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NEW LOCAL CONVERGENCE THEOREMS FOR THE INVERSE WEIERSTRASS METHOD FOR SIMULTANEOUS APPROXIMATION OF POLYNOMIAL ZEROS*

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Abstract

In this work we establish new local convergence theorems with error estimates for the Inverse Weierstrass iterative method for simultaneous approximations of polynomial zeros. Our approach enlarges the convergence radius and improves the known local convergence results. Numerical examples are also provided.

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keywords: Polynomial zeros, Simultaneous method, Weierstrass method, Durand-Kerner method, Inverse Weierstrass method, Local convergence.

1 Introduction

Let $P(z)$ be a monic polynomial

$$P(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + z^n \, , \tag{1}$$

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of degree $n \geq 2$, with simple real or complex zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$, and let $z_1^{(0)}, z_2^{(0)}, \ldots, z_n^{(0)}$ be distinct reasonable close approximations of these zeros.

In this study we consider a simultaneous iterative method defined by

$$\mathbf{z}^{(k+1)} = \mathbf{G}\left(\mathbf{z}^{(k)}\right) = \mathbf{G}^{k+1}\left(\mathbf{z}^{(0)}\right), \quad k = 0, 1, 2, \ldots, \tag{2}$$

where $\mathbf{G} : \mathbf{C}^n \to \mathbf{C}^n$ is a vector valued function with components

$$G_i = G_i(\mathbf{z}) = \frac{z_i^2}{z_i + W_i(\mathbf{z})}, \; \mathbf{z} = (z_1, \ldots, z_n), \; i = 1, \ldots, n, \tag{3}$$

and the term

$$W_i(\mathbf{z}) = \frac{P(z_i)}{\prod_{j \neq i}^{n}(z_i - z_j)}, \quad (i = 1, \ldots, n) \tag{4}$$

is called *Weierstrass' correction.*

The iteration method (2)-(3) is a modification of the famous Weierstrass' iterative method for simultaneously finding all the zeros of polynomials

$$z_i^{(k+1)} = z_i^{(k)} - W_i(\mathbf{z}^{(k)}), \quad i = 1, 2, \ldots, n, \quad k \geq 0, \tag{5}$$

which is also called *Durand-Kerner*, *Weierstrass-Dochev*, or shorter the *WDK-method.* It is originally proposed by Weierstrass in 1891 [1], rediscovered later by Durand [2], Dochev [3], Kerner [4], Prešić [5], and since then it has been investigated by many authors (see [6, 8, 9, 10, 11, 12, 14, 15, 16]).

The modified method (2)-(3) was firstly introduced in [18], and some recent results were obtained in [19, 20, 21, 22].

Throughout this paper, we will use only the maximum vector norm defined by

$$\|z\|_\infty = \max_i |z_i|,$$

and we will follow the usual convention that a summation over the empty set of indices equals 0, while a product over the same set equals 1.

2 Local Convergence Analysis

First, we prove some auxiliary results.

Lemma 1 *Let $d > 0$, $c \geq 0$, $q > 1$, $n \geq 2$ and*

$$\left(\frac{d-c}{d-2c}\right)^{n+1} < q. \tag{6}$$

Then the following relations hold true

$$(i) \quad \left(\frac{d-c}{d-2c}\right)^{n-1} < q^{\frac{n-1}{n+1}}\,; \tag{7}$$

$$(ii) \quad c < kd\,, \quad where \quad k = \frac{q-1}{2q-1}\,; \tag{8}$$

$$(iii) \quad \frac{c}{d-c} < \frac{q^{\frac{1}{n+1}}-1}{q^{\frac{1}{n+1}}} < \frac{q-1}{q}\,. \tag{9}$$

Proof. (i) The claim (7) is a direct consequence of the assumption (6) and the choice of q.

(ii) The inequality (6) is equivalent to the following two inequalities

$$\frac{d-c}{d-2c} < q^{\frac{1}{n+1}} \tag{10}$$

and consequently

$$c < \frac{q^{\frac{1}{n+1}}-1}{2q^{\frac{1}{n+1}}-1}d\,. \tag{11}$$

The assertion (8) follows from relation (11), assumption $n \geq 2$ and the choice of q.

