ISSN 2066-6594 Ann. Acad. Rom. Sci. Ser. Math. Appl. Vol. 10, No. 2/2018

COMPUTING THE NASH EQUILIBRIUM FOR LQ GAMES ON POSITIVE SYSTEMS ITERATIVELY*

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Abstract

We consider the nonsymmetric Riccati equation arising in two player linear quadratic differential games for positive linear systems with an open loop information structure. The Newton method for computing the Nash equilibrium point is derived by Jank and Kremer in 2004. We transform the Newton method into the alternately linearized implicit Newton iteration following the existing papers in this subject. However, we reorganize this alternately linearized implicit Newton iteration based on the specific characteristic of the considered Riccati equation. Thus, we derive an alternately linearized implicit decoupled iteration. The convergence properties of the proposed alternately linearized implicit decoupled iteration are investigated and a sufficient condition to apply the method is derived. The performance of the proposed algorithm is illustrated on some numerical examples.

MSC: 15A24, 15A45, 60H35, 65C20.

keywords: Open loop Nash equilibrium, generalized Riccati equation, stabilizing solution, nonnegative solution.

*Accepted for publication on March 5, 2018

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1 Introduction

Many researchers are focused their investigations on nonsymmetric matrix Riccati equations associated with M-matrices [6, 7, 3, 11, 15]. The general nonsymmetric Riccati equation

$$-DX - XA + XCX + Q = 0\,, \tag{1}$$

where D, Q, C and A are real matrices of dimensions $m \times m, m \times n, n \times m$ and $n \times n$, respectively, has many applications - in the Markov chains [7], in the transport theory [11], in the game theory [1, 10]. The matrix coefficients of (1) are arranged in the following block matrix

$$K = \begin{pmatrix} A & -C \\ -Q & D \end{pmatrix}. \tag{2}$$

The matrix K for matrix equation (1) is an M-matrix.

In this investigation we exploit the properties of nonnegative matrices. A matrix $A = (a_{ij}) \in \mathbf{R}^{n\times m}$ is a nonnegative matrix if the inequalities $a_{ij} \geq 0$ are satisfied for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The notation $\mathbf{R}^{n\times m}$ stands for $n \times m$ real matrices. The inequality $A \geq B (A > B)$ for $A = (a_{ij}), B = (b_{ij})$ means that $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ for all indexes i and j. A matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is said to be a Z-matrix if it has nonpositive off-diagonal entries. Any Z-matrix A can be written in the form $A = \alpha I - N$ with N being a nonnegative matrix. Each M-matrix is a Z-matrix with if $\alpha \geq \rho(N)$, where $\rho(N)$ is the spectral radius of N. It is called a nonsingular M-matrix if $\alpha > \rho(N)$ and a singular M-matrix if $\alpha = \rho(N)$.

Equation (1) always has a minimal nonnegative solution, if the matrix K from (2) is a nonsingular M-matrix [6]. The matrix $\tilde{X}$ is the minimal nonnegative solution to (1) if $\tilde{X} \leq X$ for any nonnegative solution X to (1).

We are interested in the linear quadratic LQ differential game described by the dynamic system:

$$x_(t) = Ax + B_1 u_1 + B_2 u_2\,, \quad x(0) = x_0\,, \tag{3}$$

with matrices $A \in \mathbf{R}^{n\times n}, B_1 \in \mathbf{R}^{m_1}, B_2 \in \mathbf{R}^{m_2}, x(t) \in \mathbf{R}^n$ is the state of the game and the control functions u_1, u_2.

We say that a system is positive, if for nonnegative inputs u_1, u_2 and nonnegative initial values x_0, the state function x is nonnegative. A sufficient condition for the above system to be a positive system is that $(-A)$ is an M-matrix and B_1, B_2 are nonnegative matrices.

Further on, the cost-functional for each player has the form

$$J_i(u_1,u_2) = \int_0^{\infty} (x^T\, Q_i\, x + u_1^T R_{i1} u_1 + u_2^T R_{i2} u_2)\, dt\,, \quad i = 1,2\,,$$

where $Q_i in \mathbf{R}^{n\times n}, R_{ij} \in \mathbf{R}^{m_i\times m_j}, i,j = 1,2$. The matrices $Q_i, R_{ij}, i,j = 1,2$ are symmetric.

