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ON SOME PERTURBATION BOUNDS FOR A MATRIX EQUATION FROM INTERPOLATION PROBLEMS [*][†]

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Abstract

Some existing perturbation bounds for a unique positive definite solution of a nonlinear matrix equation connected to the interpolation theory are analyzed and compared. We examine the behavior of the perturbation bounds, considered in five sources, through experiments with five numerical examples.

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keywords: nonlinear matrix equation, perturbation bounds.

1 Introduction

Throughout this paper, $\mathcal{C}^{p\times q}$ denotes the set of $p\times q$ complex matrices, and $\mathcal{H}^n$ the set of $n\times n$ Hermitian matrices. A^* stands conjugate transpose of a matrix A, $A>0$ ($A\geq 0$) means that A is a Hermitian positive definite (semidefinite) matrix. If $A-B>0$ (or $A-B\geq 0$) we write $A>B$ (or $A\geq B$). I (or I_n) is the identity matrix of order n. The symbols $\|\cdot\|$, $\|\cdot\|_F$

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and $\rho(\cdot)$ denote the spectral norm, the Frobenius norm, and the spectral radius, respectively. For the matrices $A = (a_{ij})$ and B, $A \otimes B = (a_{ij}B)$ is a Kronecker product. Finally, for a matrix Z, we denote with $\widehat{Z}$ the $m \times m$ block diagonal matrix with Z on its diagonal, i.e. $\widehat{Z} = I_m \otimes Z$.

Consider the nonlinear matrix equation

$$X - A^* \widehat{X}^{-1} A = Q\,, \tag{1}$$

where $A \in \mathcal{C}^{mn \times n}$, $Q \in \mathcal{H}^n$, and $Q > 0$.

Eq. (1) is a special case of the equation

$$X - A^* (\widehat{X} - C)^{-1} A = Q\,, \tag{2}$$

where $C \in \mathcal{H}^{mn}$ and $C \geq 0$. Eq. (2) is connected with certain interpolation problems [1, 2, Chapter 7]. Ran and Reurings [1] have proved that, under the condition $\widehat{Q} > C$, Eq. (2) has a unique positive definite solution X_+, satisfying $\widehat{X_+} > C$. Hence, Eq. (1) has a unique positive definite solution. Moreover, Liu and Zhang [3, Lemma 2.1] have proved that, under the condition $\widehat{Q} > C$, Eq. (2) is equivalent to an equation in the form of Eq. (1) and used the Newton's method for solving of Eq. (1).

In this paper, the unique positive definite solution of a matrix equation we will call *maximal* solution.

Let

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix} \in \mathcal{C}^{mn \times n}, \quad A_i \in \mathcal{C}^{n \times n}, \quad i = 1, 2, \ldots, m.$$

Then Eq. (1) can be written as

$$X - \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q\,. \tag{3}$$

Duan et al. [4] have investigated Eq. (3) with $Q = I$, using the Thompson metric they proved that the matrix equation always has a unique positive definite (maximal) solution. Moreover, in [4] on the matrix differentiation have been given a perturbation bound for the maximal solution. Sun [5] have obtained perturbation bounds, condition numbers for the maximal solution of Eq. (2), and residual bounds for an approximate solution to the maximal solution. In [6] two perturbation bounds and an explicit expression of the condition number for the maximal solution of Eq. (3) have been obtained. Residual bound for an approximate solution, calculated by an iterative algorithm have been derived in [7]. In [8, 9] some modifications of a perturbation bound from [6] are derived.

Eq. (1) (or (3)) for $m = 1$ arises in the analysis of stationary Gaussian reciprocal processes over a finite interval [10, 11]. It has been investigated for the existence a positive definite solution in [10, 12, 13] and it has been executed the perturbation analysis in [14, 15, 16, 17].

In addition, there are many contributions in the literature to the solvability, numerical solutions, and perturbation analysis for the matrix equations $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q$ [18, 19, 20, 21], $X - \sum_{i=1}^{m} A_i^* \mathcal{F}(X) A_i = Q$ [22], $A_0 + \sum_{i=1}^{k} \sigma_i A_i^* X^{p_i} A_i = 0$ [23, 24], $X \pm A^* X^{-n} A = Q$ [25].

Motivated by investigation of Popchev and Angelova in [17], in this paper we analyze and compare the effectiveness and the accuracy of the existing perturbation bounds for the maximal solution of Eq. (1).