(iii) The first inequality in (9) follows from (11) and the choice of d. The second inequality in (9) follows from (8) and the inequality

$$\frac{c}{d-c} < \frac{kd}{d-kd} = \frac{k}{1-k}\,.$$

Corollary 1 *Let $q \in (1,2]$, then from the relations (7), (8) and (9) of Lemma 1 it follows that*

$$c < \frac{d}{3} \tag{12}$$

and

$$\frac{c}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1} < 1\,. \tag{13}$$

Lemma 2 *Let the assumptions of Lemma 1 hold true and $q \in (1,2]$. Then*

$$0 < \frac{\frac{d}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1}-1}{1-\frac{c}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1}} < 1\,. \tag{14}$$

Proof. It is easy to show that

$$\frac{d}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1} > 1\,,$$

and using (13) we get the first inequality in (14).

The second inequality in (14) holds true if and only if

$$\frac{d}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1} - 1 < 1 - \frac{c}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1}$$

or equivalently

$$g(c,d,n) = \left(\frac{d-c}{d-2c}\right)^{n-1} + 2\frac{c}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1} < 2\,. \tag{15}$$

We can bound $g(c,d,n)$ by using the relations (7) and (9), as follows

$$g(c,d,n) < q^{\frac{n-1}{n+1}} + 2\frac{q^{\frac{1}{n+1}} - 1}{q^{\frac{1}{n+1}}} q^{\frac{n-1}{n+1}} = 3q^{\frac{n-1}{n+1}} - 2q^{\frac{n-2}{n+1}} = h(n)\,. \tag{16}$$

Now, we will prove that $h(n) \le 2$. For $n = 2$ from the choice of q, we have

$$h(2) = 3q^{\frac{1}{3}} - 2 < 2\,.$$

It is easy to prove that the function $h(n)$ is monotonically increasing and also

$$\lim_{n\to\infty} h(n) = q \le 2\,.$$

From the last expressions, (15) and (16), it follows the statement (14).

Further, in order to prove the main result we use the identity (given by [9])

$$\prod_{j=1}^{n-1}\frac{u_n - v_j}{u_n - u_j} - 1 = \sum_{s=1}^{n-1}\frac{u_s - v_s}{u_n - u_s}\prod_{j=1}^{s-1}\frac{u_n - v_j}{u_n - u_j}\,, \tag{17}$$

which is valid for any $2n$ numbers u_i, v_i, such that $u_i \ne u_j$ for $i \ne j$ $(i,j = 1,\ldots,n)$.

Theorem 1 *Let $P \in \mathcal{C}[z]$ be a polynomial of degree $n \ge 2$, where*

$$\alpha = \{\alpha \in \mathbf{C}^n : \alpha_i \ne 0 \text{ and } \alpha_i \ne \alpha_j \text{ for } i,j = 1,\ldots,n\}$$

is the root vector of P, and let

$$d = \min\{\delta, \gamma\}\,, \text{ where } \delta = min_{j\neq i}|\alpha_i - \alpha_j| \text{ and } \gamma = min_i|\alpha_i|\,.$$

If the initial guess $\mathbf{z}^{(0)} \in \mathbf{C}^n$, satisfies the inequality

$$\|\mathbf{z}^{(0)} - \alpha\| \leq \rho(n,d) := \frac{2^{\frac{1}{n+1}} - 1}{2.2^{\frac{1}{n+1}} - 1}\, d\,, \tag{18}$$

then

(i) the modified Waierstrass' iteration (2)-(3) is well defined and converges to α quadratically;

(ii) the asymptotic convergence rate satisfies

$$\overline{\lim}_{k\to\infty} \frac{\|\mathbf{z}^{(k+1)} - \alpha\|}{\|\mathbf{z}^{(k)} - \alpha\|^2} \leq \frac{n}{d}\,. \tag{19}$$