We say that (u_1^*, u_2^*) is a open loop Nash equilibrium if for each player $i = 1,2$, the inequalities

$$J_1(u_1^*,u_2^*) \ge J_1(u_1,u_2^*) \;\text{ and }\; J_2(u_1^*,u_2^*) \ge J_2(u_1^*,u_2)$$

hold [10, 1]. According to the above inequalities the aim of each player is to maximize his own utility function. The notation an 'open loop' strategy means that the players have to choose their strategies u_1 and u_2 prior to the game start and that their only information on the state of the game is the initial state x_0. A sufficient condition to exits unique Nash equilibrium is the matrices R_{11}, R_{22} are negative definite (see [[10], Theorem 4]).

Based on the analysis of the introduced game we have to find a stabilizing nonnegative solution of the following nonsymmetric matrix Riccati equation

$$\mathfrak{R}(\mathcal{X}) = -\mathcal{D}\mathcal{X} - \mathcal{X}A + \mathcal{X}\mathcal{S}\mathcal{X} - \mathcal{Q} \,=\, 0\,, \tag{4}$$

where the matrix coefficients are:

$(-A)$ is an $n\times n$ M-matrix,

$\mathcal{D} = \begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix}$,

$\mathcal{S} = (S_1 \quad S_2)$ with $S_j = B_j\, R_{jj}^{-1}\, B_j^T$ is an $n\times n$ nonpositive matrix, $j = 1,2$,

$\mathcal{Q} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ and Q_j is an $n\times n$ symmetric nonnegative matrix, and the unknown matrix $\mathcal{X}$ has the form $\mathcal{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$.

2 Iterative methods

The Newton method is given by the following set of recursive equations [10, 1]:

$$\begin{aligned} &-\mathcal{X}_{k+1}(A - \mathcal{S}\mathcal{X}_k) - (D - \mathcal{X}_k\mathcal{S})\,\mathcal{X}_{k+1} = \mathcal{Q} + \mathcal{X}_k\,\mathcal{S}\,\mathcal{X}_k \\ &k = 0,1,2,\dots\,, \end{aligned} \tag{5}$$

The following theorem proves that the Newton matrix sequence monotonically increasing.

Theorem 1 *(Theorem 5, [10]) Suppose additionally for the nonsymmetric Riccati equation positive system (3) that the matrix $-A$ is an M-matrix and $Q_i \geq 0, i = 1, 2$ and $\mathcal{S} \leq 0$. Assume further that there exists a $\mathcal{P} \geq 0$, such that $\mathbb{R}(\mathcal{P}) > 0$, then the Newton sequence $\{\mathcal{X}_k\}_{k=0}^{\infty}$ initialized with $\mathcal{X}_0 = 0$ is well defined and converges monotonically to a solution $\tilde{\mathcal{X}} \geq 0$. $\tilde{\mathcal{X}}$ is the smallest solution in the set of all nonnegative solutions.*

In fact, Theorem 4 [10] statements that the Newton sequence $\{\mathcal{X}_k\}_{k=0}^{\infty}$ converges to a solution $\tilde{\mathcal{X}} = \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}$ and the component matrices $\tilde{X}_1$ and $\tilde{X}_2$ defines the Nash equilibrium strategy (u_1^*, u_2^*) as follows: $u_i^* == -R_{ii}^{-1} B_i^T \tilde{X}_i x^*$, $i = 1, 2$, with x^* being the solution of the closed loop equation:

$$\dot{x} = (A - S_1\tilde{X}_1 - S_2\tilde{X}_2)x\,, \quad x(0) = x_0\,.$$

We apply some properties of the matrix equation $AXB = C$, i.e. it is equivalent to the linear system $(B^T \otimes A)\, vec(X) = vec(C)$, where the sign $\otimes$ denotes the Kronecker matrix product and the *vec* operator arranges the columns of a matrix into a column vector. An usual Gaussian elimination technique for solving this system requires $O(n^6)$ operations.

According to the approach investigated by Ma and Lu in [15] we introduce the following modification of the Newton method (5):

$$\begin{aligned} &\mathcal{Y}_k(\gamma I_n + A - \mathcal{S}\,\mathcal{X}_k = (\gamma I_{2n} - \mathcal{D})\mathcal{X}_k - \mathcal{Q} \\ &(\gamma I_{2n} + \mathcal{D} - \mathcal{Y}_k\mathcal{S})\mathcal{X}_{k+1} = \mathcal{Y}_k(\gamma I_n - A) - \mathcal{Q} \\ &\mathcal{X}_0 = 0\,, \quad k = 0, 1, 2, \ldots\,, \quad \gamma < 0\,. \end{aligned} \tag{6}$$

We call the above iteration the Linearized Implicit Newton Iteration (LINI).