2 Perturbation bounds

Consider the perturbed equation

$$\tilde{X} - \tilde{A}^* \widehat{\tilde{X}}^{-1} \tilde{A} = \tilde{Q}\,, \tag{4}$$

where $\tilde{A}$ and $\tilde{Q}$ ($\tilde{Q} > 0$) are small perturbations of A and Q in (1), respectively. We note that

$$\tilde{A} = \begin{pmatrix} \tilde{A}_1 \\ \vdots \\ \tilde{A}_m \end{pmatrix} \in \mathcal{C}^{mn \times n},$$

where $\tilde{A}_i$, $i = 1, 2, \ldots, m$ are small perturbations of A_i in (3), respectively. Then, we write Eq. (4) as follows

$$\tilde{X} - \sum_{i=1}^{m} \tilde{A}_i^* \tilde{X}^{-1} \tilde{A}_i = \tilde{Q}\,. \tag{5}$$

Let X_+ and $\tilde{X}_+$ be the maximal solutions of Eqs. (1) and (4), respectively.

Denote $\Delta X_+ = \tilde{X}_+ - X_+$, $\Delta Q = \tilde{Q} - Q$, $\Delta A = \tilde{A} - A$, and $\Delta A_i = \tilde{A}_i - A_i$.

We consider the perturbation bounds for equations (1) (or (3)) and (2) proposed by Sun [2], Konstantinov et al. [23], Yin and Fang [6], and Hasanov [8, 9].

We use some general notations. Let

$$L = I_{n^2} + \sum_{i=1}^{m} (X_+^{-1} A_i)^T \otimes (X_+^{-1} A_i)^*, \tag{6}$$

$$\Pi_{p,q} = \sum_{j=1}^{p} \sum_{k=1}^{q} e_j^{(p)} e_k^{(q)^T} \otimes e_k^{(q)} e_j^{(p)^T}, \tag{7}$$

where $e_j^{(p)}$ denotes the jth column of I_p.

2.1 The bound of Sun [2]

Sun [2] has obtained a perturbation bound (see [2, Theorem 2.1]) for the maximal solution of Eq. (2). We formulate Sun's results in case of $C = 0$. Let

$$L^{-1}(I_n \otimes (\widehat{X_+^{-1} A})^*) = U_1 + \mathrm{i}\Omega_1, \quad L^{-1}((\widehat{X_+^{-1} A})^T \otimes I_n)\Pi_{mn,n} = U_2 + \mathrm{i}\Omega_2,$$

where $U_1, \Omega_1, U_2, \Omega_2$ are real $n^2 \times mn^2$ matrices, $\mathrm{i} = \sqrt{-1}$.

Let

$$l = \|L^{-1}\|^{-1}, \quad p = \left\| \begin{pmatrix} U_1 + U_2 & \Omega_2 - \Omega_1 \\ \Omega_2 + \Omega_1 & U_1 - U_2 \end{pmatrix} \right\|, \tag{8}$$

$$\gamma = \|X_+^{-1}\|, \quad \beta = \|\widehat{X_+^{-1} A}\|, \quad \tilde{\beta} = \|\widehat{X_+^{-1} \tilde{A}}\|, \tag{9}$$

$$\epsilon = p\|\Delta A\|_F + \frac{1}{l}\big(\|\Delta Q\|_F + \gamma\|\Delta A\|\|\Delta A\|_F\|\big), \tag{10}$$

$$\tau = \sqrt{m}\tilde{\beta}^2\gamma, \quad \delta_1 = \gamma(\beta + \tilde{\beta})\|\Delta A\|_F. \tag{11}$$

Theorem 1. (Theorem 2.1 from [2] in case of $C = 0$) *Let X_+ be the maximal solution to Eq. (1). Define l, p, γ, β, $\tilde{\beta}$ by (8)–(11). If*

$$\begin{aligned} &\tilde{Q} > 0, \qquad l > \delta_1\,, \\ &\epsilon \le \frac{(l-\delta_1)^2}{l\big[2\tau + (l-\delta_1)\gamma + 2\sqrt{\tau[\tau + (l-\delta_1)\gamma]}\big]}, \end{aligned} \tag{12}$$

then the perturbed equation (4) *has a unique maximal solution $\tilde{X}_+$ and*

$$\|\Delta X_+\|_F \le \frac{2l\epsilon}{l - \delta_1 + l\gamma\epsilon + \sqrt{(l-\delta_1+l\gamma\epsilon)^2 - 4[\tau + (l-\delta_1)\gamma]l\epsilon}} =: est_{sun03}. \tag{13}$$

2.2 The bound of Konstantinov et al. [23]

Konstantinov et al. [23] have obtained local and non-local perturbation bounds for the matrix equation

$$A_0 + \sum_{i=1}^{k} \sigma_i A_i^* X^{p_i} A_i = 0. \tag{14}$$

One particular case of this equation is $k = m+1$, $A_0 = Q$, $A_{m+1} = I$, $\sigma_{m+1} = p_i = -1$, $p_{m+1} = \sigma_i = 1$, $i = 1, 2, \ldots, m$, i.e. Eq. (1).

Now, we formulate the results from [23] in this particular case.