Proof. The $(k+1)^{th}$ iteration stage of the algorithm (2) is

$$z_i^{(k+1)} = \frac{(z_i^{(k)})^2}{z_i^{(k)} + W_i(z^{(k)})}\,, \quad k \geq 0, \quad i = 1, 2, \ldots, n\,. \tag{20}$$

For easy of later comparisons, we will use the following equivalent form of (20) (see [22])

$$z_i^{(k+1)} = z_i^{(k)} - \frac{W_i(z^{(k)})}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}}\,, \quad i = 1, 2, \ldots, n\,, \tag{21}$$

which implies

$$z_i^{(k+1)} - \alpha_i = z_i^{(k)} - \alpha_i - \frac{W_i(z^{(k)})}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}}$$

and consequently

$$z_i^{(k+1)} - \alpha_i = (z_i^{(k)} - \alpha_i)\left[1 - \frac{\prod_{j\neq i}^n \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}}}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}}\right]$$

$$= (z_i^{(k)} - \alpha_i)\left[\frac{1 - \prod_{j\neq i}^n \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} + \frac{W_i(z^{(k)})}{z_i^{(k)}}}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}}\right]\,.$$

For the error in each component we get

$$|z_i^{(k+1)} - \alpha_i| = |z_i^{(k)} - \alpha_i| \left| \frac{1 - \prod_{j\neq i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} + \frac{W_i(z^{(k)})}{z_i^{(k)}}}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}} \right| ,$$

which implies

$$\|z^{(k+1)} - \alpha\| \leq \|z^{(k)} - \alpha\| \max_i \left| \frac{1 - \prod_{j\neq i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} + \frac{W_i(z^{(k)})}{z_i^{(k)}}}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}} \right| . \tag{22}$$

Further, we will bound the amplification factor for the i^{th} component. Let for fixed k and i denote

$$A_i^{(k)} := \left| \frac{1 - \prod_{j\neq i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} + \frac{W_i(z^{(k)})}{z_i^{(k)}}}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}} \right| .$$

Then the following inequality is valid

$$A_i^{(k)} \leq \frac{\left| \prod_{j\neq i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} - 1 \right| + \left| \frac{z_i^{(k)} - \alpha_i}{z_i^{(k)}} \right| \left| \prod_{j\neq i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} \right|}{1 - \left| \frac{z_i^{(k)} - \alpha_i}{z_i^{(k)}} \right| \left| \prod_{j\neq i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} \right|} . \tag{23}$$

We next establish the following inequalities

$$|z_i^{(k)} - z_j^{(k)}| \geq |\alpha_i - \alpha_j| - |z_i^{(k)} - \alpha_i| - |z_j^{(k)} - \alpha_j| \geq d - 2\|z^{(k)} - \alpha\| , \tag{24}$$

$$|z_i^{(k)}| \geq |\alpha_i| - |z_i^{(k)} - \alpha_i| \geq d - \|z^{(k)} - \alpha\| \tag{25}$$

and

$$\left| \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} \right| = \left| 1 + \frac{z_j^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} \right| \leq 1 + \frac{\|z^{(k)} - \alpha\|}{d - 2\|z^{(k)} - \alpha\|} = \frac{d - \|z^{(k)} - \alpha\|}{d - 2\|z^{(k)} - \alpha\|} . \tag{26}$$

Substituting

$$\begin{cases} u_s = z_s^{(k)} & 1 \leq s < i-1 \\ u_s = z_{s+1}^{(k)} & i <= s \leq n-1 \\ u_s = z_i^{(k)} , & s = n \end{cases} \quad \mathit{and} \quad \begin{cases} v_s = \alpha_s & 1 \leq s < i-1 \\ v_s = \alpha_{s+1} & i <= s \leq n-1 \\ v_s = \alpha_i , & s = n \end{cases}$$