We formulate two lemmas, which we will exploit in this investigation:

Lemma 1 *The following statements are equivalent for a Z-matrix (-A):*

(a) $-A$ is a nonsingular M-matrix;

(b) $(\theta I_n - A)$ is a nonsingular M-matrix, where $\theta < 0$ and I_n is the $n \times n$ unit matrix;

(c) A is asymptotically stable.

Lemma 2 *[8] Let $A = (a_{ij})$ be an $n \times n$ M-matrix. If the elements of $B = (b_{ij})$ satisfy the relations:*

$$a_{ii} \geq b_{ii}\,, \qquad (a_{ij}) \leq (b_{ij}) \leq 0\,, i \neq j,\ i, j = 1, \ldots, n\,,$$

then B is also an M-matrix.

We rewrite the matrix function $\mathfrak{R}(\mathcal{X})$ in the form $\mathfrak{R}(\mathcal{X}) = \begin{pmatrix} \mathcal{R}_1(X_1, X_2) \\ \mathcal{R}_2(X_1, X_2) \end{pmatrix}$.
where

$$\mathcal{R}_1(X_1, X_2) = -A^T X_1 - X_1 A + X_1 S_1 X_1 + X_1 S_2 X_2 - Q_1 \,,$$
$$\mathcal{R}_2(X_1, X_2) = -A^T X_2 - X_2 A + X_2 S_1 X_1 + X_2 S_2 X_2 - Q_2 \,.$$

The equation $\mathfrak{R}(\mathcal{X}) = 0$ is equivalent to the set of Riccati equations

$$\mathcal{R}_1(X_1, X_2) = 0 \,, \mathcal{R}_2(X_1, X_2) = 0.$$

Using the block structure of the coefficient matrices in (6) we derive the Alternately Linearized Implicit Decoupled Iteration (ALIDI) based on iteration (6):

$$Y_1^{(k)}(\gamma I_n + A - S_1 X_1^{(k)} - S_2 X_2^{(k)}) = (\gamma I - A^T) X_1^{(k)} - Q_1 \tag{7}$$

$$Y_2^{(k)}(\gamma I_n + A - S_1 X_1^{(k)} - S_2 X_2^{(k)}) = (\gamma I - A^T) X_2^{(k)} - Q_2 \tag{8}$$

$$(\gamma I_n + A^T - Y_1^{(k)} S_1) X_1^{(k+1)} = Y_1^{(k)}(\gamma I - A + S_2 X_2^{(k)}) - Q_1 \tag{9}$$

$$(\gamma I_n + A^T - Y_2^{(k)} S_2) X_2^{(k+1)} = Y_2^{(k)}(\gamma I - A + S_1 X_1^{(k)}) - Q_2 \,, \tag{10}$$

$X_1^{(0)} = X_2^{(0)} = 0 \,, \quad k = 0, 1, 2, \ldots \,, \ \gamma < 0 \,.$

According to approaches applied in [8, 12, 13] we establish the following properties of recursive equations (7)-(10):

Lemma 3 *We construct the matrix sequences* $\{{X_1}^k, {X_2}^k, {Y_1}^k, {Y_2}^k\}_{k=0}^{\infty}$ *using (7)-(10) with initial values* $X_1^{(0)} = X_2^{(0)} = 0$. *The following properties hold:*

(i) $\mathcal{R}_i({X_1}^{(k)}, {X_2}^{(k)}) = (Y_i^{(k)} - X_i^{(k)})(\gamma I_n + A - S_1 X_1^{(k)} - S_2 X_2^{(k)}) \,,$
$i = 1, 2$

(ii) $\mathcal{R}_i({Y_1}^{(k)}, {Y_2}^{(k)}) = (\gamma I_n - A^T$
$+ Y_i^{(k)} S_i)(Y_i^{(k)} - X_i^{(k)}) + Y_i^{(k)} S_j (Y_j^{(k)} - X_j^{(k)}) \,, \ i, j = 1, 2 \,, \ j \neq i \,,$

