{2^3},$$

....

$$M_j\Big(\|\frac{u_k\Delta^m_{(v)}x_j}{\gamma_j\gamma\rho},z_1,\cdots,z_{n-1}\|\Big)^{p_k}<\frac{\epsilon}{2^{j+1}}.$$

If $\lambda_0=\max\{\gamma,\gamma_1\gamma,....,\gamma_j\gamma\}$, then for all λ with $\lambda\geq\lambda_0$ we have

$$\begin{aligned}\sum_{k=1}^{\infty}M_k\Big(\|\frac{u_k\Delta^m_{(v)}x_k}{\lambda\rho},z_1,\cdots,z_{n-1}\|\Big)^{p_k}&\leq\sum_{k=1}^{j}M_k\Big(\|\frac{u_k\Delta^m_{(v)}x_k}{\gamma_k\gamma\rho},z_1,\cdots,z_{n-1}\|\Big)^{p_k}\\&+\sum_{k=j+1}^{\infty}M_k\Big(\|\frac{u_k\Delta^m_{(v)}x_k}{\gamma\rho},z_1,\cdots,z_{n-1}\|\Big)^{p_k}\\&<\frac{\epsilon}{2}+\frac{\epsilon}{2}\\&=\epsilon.\end{aligned}$$

Hence $x \in \lambda\rho B_{\mathbf{M}}(\epsilon)(p, u, \|\cdot, \cdots, \cdot\|)$.
(ii) Let $\lambda \in (0, 1]$ and $(x_k) \in \lambda U$, i.e. let

$$\sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{u_k \Delta^m_{(v)} x_k}{\lambda\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k} < \epsilon$$

be satisfied. Then because of $|\lambda|\rho \leq \rho$ we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{u_k \Delta^m_{(v)} x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k} \\
&\leq \sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{u_k \Delta^m_{(v)} x_k}{\lambda\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k} < \epsilon,
\end{aligned}$$

i.e. $(x_k) \in \rho B_{\mathbf{M}}(\epsilon)(p, u, \|\cdot, \cdots, \cdot\|) = U$.
(iii) Let $x, y \in \frac{\rho}{2} B_{\mathbf{M}}\big(\frac{\epsilon}{2}\big)(p, u, \|\cdot, \cdots, \cdot\|)$. Then

$$\begin{aligned}
&\sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{u_k \Delta^m_{(v)} (x_k + y_k)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k} \\
&\leq \sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{2u_k \Delta^m_{(v)} x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k} \\
&\leq \sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{2u_k \Delta^m_{(v)} y_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore, $x + y \in \rho B_{\mathbf{M}}(\epsilon)(p, u, \|\cdot, \cdots, \cdot\|)$.
(iv) It is evident. □

From the preceding Theorems and Lemma 1.2 one obtains the following

Corollary 3.1. $(l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|), \tau_{\mathbf{M}})$ *is a Hausdroff topological vector space, where the linear topology* $\tau_{\mathbf{M}}$ *on* $l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$ *is generated by* $\beta_{\mathbf{M}}(p, u, \|\cdot, \cdots, \cdot\|)$*, for simplicity we shall denote* $\tau_{\mathbf{M}}(p, u, \|\cdot, \cdots, \cdot\|)$ *by* $\tau_{\mathbf{M}}$.

In fact we have

Theorem 3.2. $(l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|), \tau_{\mathbf{M}})$ *is a metrizable topological vector space.*

Proof. Consider the family

$$\beta' = \{\rho B(\epsilon)(p,u,\|\cdot,\cdots,\cdot\|) : \rho,\epsilon > 0 \text{ and } \rho,\epsilon \text{ are rational numbers }\}$$
$$\subset \beta_{\mathbf{M}}(p,u,\|\cdot,\cdots,\cdot\|).$$

This family of neighbourhoods of 0 is countable and generates the same topology $\tau_{\mathbf{M}}$ on $l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$. Therefore, $\tau_{\mathbf{M}}$ is a metrizable topology. From the $K-$character of $\tau_{\mathbf{M}}$, monotonicity and continuity of K-functions $M_k, k\in\mathbb{N}$, it follows $(l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|),\tau_{\mathbf{M}})$ is a $K-$space and hence complete.

□

Now we have the following propositions.