in the identity (17) and using (23), (24), (25) we obtain

$$A_i^{(k)} \leq \frac{\sum_{s=1}^{n-1} \frac{\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|} \prod_{j=1}^{s-1}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right) + \frac{\|z^{(k)}-\alpha\|}{d-\|z^{(k)}-\alpha\|} \prod_{j\neq i}^{n}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)}{1-\frac{\|z^{(k)}-\alpha\|}{d-\|z^{(k)}-\alpha\|} \prod_{j\neq i}^{n}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)},$$

which implies

$$A_i^{(k)} \leq \frac{\frac{\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|} \sum_{s=0}^{n-2}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)^s + \frac{\|z^{(k)}-\alpha\|}{d-\|z^{(k)}-\alpha\|}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)^{n-1}}{1-\frac{\|z^{(k)}-\alpha\|}{d-\|z^{(k)}-\alpha\|}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)^{n-1}}$$

and consequently

$$A_i^{(k)} \leq \frac{\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)^{n-1} - 1 + \frac{\|z^{(k)}-\alpha\|}{d-\|z^{(k)}-\alpha\|}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)^{n-1}}{1-\frac{\|z^{(k)}-\alpha\|}{d-\|z^{(k)}-\alpha\|}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)^{n-1}}. \tag{27}$$

Finally, from the last expression, it follows that the amplification factor for the error norm in (22) can be bounded as follows:

$$A^{(k)} := \max_i A_i^{(k)} \leq \frac{\frac{d}{d-\|z^{(k)}-\alpha\|}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)^{n-1} - 1}{1-\frac{\|z^{(k)}-\alpha\|}{d-\|z^{(k)}-\alpha\|}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)^{n-1}} = \xi_k(n,d)\,.$$

Substituting $c = \|z^{(k)} - \alpha\|$ in Lemma 2 and using the assumption (18) and Lemma 1, we obtain

$$\xi_k(n,d) < 1\,,$$

whenever

$$\|z^{(k)} - \alpha\| < \rho(n,d)\,.$$

Since this bound does not depend on k and it follows by induction from the assumption (18). Then $\xi_0 < 1$ is a uniform upper bound for all the ξ_k. This completes the proof of the claim (i).

(ii) From (22) and the derivation of (27), it follows that

$$\frac{\|z^{(k+1)}-\alpha\|}{\|z^{(k)}-\alpha\|^2} \leq \frac{\frac{1}{d-2\|z^{(k)}-\alpha\|} \sum_{s=0}^{n-2}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)^s + \frac{1}{d-\|z^{(k)}-\alpha\|}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)^{n-1}}{1-\frac{\|z^{(k)}-\alpha\|}{d-\|z^{(k)}-\alpha\|}\left(\frac{d-\|z^{(k)}-\alpha\|}{d-2\|z^{(k)}-\alpha\|}\right)^{n-1}}$$

and giving in the limit, as $k \to \infty$, we get $\|z^{(k)} - \alpha\| \to 0$ and the claim (19).

In the next theorem we consider the case $a_0 = 0$ in (1), i.e. there exists s such that $\alpha_s = 0$ for $s = 1, \ldots, n$. Without loss of generality we can assume that $\alpha_n = 0$. We state the following theorem without proof (see [20] for the proof).

Theorem 2 *Let $P \in \mathcal{C}[z]$ be a polynomial of degree $n \geq 2$ with simple zeros, such that*

$$\alpha_i \text{ is } \begin{cases} \neq 0, & i = 1, \ldots, n-1 \\ = 0, & i = n \end{cases}$$

and let $d = \{min_{j \neq i} |\alpha_i - \alpha_j| : i, j = 1, \ldots, n\}$. If the initial guess $\mathbf{z}^{(0)} \in \mathbf{C}^n$, satisfies the initial condition (18) from Theorem 2

$$\|\mathbf{z}^{(0)} - \alpha\| \leq \rho(n, d)\,,$$

the modified Weierstrass iteration (2)-(3) is well defined and converges
(i) quadratically to $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$;
(ii) at least linearly to $\alpha_n = 0$.