(iii) $\mathcal{R}_i({Y_1}^{(k)}, {Y_2}^{(k)}) = (\gamma I_n + A^T - Y_i^{(k)} S_i)(X_i^{(k+1)} - Y_i^{(k)})$
$+ Y_i^{(k)} S_j (Y_j^{(k)} - X_j^{(k)}) \,, \ i, j = 1, 2 \,, \ j \neq i \,,$

(iv) $\mathcal{R}_i({X_1}^{(k)}, {X_2}^{(k)}) = (X_i^{(k+1)} - Y_i^{(k)})(\gamma I_n - A + S_1 X_1^{(k+1)} + S_2 X_2^{(k+1)}) \,,$
$i = 1, 2 \,.$

In addition, the following equalities true for any symmetric nonnegative matrices $\hat{X}_1, \hat{X}_2$:

$$\begin{array}{lll} & & (Y_i^{(k)} - \hat{X}_i)(\gamma I_n + A - S_1 X_1^{(k)} - S_2 X_2^{(k)}) \\ \text{(v)} & = & (\gamma I_n - A + \hat{X}_i S_i)(X_i^{(k)} - \hat{X}_i) + \hat{X}_i S_j (X_j^{(k)} - \hat{X}_j) + \mathcal{R}_i(\hat{X}_1, \hat{X}_2)\,, \\ & & i, j = 1, 2\,,\ j \neq i\,, \end{array}$$

$$\begin{array}{ll} & (\gamma I_n + A^T - Y_i^{(k)} S_i) = (Y_i^{(k)} - \hat{X}_i)(\gamma I_n - A + S_1 \hat{X}_1 + S_2 \hat{X}_2) \\ \text{(vi)} & +Y_i^{(k)} S_j (X_j^{(k)} - \hat{X}_j) + \mathcal{R}_i(\hat{X}_1, \hat{X}_2)\,, \\ & i, j = 1, 2\,,\ j \neq i\,. \end{array}$$

Proof: The proof is completed by a direct calculation. □

3 Convergence properties of the alternately linearized implicit decoupled iteration

In this section we establish the convergence of the alternately linearized implicit decoupled iteration (7)-(10). The convergence properties are derived under the some assumptions and the fact that the inequalities $\mathcal{R}_i(X_1, X_2) \geq 0\,, i = 1, 2$ are solvable. We combine the approaches applied in [12, 13] and [3]. We begin with a preliminary lemma.

Lemma 4 *Assume the matrix $(-A)$ is an M-matrix and $\mathcal{Q} \geq 0$ and $\mathcal{S} < 0\,, \gamma < 0$, such that $(-\gamma I - A)$ is an M-matrix and $(\gamma I - A)$ is nonpositive. Assume there exist symmetric nonnegative matrices $\hat{X}_1, \hat{X}_2$, such that $\mathcal{R}_i(\hat{X}_1, \hat{X}_2) \geq 0\,, i = 1, 2$ and $-A + S_1\hat{X}_1 + S_2\hat{X}_2$ is an M-matrix. We construct the matrix sequences $\{{X_1}^k, {X_2}^k, {Y_1}^k, {Y_2}^k\}_{k=0}^{\infty}$ using (7)-(10) with initial values $X_1^{(0)} = X_2^{(0)} = 0$. The following properties are satisfied:*

(a) $0 \leq Y_i^{(k)} \leq \hat{X}_i$ *for* $i = 1, 2\,, \quad k = 0, 1, \ldots$*;*

(b) $0 \leq X_i^{(k)} \leq \hat{X}_i$ *for* $i = 1, 2\,, \quad k = 0, 1, \ldots$*;*

Proof: For $k = 0$, we have $Y_i^{(0)} = -Q_i\,(\gamma I_n + A)^{-1} \geq 0$, because $(\gamma I_n + A)^{-1}$ is nonpositive. Using Lemma 3 (v) we get

$$(Y_i^{(0)} - \hat{X}_i) = [-(\gamma I_n - A^T)\hat{X}_i - \hat{X}_i S_i (X_i^{(k)} - \hat{X}_i S_j (X_j^{(k)} + \mathcal{R}_i(\hat{X}_1, \hat{X}_2)](\gamma I_n + A)^{-1}\,,$$

with $i, j = 1, 2\,,\ j \neq i\,.$ Note that $\gamma I_n - A^T, S_1, S_2$ are nonpositive and thus $(Y_i^{(0)} - \hat{X}_i) \leq 0$, and $Y_i^{(0)} \leq \hat{X}_i\,, i = 1, 2.$