Proposition 3.1. $(l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|),\tau_{\mathbf{M}})$ *is a Fréchet $K-$ space.*

By imposing the Δ_2- condition, on each $K-$ functions M_k, we show that the Fréchet space $l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$ becomes an $AK-$space (see [23]). In this connection we define the following space:
$h^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|) =$

$$\Big\{\mathbf{x}\in w(X{-}n) : \sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{u_k\Delta^m_{(v)}x_k}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)^{p_k} < \infty, \text{ for all } \rho>0\Big\}.$$

Clearly, $h^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$ is a subspace of $l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$.

Proposition 3.2. $h^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$ *is an $AK-$space*

Proof. Let $\mathbf{x}=(x_k)\in h^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$ and $\epsilon>0$ be arbitrary chosen. Then

$$\sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{u_k\Delta^m_{(v)}x_k}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)^{p_k} < \infty, \text{ for every } \rho>0.$$

Hence, we can find an integer s_0 such that

$$\sum_{k=s+1}^{\infty} M_k\Big(\Big\|\frac{u_k\Delta^m_{(v)}x_k}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)^{p_k} \le \epsilon, \text{ for all } s\ge s_0.$$

It implies that $x^{[s]}-\mathbf{x}\in \rho B_{\mathbf{M}}(\epsilon)(p,u,\|\cdot,\cdots,\cdot\|)$. Here $x^{[s]}$ denotes the s-section of $\mathbf{x}$ that is, $x^{[s]}=\sum_{k=1}^{\infty} x_k e^{(k)},\ e^{(k)}_k=1,\ e^{(k)}_t=0$ for $t\neq k$. Since ρ and $\epsilon>0$ was arbitrary, it follows that $x^{(s)}\to\mathbf{x}$ in the topology $\tau_{\mathbf{M}}$. □

Proposition 3.3. *If each $K-$function of the sequence $\mathbf{M} = (M_k)$ satisfies the Δ_2-condition, then $h^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|) = l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$.*

Proof. Let $\mathbf{x} = (x_k) \in l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$. Then

$$\sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{u_k\Delta^m_{(v)}x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k} < \infty,$$

for some $\rho > 0$. Let us choose an arbitrary $r > 0$. Then

$$\begin{aligned}&\sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{u_k\Delta^m_{(v)}x_k}{r}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k}\\ &= \sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{u_k\Delta^m_{(v)}x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k} \frac{M_k(\frac{\rho y_k}{r})}{M_k(y_k)},\end{aligned}$$

where

$$y_k = \Big\|\frac{u_k\Delta^m_{(v)}x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|.$$

Since each M_k satisfies the Δ_2-condition we have

$$\begin{aligned}&\sum_{k=1}^{\infty} M_k\Big(\Big\|\frac{u_k\Delta^m_{(v)}x_k}{r}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k}\\ &\leq \sum_{k=1}^{\infty} K_{M_k}, \frac{\rho}{r} M_k\Big(\Big\|\frac{u_k\Delta^m_{(v)}x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k}.\end{aligned}$$

Thus, $x \in h^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$ and so

$$h^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|) = l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|).$$

□

Combining Propositions 3.4, 3.5 and 3.6 we get the most predictable result in the following proposition.

Proposition 3.4. *If each $K-$function of the sequence $\mathbf{M} = (M_k)$ satisfies the Δ_2-condition, then $l^{\mathbf{M}}(\Delta^m_{(v)}, p, u, \|\cdot, \cdots, \cdot\|)$ is an $AK-$space.*

Definition of pseudonorm and Lemma 1.3 support us to talk about $\tau_{\mathbf{M}}$ in terms of pseudonorms which generate this topology. For each ρ and $\epsilon > 0$, let us define

$$P_{\rho,\epsilon}(\mathbf{x})(p, u, \|\cdot, \cdots, \cdot\|) = \inf\{\alpha > 0, \mathbf{x} \in \alpha\rho B_{\mathbf{M}}(\epsilon)(p, u, \|\cdot, \cdots, \cdot\|)\}.$$

Clearly, $P_{\rho,\epsilon}(\lambda\mathbf{x})(p,u,\|\cdot,\cdots,\cdot\|) = |\lambda| P_{\rho,\epsilon}(\mathbf{x})(p,u,\|\cdot,\cdots,\cdot\|)$ and $P_{\rho,\epsilon}(\mathbf{x}+\mathbf{y})(p,u,\|\cdot,\cdots,\cdot\|) \leq P_{\frac{\rho}{2},\frac{\epsilon}{2}}(\mathbf{x})(p,u,\|\cdot,\cdots,\cdot\|) + P_{\frac{\rho}{2},\frac{\epsilon}{2}}(\mathbf{y})(p,u,\|\cdot,\cdots,\cdot\|)$ for all $\mathbf{x},\mathbf{y} \in l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$ and $\lambda \in \mathbb{R}$.
Hence, we have the following proposition.