Let

$$\begin{aligned}
&\delta = \left(\|\Delta Q\|_F, \|\Delta A_1\|_F, \cdots, \|\Delta A_m\|_F\right)^T, \\
&W_Q = L^{-1} = W_{Q0} + \mathrm{i}W_{Q1}, \\
&W_{A_i} = \sigma_i L^{-1}(I_n \otimes (X_+^{p_i} A_i)^*) = W_{A_i 0} + \mathrm{i}W_{A_i 1}, \\
&W_{\bar{A}_i} = \sigma_i L^{-1}((X_+^{p_i} A_i)^T \otimes I_n)\Pi_{n,n} = W_{\bar{A}_i 0} + \mathrm{i}W_{\bar{A}_i 1}, \\
&M_{A_i} = \begin{pmatrix} W_{A_i 0} + W_{\bar{A}_i 0} & W_{\bar{A}_i 1} - W_{A_i 1} \\ W_{\bar{A}_i 1} + W_{A_i 1} & W_{A_i 0} - W_{\bar{A}_i 0} \end{pmatrix}, \\
&W_Q^{\mathcal{R}} = \begin{pmatrix} W_{Q0} & -W_{Q1} \\ W_{Q1} & W_{Q0} \end{pmatrix}, \quad k_Q = \|W_Q\|, \\
&k_{A_i} = \|M_{A_i}\|, \quad i = 1, 2, \ldots, m, \\
&\Gamma = \left(\Gamma_1, \Gamma_2, \ldots, \Gamma_{m+2}\right) = \left(W_Q^{\mathcal{R}}, M_{A_1}, \ldots, M_{A_{m+1}}\right),
\end{aligned} \tag{15}$$

where L, $\Pi_{p,q}$ are from (6), (7).

Konstantinov et al. [23] have obtained the local perturbation bounds:

$$\begin{aligned}
&est_1(\delta) = k_Q \|\Delta Q\|_F + \sum_{i=1}^{m} k_{A_i} \|\Delta A_i\|_F, \\
&est_2(\delta) = \|\Gamma\| \|\delta\|, \quad est_3(\delta) = \sqrt{\delta^T R \delta}, \\
&est(\delta) = \min\{est_2(\delta), est_3(\delta)\},
\end{aligned}$$

where R is an $(m+1) \times (m+1)$ real symmetric matrix with non-negative entries $r_{ij} = \|\Gamma_i^T \Gamma_j\|$, $i, j = 1, 2, \ldots, m+1$.

We note that, in case of real matrix coefficients in Eq. (1), the above formulas are more simple (see [23]).

Let

$$a_0(\delta) = est(\delta) + \|L^{-1}\|\|X_+^{-1}\|\sum_{i=1}^{m}\|\Delta A_i\|_F^2, \tag{16}$$

$$a_1(\delta) = \|L^{-1}\|\|X_+^{-1}\|^2\sum_{i=1}^{m}(2\|A_i\| + \|\Delta A_i\|_F)\|\Delta A_i\|_F, \tag{17}$$

$$a_2(\delta) = \|L^{-1}\|\|X_+^{-1}\|^3\sum_{i=1}^{m}(\|A_i\| + \|\Delta A_i\|_F)^2, \tag{18}$$

$$\Omega = \left\{\delta \text{ from (15)} : a_1(\delta) + 2\sqrt{a_0(\delta)a_2(\delta)} \le 1\right\}. \tag{19}$$

The following non-local perturbation bound was obtained in [23].

Theorem 2. ([23, Theorem 5.1]) *Let $\delta \in \Omega$, where Ω is given in (19). Then the non-local perturbation bound*

$$\|\Delta X_+\|_F \le \frac{2a_0(\delta)}{1 - a_1(\delta) + \sqrt{(1-a_1(\delta))^2 - 4a_0(\delta)a_2(\delta)}} =: est_{konppa11} \tag{20}$$

is valid for Eq. (1), where $a_i(\delta)$, $i = 0,1,2$ are determined by (16)-(18).

2.3 The bound of Yin and Fang [6]

Theorem 3. ([6, Theorem 2.1]) *Let A, Q and $\tilde{A}$, $\tilde{Q}$ with $Q, \tilde{Q}$ positive definite be coefficient matrices for Eqs. (1) and (4), respectively. Denote*

$$\begin{aligned} b &= 1 - \|A\|^2\|X_+^{-1}\|^2 + \|X_+^{-1}\|\|\Delta Q\|\,, \\ c &= \|\Delta Q\| + 2\,\|A\|\|X_+^{-1}\|\|\Delta A\| + \|X_+^{-1}\|\|\Delta A\|^2\,, \\ D &= b^2 - 4c\|X_+^{-1}\|. \end{aligned}$$

If $\|A\|\|X_+^{-1}\| < 1$ and

$$2\,\|\Delta A\| + \|\Delta Q\| \le \frac{\left(1 - \|A\|\|X_+^{-1}\|\right)^2}{\|X_+^{-1}\|}\,, \tag{21}$$

then the maximal solutions X_+ and $\tilde{X}_+$ the respective matrix equations (1) and (4) satisfy

$$\|\Delta X_+\| \le \frac{b - \sqrt{D}}{2\|X_+^{-1}\|} =: est_{yinf13}. \tag{22}$$

2.4 The bounds of Hasanov [8, 9]

Applying the technique developed in [16], Hasanov have obtained the following result in [8].