Further on, we apply equality (vi) from Lemma 3 for $k = 0$ and we obtain:

$$
\begin{aligned}
&(\gamma I_n + A^T - Y_i^{(0)} S_i)(X_i^1 - \hat{X}_i) = (Y_i^{(0)} - \hat{X}_i) \\
&\times(\gamma I_n - A + S_1 \hat{X}_1 + S_2 \hat{X}_2) - Y_i^{(0)} S_j \hat{X}_j + \mathcal{R}_i(\hat{X}_1, \hat{X}_2) \\
&i, j = 1, 2,\ j \neq i\,.
\end{aligned}
$$

The matrix $\gamma I_n - A + S_1 \hat{X}_1 + S_2 \hat{X}_2$ in the nonpositive one. Thus the right hand of the last equality is nonpositive. We know that $-A + \hat{X}_i S_i\,, i = 1, 2$ is an M-matrix and

$$
-A^T + \hat{X}_1 S_1 + \hat{X}_2 S_2 \leq -A^T + \hat{X}_i S_i \leq -A^T + Y_i^{(0)} S_i\,,
$$

because $Y_i^{(0)} \leq \hat{X}_i\,, i = 1, 2$. The matrix $Y_i^{(0)} S_i\,, i = 1, 2$ is a Z-matrix and according to Lemma 2 we infer that the following series of statements: $-A^T + Y_i^{(0)} S_i\,, i = 1, 2$ is an M-matrix; $-\gamma I_n - A^T + Y_i^{(0)} S_i\,, i = 1, 2$ is an M-matrix and $(\gamma I_n + A^T - Y_i^{(0)} S_i)^{-1} \leq 0\,, i = 1, 2$. Thus, we conclude $X_i^{(1)} - \hat{X}_i \leq 0\,, i = 1, 2$. The statements (a) and (b) in the case of $k = 0$ are proved.

We assume that the inequalities are true: $0 \leq Y_i^{(r)} \leq \hat{X}_i\,, i = 1, 2$, and $0 \leq X_i^{(r)} \leq \hat{X}_i\,, i = 1, 2$, for $k = r$.

We will prove the following inequalities: $0 \leq Y_i^{(r+1)} \leq \hat{X}_i\,, 0 \leq X_i^{(r+1)} \leq \hat{X}_i\,, i = 1, 2$. We compute $Y_i^{(r)}\,, i = 1, 2$ using (7)-(8) with $k = r$. We have

$$
Y_i^{(r)} Z^{(r)} = W_i^{(r)} := [(\gamma I_n - A^T) X_i^{(r)} - Q_i]\,,\ i = 1, 2,
$$

where

$$
Z^{(r)} = \gamma I_n + A - S_1 X_1^{(r)} - S_2 X_2^{(r)}\,.
$$

We know that $X_i^{(r)}$ is nonnegative, and Q_i is nonnegative for $i = 1, 2$, and $\gamma I_n - A^T$ is nonpositive. We conclude the matrices $W_1^{(r)}, W_2^{(r)}$ are nonpositive. Moreover, we have

$$
-A + S_1 \hat{X}_1 + S_2 \hat{X}_2 \leq -A + S_1 X_1^{(r)} + S_2 X_2^{(r)}\,,
$$

because $X_i^{(r)} \leq \hat{X}_i\,, i = 1, 2$. Thus the matrix $(-Z^{(r)})$ is an M-matrix. Therefore, $Y_i^{(r)} = W_i^{(r)} (Z^{(r)})^{-1} \geq 0, i = 1, 2$. According to Lemma 3 (v) with $k = r$ we get

$$\begin{aligned}(Y_i^{(r)} - \hat{X}_i)Z^{(r)} = \; & [\gamma I_n - A^T + \hat{X}_i S_i](X_i^{(r)} - \hat{X}_i) \\ & + \hat{X}_i S_j (X_j^{(r)} - \hat{X}_j) + \mathcal{R}_i(\hat{X}_1, \hat{X}_2) \\ i, j = 1, 2, \; j \neq i \, . & \end{aligned}$$