3 Comparison with earlier convergence results

In our previous work [20], we show that if the initial approximation $z^{(0)} \in \mathbf{C}^n$ satisfies

$$\|\mathbf{z}^{(0)} - \alpha\| \leq \tau(n, d) = \frac{\sqrt[n-1]{4/3} - 1}{2\,\sqrt[n-1]{4/3} - 1}\, d\,, \tag{28}$$

then the iteration (2)-(3) converges to α quadratically (see also [21]).

In [22], we have proved that if the initial approximation $z^{(0)} \in \mathbf{C}^n$ satisfies

$$\|\mathbf{z}^{(0)} - \alpha\| \leq \sigma(n, d) = \frac{1}{an + 4}\, d\,, \tag{29}$$

where $a \approx 1.76$ is the unique root of the equation $t = \exp(1/t)$, then the iteration (2)-(3) converges to α quadratically.

The radius of convergence $\rho(n, d)$ defined by (18) (in Theorem 1) is larger than $\tau(n, d)$ and $\sigma(n, d)$. Indeed, the ratios $\rho(n, d)/\tau(n, d)$ and $\rho(n, d)/\sigma(n, d)$ do not depend on d, and we have the limits

$$\lim_{n \to \infty} \frac{\rho(n, d)}{\tau(n, d)} \approx 2.4$$

and

$$\lim_{n \to \infty} \frac{\rho(n, d)}{\sigma(n, d)} \approx 1.2\,.$$

The comparative values of the asserted radiuses are included in Table 1.

Table 1: Some comparative values of $\rho(n,d)$, $\tau(n,d)$ and $\sigma(n,d)$.

n	$\frac{\rho(n,d)}{d}$	$\frac{\tau(n,d)}{d}$	$\frac{\sigma(n,d)}{d}$	$\frac{\rho(n,d)}{\tau(n,d)}$	$\frac{\rho(n,d)}{\sigma(n,d)}$
2	0.17	0.2	0.13	0.85	1.28
3	0.13	0.11	0.10	1.16	1.27
4	0.11	0.08	0.09	1.36	1.26
5	0.09	0.06	0.07	1.51	1.25
6	0.08	0.05	0.06	1.62	1.25
10	0.05	0.03	0.04	1.88	1.24
20	0.03	0.01	0.02	2.12	1.23
50	0.01	0.005	0.01	2.28	1.22
100	0.006	0.002	0.005	2.34	1.21

4 Numerical Examples

In this section, we provide several numerical examples to show the local convergence properties of the considered method (2)-(3).

Example 1 *We take the polynomial*

$$p(z) = z^5 - 15.5z^4 + 77.5z^3 - 155z^2 + 124z - 32$$

with root vector $\alpha = (0.5, 1, 2, 4, 8)$, *which was studied in Niell ([17], Ex.7.3). We use the same initial approximation* $z^{(0)} = (0.45, 0.9, 1.8, 3.6, 7.2)$.

Table 2: Numerical results for Example 1.

$iter(i)$	$z_1^{(i)}$	$z_2^{(i)}$	$z_3^{(i)}$	$z_4^{(i)}$	$z_5^{(i)}$
0	0.45	0.9	1.8	3.6	7.2
2	0.5150080103196240	1.002739480385864	1.996805551378158	3.994885604749412	8.003591516944205
6	0.5000000000000002	0.999999999999998	1.999999999999994	3.999999999999996	8.000000000000002

In this case the radius of convergence (from Theorem 1) is

$$\rho(n,d) = \rho(5, 0.5) \approx 0.049$$

and the condition (18) is satisfied for $z^{(2)}$, i.e.

$$\|z^{(2)} - \alpha\| \approx 0.015 < \rho(5, 0.5)\,.$$

The stopping criteria $\|z^{(i)} - \alpha\| \leq 10^{-15}$ is reached after six iterations, see Table 2.

For the same polynomial with the classical Weierstrass' iterative method (5), if we use the same initial vector and stopping criteria, we get the root vector after eight iterations (see [17]).