Theorem 4. ([8, Theorem 2]) *Let A, Q and $\tilde{A}$, $\tilde{Q}$ with Q, $\tilde{Q}$ positive definite be coefficient matrices for Eqs. (1) and (4), respectively, P is a positive definite matrix. Denote $\alpha_p = \|\widehat{PX_+^{-1}AP^{-1}}\|$, $\beta_p = \|PX_+^{-1}P\|$, where X_+ is the maximal solution of Eq. (1),*

$$
\begin{aligned}
b_p &= 1 - \alpha_p^2 + \beta_p \|P^{-1}\Delta Q P^{-1}\|\,, \\
c_p &= \|P^{-1}\Delta Q P^{-1}\| + 2\alpha_p \|\widehat{P^{-1}}\Delta A P^{-1}\| + \beta_p \|\widehat{P^{-1}}\Delta A P^{-1}\|^2\,.
\end{aligned}
$$

If $\alpha_p < 1$ and

$$
2\|\widehat{P^{-1}}\Delta A P^{-1}\| + \|P^{-1}\Delta Q P^{-1}\| \le \frac{(1-\alpha_p)^2}{\beta_p}\,, \tag{23}
$$

then $D_p = b_p^2 - 4c_p\beta_p \ge 0$ and

$$
\|\Delta X_+\| \;\le\; \|P\|^2 \frac{b_p - \sqrt{D_p}}{2\beta_p} =: est_{hasP17a}\,. \tag{24}
$$

We note that in [8] for the maximal solution X_+ of Eq. (1) we have

$$
\begin{aligned}
&\rho\Big(\big(X_+^{-T} \otimes X_+^{-1}\big) \sum_{i=1}^{m} A_i^T \otimes A_i^*\Big) = \rho\Big(\sum_{i=1}^{m} (X_+^{-1}A_i)^T \otimes (X_+^{-1}A_i)^*\Big) < 1, \\
&\rho\Big(\sum_{i=1}^{m} (X_+^{-1}A_i)^T \otimes (X_+^{-1}A_i)^*\Big) \le \Big\|\sum_{i=1}^{m} A_i^* X_+^{-2} A_i\Big\| = \big\|\widehat{X_+^{-1}A}\big\|^2\,.
\end{aligned}
$$

In case of $\big\|\widehat{X_+^{-1}A}\big\| < 1$, we can apply Theorem 4 with $P = I$. In this case, the estimate $est_{hasP17a}$ in (24) we denote $est_{hasI17a}$. But, in case of $\big\|\widehat{X_+^{-1}A}\big\| \ge 1$ appears the question "How to choose the matrix P, such that $\|\widehat{PX_+^{-1}AP^{-1}}\| < 1$". In [9] have been proven that $\Big\|\widehat{\sqrt{X_+^{-1}}A\sqrt{X_+^{-1}}}\Big\| < 1$. Hence, Theorem 4 is applicable with $P = \sqrt{X_+}$. In this case the estimate $est_{hasP17a}$ in (24) we will denote by $est_{has\sqrt{X}17a}$. Moreover, in [9] have been obtained an alternative result to Theorem 4 with $P = \sqrt{X_+}$.

Theorem 5. ([9, Theorem 3]) *Let A, Q and $\tilde{A}$, $\tilde{Q}$ with Q, $\tilde{Q}$ positive definite be coefficient matrices for Eqs. (1) and (4), respectively. Let*

$$b_1 = 1 - \|\widehat{\sqrt{X_+^{-1}}A}\sqrt{X_+^{-1}}\|^2 + \|X_+^{-1}\|\|\Delta Q\|\,,$$

$$c_1 = \|\Delta Q\| + 2\|\widehat{\sqrt{X_+^{-1}}A}\sqrt{X_+^{-1}}\|\|\Delta A\| + \|X_+^{-1}\|\|\Delta A\|^2\,,$$

where X_+ is the maximal solution of Eq. (1). If

$$2\|\Delta A\| + \|\Delta Q\| \le \frac{\left(1 - \|\widehat{\sqrt{X_+^{-1}}A}\sqrt{X_+^{-1}}\|\right)^2}{\|X_+^{-1}\|}\,, \tag{25}$$

then $D_1 = b_1^2 - 4c_1\|X_+^{-1}\| \ge 0$ and

$$\|\Delta X_+\| \;\le\; \|X_+\|\frac{b_1 - \sqrt{D_1}}{2} =: est_{has\sqrt{X}17b}\,. \tag{26}$$

In case of $\|\widehat{X_+^{-1}A}\| \ge 1$, but if $\|AX_+^{-1}\| < 1$, then Theorem 4 is applicable with $P = X_+$. In this case, the estimate $est_{hasP17a}$ in (24) we will denote by $est_{hasX17a}$. Hasanov [9] has obtained following alternative result for this case.