Proposition 3.5. *The family* $\{P_{\rho,\epsilon}(\lambda\mathbf{x})(p,u,\|\cdot,\cdots,\cdot\|) : \rho,\epsilon > 0\}$ *of pseudonorms on* $l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$ *generates the topology* $\tau_{\mathbf{M}}$.

Next suppose each $K-$function of the sequence $\mathbf{M} = (\mathbf{M_k})$ satisfies the Δ_2-condition and let us define the function $P_\epsilon(p,u,\|\cdot,\cdots,\cdot\|)$ on $l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$ as follows

$$P_\epsilon(p,u,\|\cdot,\cdots,\cdot\|) = \inf\Big\{\alpha > 0 : \sum_{k=1}^{\infty} M_k\Big(\|\frac{u_k\Delta^m_{(v)}x_k}{\alpha}, z_1,\cdots,z_{n-1}\|\Big)^{p_k} \leq \epsilon,\Big\}.$$

Then $P_\epsilon(p,u,\|\cdot,\cdots,\cdot\|)$ is a pseudonorm on $l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$.
For the next results we shall assume that each $K-$function of the sequence $\mathbf{M} = (M_k)$ satisfies the Δ_2-condition.

Proposition 3.6. *The family* $\{P_{\rho,\epsilon}(\lambda\mathbf{x})(p,u,\|\cdot,\cdots,\cdot\|) : \rho,\epsilon > 0\}$ *of pseudonorms on* $l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$ *generates the topology* $\sigma_{\mathbf{M}}(p,u,\|\cdot,\cdots,\cdot\|)$ *on* $l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$.

Proposition 3.7. *If* $\mathbf{x} \in l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$, *then*

$$P_{\rho,\epsilon}(\mathbf{x})(p,u,\|\cdot,\cdots,\cdot\|) = \frac{1}{\rho}P_\epsilon(\mathbf{x})(p,u,\|\cdot,\cdots,\cdot\|)$$

for each $\rho > 0$ *and* $\epsilon > 0$.

Proof. Let $\mathbf{x} = (x_k) \in l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$. Then

$$\begin{aligned}
P_{\rho,\epsilon}(\mathbf{x})(p,u,\|\cdot,\cdots,\cdot\|) &= \inf\{\alpha > 0, \mathbf{x} \in \alpha\rho B_{\mathbf{M}}(\epsilon)(p,u,\|\cdot,\cdots,\cdot\|)\} \\
&= \frac{1}{\rho}\inf\Big\{\alpha\rho > 0 : \sum_{k=1}^{\infty} M_k\Big(\|\frac{u_k\Delta^m_{(v)}x_k}{\alpha\rho}, z_1,\cdots,z_{n-1}\|\Big)^{p_k} \leq \epsilon,\Big\} \\
&= \frac{1}{\rho}\inf\Big\{r > 0 : \sum_{k=1}^{\infty} M_k\Big(\|\frac{u_k\Delta^m_{(v)}x_k}{r}, z_1,\cdots,z_{n-1}\|\Big)^{p_k} \leq \epsilon,\Big\} \\
&= \frac{1}{\rho}P_\epsilon(\mathbf{x})(p,u,\|\cdot,\cdots,\cdot\|).
\end{aligned}$$

Thus, $P_{\rho,\epsilon}(\mathbf{x})(p,u,\|\cdot,\cdots,\cdot\|) = \frac{1}{\rho}P_\epsilon(\mathbf{x})(p,u,\|\cdot,\cdots,\cdot\|)$ for each $\mathbf{x} \in l^{\mathbf{M}}(\Delta^m_{(v)},p,u,\|\cdot,\cdots,\cdot\|)$. □

Hence, we have the following proposition.

Proposition 3.8. *The topologies* $\tau_{\mathbf{M}}$ *and* $\sigma_{\mathbf{M}}(p, u, \|\cdot, \cdots, \cdot\|)$ *are equivalent.*

4 Some Spaces of double sequences

Some initial results on double sequences can be found in ([1], [12], [13], [29]). For other useful results on double sequences, one may refer to Moricz and Rhoades [25].