The right hand side of the above equality is nonnegative. Since $(Z^{(r)})^{-1} \leq 0$ we conclude that $Y_i^{(r)} - \hat{X}_i \leq 0 \, , i = 1, 2$. Further on, we compute $X_i^{(r+1)} \, , i = 1, 2$ using (9)-(10) with $k = r$. We obtain:

$$(\gamma I_n + A^T - Y_i^{(r)} S_i) \, X_i^{(r+1)} = Y_i^{(r)} (\gamma I_n - A + S_j X_j^{(r)}) - Q_i \, ,$$

$i, j = 1, 2 \, , j \neq i$. Since the matrix $(\gamma I_n - A + S_j X_j^{(r)}) \, , j = 1, 2$ is nonpositive the right hand side of the above equality is nonpositive and thus

$$X_i^{(r+1)} = (\gamma I_n + A^T - Y_i^{(r)} S_i)^{-1} [Y_i^{(r)} (\gamma I_n - A + S_j X_j^{(r)}) - Q_i] \geq 0 \, ,$$

$i = 1, 2$. According to Lemma 3 (v) with $k = r$ we obtain:

$$\begin{aligned} & (\gamma I_n + A^T - Y_i^{(r)} S_i)(X_i^{(r+1)} - \hat{X}_i) \\ = \; & (Y_i^{(r)} - \hat{X}_i)(\gamma I_n - A + S_1 \hat{X}_1 + S_2 \hat{X}_2) + Y_i^{(r)} S_j (X_j^{(r)} - \hat{X}_j) \\ & + \mathcal{R}_i(\hat{X}_1, \hat{X}_2) \, , \quad i, j = 1, 2, \; j \neq i \, . \end{aligned}$$

The right hand side of the above equality is nonpositive and $(\gamma I_n + A^T - Y_i^{(r)} S_i) \leq 0 \, , i = 1, 2$. Thus $X_i^{(r+1)} - \hat{X}_i) \leq 0 \, , i = 1, 2$. Therefore the statements (a) and (b) are proved of $k = r + 1$. This ends the proof. □

In the next theorem we derive a sufficient condition for the convergence of the introduced ALIDI.

Theorem 2 *Assume the matrix the matrix $-A$ is an M-matrix and $\mathcal{Q} \geq 0$, and $\mathcal{S} \leq 0 \, , \gamma < 0$, such that $(-\gamma I - A)$ is an M-matrix and $\gamma I - A$ is non-positive. Assume there exist symmetric nonnegative matrices $\hat{X}_1, \hat{X}_2$, such that $\mathcal{R}_i(\hat{X}_1, \hat{X}_2) \geq 0 \, , i = 1, 2$ and $-A + S_1 \hat{X}_1 + S_1 \hat{X}_2$ is an M-matrix. The matrix sequences $\{\mathcal{X}_1^{(k)} \, , \mathcal{X}_2^{(k)}\}_{k=0}^{\infty}$ defined by (7)-(10) satisfy the following properties:*

(i) $\tilde{X}_i, \geq X_i^{(k+1)} \geq Y_i^{(k)} \geq X_i^{(k)}$ *for* $i = 1, 2 \, , \quad k = 0, 1, \ldots;$

(ii) $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \le 0\,, \quad \mathcal{R}_i(Y_1^{(k)}, Y_2^{(k)}) \le 0\,,$
$\mathcal{R}_i(X_1^{(k+1)}, X_2^{(k+1)}) \le 0\,, \quad i = 1, 2\,, \quad k = 0, 1, \ldots$

(iii) *The matrix sequences* $\{\mathcal{X}_1^{(k)}, \mathcal{X}_2^{(k)}\}_{k=0}^{\infty}$ *converge to the nonnegative minimal solution* $\tilde{X}_1, \tilde{X}_2$ *to the set of Riccati equations* $\mathcal{R}_1(X_1, X_2)) = 0\,, \mathcal{R}_2(X_1, X_2)) = 0$ *with* $\tilde{X}_i \le \hat{X}_i$ *and the matrix* $A - S_1\tilde{X}_1 - S_2\tilde{X}_2$ *is asymptotically stable.*