Example 2 *Consider the polynomial*

$$p(z) = z^3 - 8z^2 - 23z + 30\,,$$

with root vector $\alpha = (-3, 1, 10)$ *and the initial guess* $z^{(0)} = (-4, 2, 9)$ *which is taken from Dochev[3](see also [16]).*

In this case the radius of convergence (from Theorem 1) is

$$\rho(n, d) = \rho(3, 1) \approx 0.1372$$

and the condition (18) is satisfied for $z^{(2)}$, i.e.

$$\|z^{(2)} - \alpha\| \approx 0.0914 < \rho(3, 1)\,.$$

The stopping criteria $\|z^{(i)} - \alpha\| \leq 10^{-15}$ is reached after six iterations, see Table 3.

Table 3: Numerical results for Example 2.

iter(i)	$z_1^{(i)}$	$z_2^{(i)}$	$z_3^{(i)}$
0	-4	2	9
2	-3.040886694525941	1.091441307965112	9.999998807826081
6	-3.000000000000000	1.000000000000000	10.0000000000000

With the classical Weierstrass' iterative method (5), we get the root vector after six iterations, for the same initial vector and same stopping criteria (see [16]).

Example 3 *Consider the polynomial*

$$f(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300$$

with the zero vector $\alpha = (2i, 2 + i, -3, -2i, -1, 1, -2 + i, 2 - i, -2 - i)$.

We use Abert's initial approximation vector $z^{(0)}$ (see [7]) given by

$$z_k^{(0)} = -\frac{a_1}{n} + r_0 \exp i\theta_k \, , \;\; \theta_k = \frac{\pi}{n}\left(2k - \frac{3}{2}\right) \, , \;\; k = 1, \ldots, n \, ,$$

where $n = 9$ and $r_0 = 10$ (see also [13]).

The radius of convergence is

$$\rho(n, d) = \rho(9, 1) \approx 0.0628$$

and the condition (18) is satisfied for $z^{(2)}$, i.e.

$$\|z^{(8)} - \alpha\| \approx 0.0207 < \rho(9, 1) \, .$$

The stopping criteria $\|z^{(i)} - \alpha\| \leq 10^{-15}$ is reached after eleven iterations, see Table 4.

Table 4: Numerical results for Example 3.

$iter(i)$	$z_1^{(i)}$	$z_2^{(i)}$
0	-1.263 +1.736i	-4.683 +7.660i
8	0.0050 + 1.9960i	1.9847 + 0.9861i
11	$2.963 \times 10^{-18} + 2i$	2+i

$z_3^{(i)}$	$z_4^{(i)}$
-11.11+10i	-17.53 +7.660i
-3.0039 - 0.0003i	- 2.0005i
$-3 + 1.009 \times 10^{-18} i$	$5.340 \times 10^{-18} - 2i$

$z_5^{(i)}$	$z_6^{(i)}$
-20.95+1.736i	-19.77 -5i
-1.0003 + 0.0005i	1.0031 - 0.0022i
$-1 + 2.222 \times 10^{-19} i$	$1 + 2.568 \times 10^{-18} i$

$z_7^{(i)}$	$z_8^{(i)}$	$z_9^{(i)}$
-14.53-9.396i	-7.690 -9.396i	-2.450-4.999i
-1.9999 + 1.0000i	2.0086 - 1.0093i	-1.9971 - 0.9993i
$-2 + i$	$2 - i$	$-2 - i$

With Weierstrass' classical iterative method (5), we get the root vector after same number of iterations, if we use the same initial vector and stopping criteria.

5 Conclusion

This work is devoted to convergence analysis of a modified Weierstrass iterative method for simultaneous approximation of polynomial zeros. Our goal was to improve the existing convergence analysis results. We establish a new radius of convergence, which enlarges the existing radiuses. We have compared all the known radiuses of convergence. Numerical results with different examples show that the Inverse Weierstrass iterative method (2)-(3) have even larger radius of convergence. The included numerical examples confirm that the proposed algorithm has very similar convergence properties with the classical Weierstrass method (5).

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