Theorem 6. ([9, Theorem 4]) *Let A, Q and $\tilde{A}$, $\tilde{Q}$ with Q, $\tilde{Q}$ positive definite be coefficient matrices for Eqs. (1) and (4), respectively. Let*

$$b_2 = 1 - \|AX_+^{-1}\|^2 + \|X_+\|\|X_+^{-1}\|^2\|\Delta Q\|\,,$$

$$c_2 = \|\Delta Q\| + 2\|AX_+^{-1}\|\|\Delta A\| + \|X_+^{-1}\|\|\Delta A\|^2\,,$$

where X_+ is the maximal solution of Eq. (1). If $\alpha_2 < 1$ and

$$2\|\Delta A\| + \|\Delta Q\| \le \frac{(1 - \|AX_+^{-1}\|)^2}{\|X_+\|\|X_+^{-1}\|^2}\,, \tag{27}$$

then $D_2 = b_2^2 - 4c_2\|X_+\|\|X_+^{-1}\|^2 \ge 0$ and

$$\|\Delta X_+\| \;\le\; \|X_+\|\frac{b_2 - \sqrt{D_2}}{2} =: est_{hasX17b}\,. \tag{28}$$

We note that, the maximal solution X_+ of Eq. (1) satisfies $X_+ \ge Q$. Hence, $\|\sqrt{Q}X_+^{-1}\sqrt{Q}\| \le 1$. Thus, if $\|\widehat{\sqrt{Q^{-1}}A}\sqrt{Q^{-1}}\| < 1$, then

$$\|\widehat{\sqrt{Q}X_+^{-1}A}\sqrt{Q^{-1}}\| \le \|\sqrt{Q}X_+^{-1}\sqrt{Q}\|\|\widehat{\sqrt{Q^{-1}}A}\sqrt{Q^{-1}}\| < 1.$$

Therefore, in case of $\|\widehat{\sqrt{Q^{-1}}A}\sqrt{Q^{-1}}\| < 1$, Theorem 4 is applicable with $P = \sqrt{Q}$. In this case, the estimate $est_{hasP17a}$ in (24) we denote $est_{has\sqrt{Q}17a}$.

3 Numerical examples

We consider some numerical examples and compare the perturbation bounds. Denote the ratio of the perturbation bounds to the estimated value as follows:

$$sun03 = \frac{est_{sun03}}{\|\Delta X_+\|_F}, \qquad konppa11 = \frac{est_{konppa11}}{\|\Delta X_+\|_F}, \qquad yinf13 = \frac{est_{yinf13}}{\|\Delta X_+\|},$$
$$hasI17a = \frac{est_{hasI17a}}{\|\Delta X_+\|}, \qquad has\sqrt{X}17a = \frac{est_{has\sqrt{X}17a}}{\|\Delta X_+\|},$$
$$hasX17a = \frac{est_{hasX17a}}{\|\Delta X_+\|}, \qquad has\sqrt{Q}17a = \frac{est_{has\sqrt{Q}17a}}{\|\Delta X_+\|},$$
$$has\sqrt{X}17b = \frac{est_{has\sqrt{X}17b}}{\|\Delta X_+\|}, \qquad hasX17b = \frac{est_{hasX17b}}{\|\Delta X_+\|},$$

where $est_{hasI17a}$, $est_{has\sqrt{X}17a}$, $est_{hasX17a}$, $est_{has\sqrt{Q}17a}$ are denoted the perturbation bound $est_{hasP17a}$ in (24) for $P = I$, $P = \sqrt{X_+}$, $P = X_+$, and $P = \sqrt{Q}$, respectively.

Example 1. ([6, Example 4.1]) *Consider Eq. (1) with coefficient matrices*

$$A = \begin{pmatrix} -0.4326 & -1.1465 \\ -1.6665 & 1.1909 \\ 0.1253 & 1.1892 \\ 0.2877 & -0.0376 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0.1376 & 0.0656 \\ 0.0656 & 0.5616 \end{pmatrix}$$

and perturbations on the matrices A and Q

$$\Delta A = 10^{-j} \begin{pmatrix} 10 & 6 \\ 2 & 4 \\ 8 & 4 \\ 6 & 1 \end{pmatrix}, \qquad \Delta Q = 10^{-j} \begin{pmatrix} 4 & 7 \\ 7 & 3 \end{pmatrix},$$

respectively.