Proof: The matrix $\gamma I_n - A$ is nonpositive. We construct the matrix sequences $\{X_1{}^{(k)}, X_2{}^{(k)}, Y_1{}^{(k)}, Y_2{}^{(k)}\}_{k=0}^{\infty}$ applying recursive equations (7)-(10) with $X_1^{(0)} = 0\,, X_2^{(0)} = 0$ and $\gamma < 0$. According to Lemma 4 we know that the statements (a) and (b) are true. We have to prove the inequalities $X_i^{(k+1)} \ge Y_i^{(k)} \ge X_i^{(k)}$ for $i = 1, 2\,, \quad k = 0, 1, \ldots$; For $k = 0$ we have $Y_i^{(0)} \ge X_i^{(0)} = 0\,, i = 1, 2$. Applying Lemma 3 (ii), we get $\mathcal{R}_i(Y_1^{(0)}, Y_2^{(0)}) = (\gamma I_n - A^T)Y_i^{(0)} + Y_i^{(0)}S_iY_i^{(0)} + Y_i^{(0)}S_jY_j^{(0)} \le 0\,, i = 1, 2$. It is easy to check the inequalities $\mathcal{R}_i(X_1^{(0)}, X_2^{(0)}) = -Q_i \le 0, i = 1, 2$. We apply equalities (ii) and (iii) from Lemma 3 in order to obtain:

$$(\gamma I_n + A^T - Y_i^{(0)}S_i)(X_i^{(1)} - Y_i^{(0)}) = (\gamma I_n - A^T - Y_i^{(0)}S_i)Y_i^{(0)}\,,$$

$i, j = 1, 2,\ j \ne i$. The right hand side of the above inequality is nonpositive because the matrix $(\gamma I_n - A^T - Y_i^{(0)}S_i)\,, i = 1, 2$ is nonpositive. In addition $(\gamma I_n + A^T - Y_i^{(0)}S_i)^{-1} \le 0\,, i = 1, 2$. Thus $X_i^{(1)} - Y_i^{(0)} \ge 0\,, i = 1, 2$. We compute $X_1^{(1)}, X_2^{(1)}$ applying the recursive equations (9)-(10). According to Lemma 3 (iv) we induce

$$\begin{aligned}\mathcal{R}_i(X_1^{(1)}, X_2^{(1)}) = \ & (X_1^{(1)} - Y_1^{(0)}) \\ & (\gamma I_n - A + S_1X_1^{(1)} + S_2X_2^{(1)}) \le 0\,, \quad i = 1, 2\,,\end{aligned}$$

because $\gamma I_n - A$ is nonpositive and $S_iX_i^{(1)}, i = 1, 2$ is nonpositive, too.

Assume that the inequalities (i) - (ii) hold for $k = r$. We know

$$X_i^{(r+1)} \ge Y_i^{(r)} \ge X_i^{(r)}\,, i = 1, 2$$

and

$$\mathcal{R}_i(X_1^{(r)}, X_2^{(r)}) \le 0\,, \mathcal{R}_i(Y_1^{(r)}, Y_2^{(r)}) \le 0\,, \mathcal{R}_i(X_1^{(r+1)}, X_2^{(r+2)}) \le 0\,, i = 1, 2.$$

Applying Lemma 3 (i) with $k = r+1$, we get

$$\begin{aligned} \mathcal{R}_i(X_1^{(r+1)}, X_2^{(r+2)}) = & \ (Y_i^{(r+1)} - X_i^{(r+1)}) \\ & (\gamma I_n + A - S_1 X_1^{(r+1)} - S_2 X_2^{(r+1)}), \quad i = 1,2 . \end{aligned}$$

We know that $\mathcal{R}_i(X_1^{(r+1)}, X_2^{(r+1)}) \leq 0\,, i = 1,2$ by the Induction assumption and $(\gamma I_n + A - S_1 X_1^{(r+1)} - S_2 X_2^{(r+1)})^{-1} \leq 0$. Thus, $Y_i^{(r+1)} \geq X_i^{(r+1)}\,, i = 1,2$. Applying Lemma 3 (ii) and the fact that $\gamma I_n - A^T + Y_i^{(r+1)} S_i,\ i = 1,2$ is a nonpositive matrix, we conclude

$$\begin{aligned} \mathcal{R}_i(Y_1^{(r+1)}, Y_2^{(r+2)}) = & \ (\gamma I_n - A^T + Y_i^{(r+1)} S_i)(Y_i^{(r+1)} - X_i^{(r+1)}) \\ & Y_i^{(r+1)} S_j (Y_j^{(r+1)} - X_j^{(r+1)}) \ \leq 0\,, \\ & \ i = 1,2\,, \quad j \neq i. \end{aligned}$$