A double real sequence $\mathbf{x} : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is usually denoted by $\mathbf{x} = (x_{mv})$ and expressed as an infinite matrix. In 1900, Pringsheim [28] introduced the concept of convergence of real double sequences: a double sequence $\mathbf{x} = (x_{mv})$ converges to $L \in \mathbb{R}$, denoted by $P - \lim x = L$ or $P - \lim x_{mv} = L$, if for every $\epsilon > 0$ there is $N_0 \in \mathbb{N}$ such that $|x_{mv} - L| < \epsilon$ for all $m, v > N_0$. The limit L is called the Pringsheim limit of $\mathbf{x}$. The notion of regular convergence of double sequence was introduced by Hardy [12] as follows. A double sequence $\mathbf{x} = (x_{mv})$ is said to converge regularly if it converges in the Pringsheim's sense and the following limits exist:
$\lim\limits_{m\to\infty} x_{mv}$ for each $n \in \mathbb{N}$ and $\lim\limits_{v\to\infty} x_{mv}$ for each $v \in \mathbb{N}$.

Example 4.1. *Consider a double sequence space* $\mathbf{x} = (x_{mv})$ *and is defined as*

$$\mathbf{x} = (x_{mv}) = \begin{cases} m, & \textit{if } v = 3; \\ v, & \textit{if } m = 5; \\ 8, & \textit{otherwise.} \end{cases}$$

Then $(x_{mv}) \to 8$ *in Pringsheim's sense (because* $P - \lim x_{mv} = 8$ *if for every* $\epsilon > 0$ *there is* $N_0 \in \mathbb{N}$ *such that* $|x_{mv} - 8| < \epsilon$ *for all* $m, v > N_0$*. The limit 8 is called the Pringsheim limit of* $\mathbf{x} = (x_{mv})$*)). But not bounded as well as not regularly convergent.*

Example 4.2. *Let* $\mathbf{x} = (x_{mv}) = 1$ *for all* $m, v \in \mathbb{N}$*. Then* (x_{mv}) *is convergent in Pringsheim's sense, bounded and regularly convergent.*

In [6] Dutta and Koćinac introduced the notion of $OK-$space of double sequences ${}_2l^{\mathbf{M}}$. Also present an idea how to use the difference operator to double sequences in order to introduce the spaces of double difference sequences extended by K-functions and give an alternative definition of the spaces $Z(\Delta^s_{(r)})$ of difference sequences. In the end they define the $OK-$spaces of double difference sequences ${}_2l^{\mathbf{M}}(\Delta^s_{(r)})$. The first order difference operator Δ can be expressed as an infinite triangular matrix

$$\Delta = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & ... \\ 0 & 1 & -1 & 0 & 0 & ... \\ 0 & 0 & 1 & -1 & 0 & ... \\ .. & .. & .. & .. & .. & ... \\ .. & .. & .. & .. & .. & ... \end{pmatrix}.$$

Let $\Delta_{(1)}$ denote the additive inverse of Δ, i.e. $\Delta + \Delta_{(1)} = 0$, the zero infinite matrix.
Define inductively $\Delta^2 = \Delta.\Delta$, $\Delta^2_{(1)} = -\Delta^2; ...; \Delta^n = \Delta.\Delta^{n-1}$, $\Delta^n_{(1)} = -\Delta^n$.
Next, Δ^2 can be considered as

$$\Delta^2 = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & ... \\ 0 & 1 & 0 & -1 & 0 & ... \\ 0 & 0 & 1 & 0 & -1 & ... \\ .. & .. & .. & .. & .. & ... \\ .. & .. & .. & .. & .. & ... \end{pmatrix},$$

and $\Delta^{(2)}$ as the additive inverse of Δ^2. Similarly, we can have $\Delta^{(r)}$ and Δ^r for each $r \geq 2$. Hence we can define $\Delta^s_{(r)}$ as

$$\Delta^s_{(r)} = \Delta_{(r)}.\Delta^{s-1}_{(r)}.$$

Now we can give another definition of the space $Z(\Delta^s_{(r)}, p, u, \|\cdot, \cdots, \cdot\|)$ of difference sequences as follows:

$$Z(\Delta^s_{(r)}, p, u, \|\cdot, \cdots, \cdot\|) = \{(x_k) : (A_i X) \in Z(p, u, \|\cdot, \cdots, \cdot\|)\}$$

where

$$X = [x_1 x_2 ... x_n ...]^\tau, \Delta^s_{(r)} = A = (a_{ik})$$

and

$$A_i X = \sum_{k=1}^{\infty} a_{ik} x_k, \text{ for each } i \geq 1.$$

This approach is very useful to construct new difference double sequences.
Let a double sequence $\mathbf{a} = (a_{mv})$ be expressed as an infinite matrix

$$(a_{mv}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1n} & ... \\ a_{21} & a_{22} & a_{23} & ... & a_{2n} & ... \\ a_{31} & a_{32} & a_{33} & ... & a_{3n} & ... \\ .. & .. & .. & .. & .. & ... \\ .. & .. & .. & .. & .. & ... \end{pmatrix}.$$