The approximation of the maximal solution

$$X_+ \approx \begin{pmatrix} 1.1572575 & 0.01971555 \\ 0.01971555 & 3.3569583 \end{pmatrix}$$

have been computed after 200 iteration by formula

$$X_{k+1} = Q + A^* \widehat{X_k^{-1}} A, \quad X_0 = Q. \tag{29}$$

The solution $\tilde{X}_+$ to the perturbed equation (4) have been computed iteratively by formula (29), also, as $\tilde{X}_+ \approx \tilde{X}_{200}$.

In Table 1 the perturbation bounds with different values of j are given. The cases when the conditions of existence of a bound are violated are denoted by an asterisk. In case of $j = 5$, $j = 6$, and $j = 7$ the perturbation bounds est_{sun03} and $est_{konppa11}$ are sharper, but in case of $j = 4.4$ are not applicable.

Table 1: Numerical results of Example 1:

j	$j = 4.4$	$j = 5$	$j = 6$	$j = 7$
$\|\Delta X_+\|$	0.0012	$3.0193e - 04$	$3.0159e - 05$	$2.9752e - 06$
$\|\Delta X_+\|_F$	0.0013	$3.2960e - 04$	$3.2929e - 05$	$3.2550e - 06$
$sun03$	$*$	10.2369	9.5045	9.5514
$konppa11$	$*$	10.5080	9.4466	9.4704
$yinf13$	$*$	$*$	$*$	$*$
$hasI17a$	$*$	$*$	$*$	$*$
$hasqQ17a$	$*$	$*$	$*$	$*$
$hasqX17a$	35.3452	31.9281	31.1718	31.5215
$hasX17a$	$*$	$*$	$*$	$*$
$hasqX17b$	45.6326	39.8713	38.6852	39.0970
$hasX17b$	$*$	$*$	$*$	$*$

Example 2. ([6, Example 4.2]) *Consider Eq. (1) with $m = 2$, $n = 4$ and matrices A and Q as follows:*

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad Q := X_+ - A^* \widehat{X_+^{-1}} A,$$

where

$$A_1 = \frac{1}{100} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 22.5 & 12 & 2 \\ 2 & 9 & 7 & 3 \\ 12 & 1 & 2 & 19 \end{pmatrix}, \quad A_2 = \frac{2\sqrt{3}}{45} \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

and

$$X_+ = \begin{pmatrix} 2.5 & 1 & 1 & 1 \\ 1 & 2.5 & 1 & 1 \\ 1 & 1 & 2.5 & 1 \\ 1 & 1 & 1 & 2.5 \end{pmatrix}.$$

Consider perturbation on the matrices A and Q:

$$\Delta A = 10^{-2j}\begin{pmatrix} C_1/\|C_1\| \\ C_2/\|C_2\| \end{pmatrix}, \quad \Delta Q := \tilde{X}_+ - \tilde{A}^*\widehat{\tilde{X}_+^{-1}}\tilde{A} - Q,$$

with $\tilde{X}_+ = X_+ + 10^{-2j}(I-E)$, E being the 4×4 matrix with all entries equal to 1, C_1, C_2 random matrices generated by MATLAB function **randn**.

In Table 2 the perturbation estimates for different values of j are given. Among the bounds considered in this example the bound $est_{hasI17a}$, followed by $est_{konppa11}$ and est_{sun03}, gives the sharpest estimates.

Table 2: Numerical results of Example 2:

j	$j=2$	$j=3$	$j=4$	$j=5$
$\|\Delta X\|$	$3.0000e-04$	$3.0000e-06$	$3.0000e-08$	$3.0000e-10$
$\|\Delta X\|_F$	$3.4641e-04$	$3.4641e-06$	$3.4641e-08$	$3.4641e-10$
$sun03$	1.3939	1.4355	1.3715	1.3958
$konppa11$	1.3736	1.4265	1.3674	1.3784
$yinf13$	1.4950	1.5490	1.4506	1.4890
$hasI17a$	1.3609	1.4201	1.3288	1.3654
$hasqQ17a$	2.9440	2.5763	2.6747	2.6290
$hasqX17a$	2.9027	2.5441	2.6388	2.6109
$hasX17a$	10.9163	9.4421	9.5722	9.6494
$hasqX17b$	4.9918	5.2088	4.8738	5.0082
$hasX17b$	18.6834	19.4593	18.2115	18.7087

In following two examples we consider Eq. (1) with complex matrices A.