Now we define the set ${}_2Z(\Delta, p, u, \|\cdot, \cdots, \cdot\|)$ of double difference sequences as follows:

$$ {}_2Z(\Delta, p, u, \|\cdot, \cdots, \cdot\|) = \{\mathbf{a} = (a_{mv}) : (\Delta\mathbf{a}) \in {}_2Z(p, u, \|\cdot, \cdots, \cdot\|)\}. $$

$$ \begin{aligned} \Delta\mathbf{a} &= \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & -1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & -1 & 0 & \ldots \\ .. & .. & .. & .. & .. & \ldots \\ .. & .. & .. & .. & .. & \ldots \end{pmatrix} ; \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & \ldots \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & \ldots \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} & \ldots \\ .. & .. & .. & .. & .. & \ldots \\ .. & .. & .. & .. & .. & \ldots \end{pmatrix} \\ &= \begin{pmatrix} a_{11} - a_{21} & a_{12} - a_{22} & \ldots & a_{1n} - a_{2n} & \ldots \\ a_{21} - a_{31} & a_{22} - a_{32} & \ldots & a_{2n} - a_{3n} & \ldots \\ a_{31} - a_{41} & a_{32} - a_{42} & \ldots & a_{3n} - a_{4n} & \ldots \\ .. & .. & .. & .. & \ldots \\ .. & .. & .. & .. & \ldots \end{pmatrix}. \end{aligned} $$

Now we define

$$ \begin{aligned} {}_2Z(\Delta^s_{(r)}, p, u, \|\cdot, \cdots, \cdot\|) &= \{\mathbf{a} = (a_{mv}) : B\mathbf{a} \in {}_2Z(p, u, \|\cdot, \cdots, \cdot\|)\} \\ &= \{\mathbf{a} = (a_{mv}) : (c_{kv}) \in {}_2Z(p, u, \|\cdot, \cdots, \cdot\|)\}, \end{aligned} $$

where

$$ B = (b_{vk}) = \Delta^s_{(r)} \text{ and } B\mathbf{a} = C = (c_{kv}) $$

with

$$ c_{kv} = \sum_{k=1}^{\infty} b_{km} a_{mv}, \text{ for each } k, v \in \mathbb{N}. $$

For a $K-$function M, in view of the above interpretations, we define the $OK-$ spaces of double difference sequences as follows:

$$ \begin{aligned} & {}_2l^{\mathbf{M}}(\Delta^s_{(r)}, p, u, \|\cdot, \cdots, \cdot\|) \\ &= \Big\{\mathbf{a} = (a_{mv}) \in {}_2w(X-n) : (\Delta^s_{(r)} a_{mv}) \in {}_2l^{\mathbf{M}}(p, u, \|\cdot, \cdots, \cdot\|)\Big\} \\ &= \Big\{(a_{mv}) \in {}_2w(X-n) : \sum_{m=1}^{\infty}\sum_{v=1}^{\infty} M_k\Big(\Big\|\frac{u_k \Delta^s_{(r)} a_{mv}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k} < \infty, \\ &\quad \text{for some } \rho > 0\} \\ &= \Big\{(c_{kv}) \in {}_2w(X-n) : \sum_{k=1}^{\infty}\sum_{v=1}^{\infty} M_k\Big(\Big\|\frac{u_k c_{kv}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_k} < \infty, \\ &\quad \text{for some } \rho > 0\}. \end{aligned} $$

One can investigate topologies on ${}_2l^{\mathbf{M}}(\Delta^s_{(r)}, p, u, \|\cdot, \cdots, \cdot\|)$ and hence prove that the spaces ${}_2l^{\mathbf{M}}(p, u, \|\cdot, \cdots, \cdot\|)$ and ${}_2l^{\mathbf{M}}(\Delta^s_{(r)}, p, u, \|\cdot, \cdots, \cdot\|)$ are topologically equivalent, for all $r, s \in \mathbb{N}$.

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