Example 3. *Consider Eq. (1) with $m=3$, $n=5$ and matrices A and Q as follows:*

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \quad Q := X_+ - A^*\widehat{X_+^{-1}}A,$$

where $A_1 = \frac{1+\mathbf{i}}{25}A_0$, $A_2 = \frac{1+\mathbf{i}}{25}A_0^T$, $A_3 = \frac{1}{70}A_0^TA_0$, *with* $\mathbf{i} = \sqrt{-1}$, *and*

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}, \; X_+ = \begin{pmatrix} 2.5 & 1 & 1 & 1 & 1 \\ 1 & 2.5 & 1 & 1 & 1 \\ 1 & 1 & 2.5 & 1 & 1 \\ 1 & 1 & 1 & 2.5 & 1 \\ 1 & 1 & 1 & 1 & 2.5 \end{pmatrix}.$$

Consider perturbation on the matrices A and Q:

$$\Delta A = \frac{10^{-2j}}{\|C\|}\begin{pmatrix} 20\mathbf{i}\, C \\ 20\mathbf{i}\, C^T \\ C^T + C \end{pmatrix}, \quad \Delta Q := \tilde{X}_+ - \tilde{A}^* \widehat{\tilde{X}_+^{-1}} \tilde{A} - Q,$$

with $\tilde{X}_+ = X_+ + 10^{-2j}(I - E)$, E being the 5×5 matrix with all entries equal to 1, C random matrix generated by MATLAB function **randn**.

In Table 3 the perturbation estimates for different values of j are given. The most effective bound is the bound $est_{hasI17a}$, followed by the bounds est_{yinf13}, $est_{konppa11}$ and est_{sun03}.

Table 3: Numerical results of Example 3:

j	$j=2$	$j=3$	$j=4$	$j=5$
$\|\Delta X\|$	$4.0000e-04$	$4.0000e-06$	$4.0000e-08$	$4.0000e-10$
$\|\Delta X\|_F$	$4.4721e-04$	$4.4721e-06$	$4.4721e-08$	$4.4721e-10$
$sun03$	3.3011	3.3570	3.2887	3.0113
$konppa11$	3.4066	3.4447	3.3836	3.0894
$yinf13$	2.8904	3.1122	3.2372	2.7202
$hasI17a$	2.4637	2.6985	2.7782	2.3140
$hasqQ17a$	7.6038	7.6915	9.1061	6.3522
$hasqX17a$	7.5700	7.6329	9.0323	6.3220
$hasX17a$	38.7663	40.4155	48.8969	32.6099
$hasqX17b$	10.3512	11.3791	11.6895	9.7180
$hasX17b$	54.2108	58.2862	60.6146	50.9264

Example 4. *Consider Eq. (1) with $m = 4$, $n = 9$ and matrices A and Q as follows:*

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}, \quad Q := X_+ - A^* \widehat{X_+^{-1}} A,$$

where

$$A_1 = \frac{1}{n}(K_n + \mathbf{i}P_n), \qquad A_2 = \frac{1}{n}(3K_n + 2\mathbf{i}P_n), \quad A_3 = \frac{1}{n}(4K_n + 3\mathbf{i}P_n),$$
$$A_4 = \frac{1}{n}(5K_n + 3\mathbf{i}P_n), \quad X_+ = nI + P_n,$$

with K_n and P_n are Kahan's and Poisson's matrices, respectively, i.e.

$$K_n = \begin{pmatrix} s & 0 & \cdots & 0 \\ 0 & s^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^n \end{pmatrix} \begin{pmatrix} 1 & -c & \cdots & -c \\ 0 & 1 & \cdots & -c \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \begin{array}{l} s = \sin(\theta) \\ c = \cos(\theta), \quad \theta = 1.2, \end{array}$$

$$P_n = (p_{ij}), \quad p_{ij} = \begin{cases} 4, & i = j, \\ -1, & |i-j| = 1 \text{ or } |i-j| = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Consider perturbation on the matrices A and Q:

$$\Delta A = 10^{-j} \begin{pmatrix} C_1/\|C_1\| \\ C_2/\|C_2\| \\ C_3/\|C_3\| \\ C_4/\|C_4\| \end{pmatrix}, \quad \Delta Q := \tilde{X}_+ - \tilde{A}^* \widehat{\tilde{X}_+^{-1}} \tilde{A} - Q,$$

with $\tilde{X}_+ = X_+ + 10^{-j-1}(I + E + C_5/\|C_5\|)$, C_i, $i = 1,2,3,4,5$ are random matrices generated by MATLAB function **randn**.

In Table 4 the perturbation estimates for different values of j are given. All the bounds considered give close estimates of the perturbation in the solution. The bound $est_{hasI17a}$ is the sharpest ones.

Table 4: Numerical results of Example 4:

j	$j = 4$	$j = 5$	$j = 6$	$j = 7$
$\|\Delta X\|$	$1.0039e-04$	$1.0002e-05$	$1.0091e-06$	$9.9406e-08$
$\|\Delta X\|_F$	$1.0552e-04$	$1.0427e-05$	$1.0543e-06$	$1.0467e-07$
$sun03$	2.6909	2.7396	2.8096	2.7030
$konppa11$	2.5130	2.5933	2.6243	2.5313
$yinf13$	2.8063	2.8001	2.7989	2.7682
$hasI17a$	2.0068	1.9932	2.0079	1.9677
$hasqQ17a$	2.6215	2.6722	2.7003	2.5668
$hasqX17a$	2.7530	2.8141	2.8438	2.7026
$hasX17a$	3.9335	4.0438	4.1108	3.9408
$hasqX17b$	3.1228	3.1016	3.1244	3.0620
$hasX17b$	4.8596	4.8266	4.8621	4.7649

Example 5. *Consider Eq. (1) with $m = 2$, $n = 4$ and matrices A and Q as follows:*

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad Q := X_+ - A^* \widehat{X_+^{-1}} A,$$

where $A_1 = 12A_2^T$,

$$A_2 = \frac{2\sqrt{3}}{45} \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \text{ and } X_+ = \frac{1}{10} \begin{pmatrix} 26 & 1 & 1 & 1 \\ 1 & 26 & 1 & 1 \\ 1 & 1 & 26 & 1 \\ 1 & 1 & 1 & 26 \end{pmatrix}.$$

Consider perturbation on the matrices A and Q:

$$\Delta A = 10^{-2j} \begin{pmatrix} C_1/\|C_1\| \\ C_2/\|C_2\| \end{pmatrix}, \quad \Delta Q := \tilde{X}_+ - \tilde{A}^* \widehat{\tilde{X}_+^{-1}} \tilde{A} - Q,$$

with $\tilde{X}_+ = X_+ + 10^{-2j}(I - E)$, E being the 4×4 matrix with all entries equal to 1, C_1, C_2 random matrices generated by MATLAB function **randn**.

In Table 5 the perturbation estimates for different values of j are given. Among the bounds considered in this example the bounds $est_{konppa11}$ and est_{sun03} give the sharpest estimates followed by $est_{hasX17a}$ and $est_{hasX17b}$.

Table 5: Numerical results of Example 5:

j	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$\|\Delta X\|$	$3.0000e-04$	$3.0000e-06$	$3.0000e-08$	$3.0000e-10$
$\|\Delta X\|_F$	$3.4641e-04$	$3.4641e-06$	$3.4641e-08$	$3.4641e-10$
$sun03$	5.1907	4.4964	5.4698	4.5634
$konppa11$	3.9230	3.3392	4.1122	3.3200
$yinf13$	32.1669	28.8418	32.9264	29.6421
$hasI17a$	29.7512	26.8199	30.6258	27.5686
$hasqQ17a$	$*$	$*$	$*$	$*$
$hasqX17a$	18.2291	16.1100	19.0277	16.2319
$hasX17a$	15.2685	13.0048	15.8923	12.8708
$hasqX17b$	18.3767	16.9686	19.4292	17.4715
$hasX17b$	15.5064	14.3843	16.5058	14.8296

Perturbation bounds depends from different parameters. In Table 6, we report some parameters to considered examples.

We denote

$$NQA = \|\widehat{\sqrt{Q^{-1}}A}\sqrt{Q^{-1}}\|,$$
$$NXA = \|\widehat{\sqrt{X_+^{-1}}A}\sqrt{X_+^{-1}}\|,$$
$$k(X_+) = \|X_+^{-1}\|\|X_+\|.$$

Table 6:

Ex.	$\|X_+^{-1}\|\|A\|$	$\|\widehat{X_+^{-1}A}\|$	NQA	NXA	$\|AX_+^{-1}\|$	$k(X_+)$
1	1.9443	1.4926	1.2253	0.9472	1.5385	2.9014
2	0.2119	0.1667	0.1672	0.1669	0.1766	3.6667
3	0.1637	0.1294	0.1234	0.1232	0.1630	4.3333
4	0.4149	0.2667	0.2667	0.2667	0.2667	1.5561
5	0.9478	0.9439	1.7048	0.8981	0.8617	1.1600

4 Conclusion

Analysing the behaviour of the perturbation bounds considered in the paper, we can point out as most effective the bounds $est_{konppa11}$ (20), est_{sun03} (13), $est_{hasP17a}$ (24) with $P = I$, $P = \sqrt{X_+}$, or $P = X_+$. The bound $est_{hasP17a}$ with different value of P is very simple for computing. It was the sharpest for $P = I$ in cases of small value of $\|\widehat{X_+^{-1}A}\|$ in considered examples. The bound $est_{hasP17a}$ for $P = \sqrt{X_+}$ is applicable always, but is not sharpest. The bounds $est_{konppa11}$ and est_{sun03} are reliable and generally give satisfactory accurate estimates. But the dependence of the bounds $est_{konppa11}$ and est_{sun03} on many parameters makes the difficult for computing in general.

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