According to Lemma 3 (i) we extract $(i = 1,2)$

$$\begin{aligned} & \mathcal{R}_i(X_1^{(r+1)}, X_2^{(r+2)}) = (Y_i^{(r+1)} - X_i^{(r+1)})(\gamma I_n + A - S_1 X_1^{(r+1)} - S_2 X_2^{(r+1)}) \\ & \mathcal{R}_i(X_1^{(r+1)}, X_2^{(r+2)}) = (Y_i^{(r+1)} - X_i^{(r+1)})\, Z^{(r+1)} \\ & \mathcal{R}_i(X_1^{(r+1)}, X_2^{(r+2)})(Z^{(r+1)})^{-1} = (Y_i^{(r+1)} - X_i^{(r+1)}) \ \geq 0\,,\ i = 1,2 . \end{aligned}$$

Since $(Z^{(r+1)})^{-1} \leq 0$, because $-Z^{(r+1)}$ is a nonsingular M-matrix, we infer $\mathcal{R}_i(X_1^{(r+1)}, X_2^{(r+2)}) \leq 0\,, i = 1,2$.

Further on, we compute $X_1^{(r+2)}, X_2^{(r+2)}$ applying the recursive equations (9)-(10). We know $\hat{X}_i \geq X_i^{(r+2)} \geq 0, i = 1,2$. We apply equalities (ii) and (iii) from Lemma 3 in order to obtain:

$$\begin{aligned} & (\gamma I_n + A^T - Y_i^{(r+1)} S_i)\, (X_i^{(r+2)} - Y_i^{(r+1)}) \\ = & \ (\gamma I_n - A^T + Y_i^{(r+1)} S_i)\, (Y_i^{(r+1)} - X_i^{(r+1)})\,, \end{aligned}$$

$i, j = 1,2, j \neq i$. The right hand side of the above equality is nonposivive. The matrix $-\gamma I_n - A^T + Y_i^{(r+1)} S_i$ is a nonsingular M-matrix, $i = 1,2$. Thus $X_i^{(r+2)} - Y_i^{(r+1)} \geq 0, i = 1,2$.

According to Lemma 3 (iv) we write down

$$\begin{aligned} \mathcal{R}_i(X_1^{(r+2)}, X_2^{(r+2)}) = & \ (X_i^{(r+2)} - Y_i^{(r+1)}) \\ & \times(\gamma I_n - A + S_1(X_1^{(r+2)} + S_2 X_2^{(r+2)})\,, \quad i = 1,2\,, \end{aligned}$$

and therefore $\mathcal{R}_i(X_1^{(r+2)}, X_2^{(r+2)}) \leq 0\,, i = 1, 2$.

Hence, the induction process has been completed. Thus the matrix sequences $\{\mathcal{X}_1^{(k)}, \mathcal{X}_2^{(k)}\}_{k=0}^{\infty}$ are nonnegative, monotonically increasing and bounded from above by $(\hat{X}_1, \hat{X}_2)$ (in the elementwise ordering). We denote $\lim_{k\to\infty}(\mathcal{X}_1^{(k)}, \mathcal{X}_2^{(k)}) = (\tilde{X}_1, \tilde{X}_2)$. By taking the limits in (7)-(10) it follows that $(\tilde{X}_1, \tilde{X}_2)$ is a solution of $\mathcal{R}_i(X_1, X_2) = 0, i = 1, 2$ with the property $\tilde{X}_i \leq \hat{X}_i, i = 1, 2$. Moreover, the matrix $-A + S_1\tilde{X}_1 + S_2\tilde{X}_2$ is a nonsingular M-matrix and therefore the matrix $A - S_1\tilde{X}_1 - S_2\tilde{X}_2$ is asymptotically stable.

Assuming there exits another nonnegative solution $\acute{\mathcal{X}} = \begin{pmatrix} \acute{X}_1 \\ \acute{X}_2 \end{pmatrix}$ to $\Re(\mathcal{X}) = 0$ with the property $\acute{\mathcal{X}} \leq \tilde{\mathcal{X}} = \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}$.

We consider

$$\begin{aligned}
& (\mathcal{D} - \tilde{\mathcal{X}}\mathcal{S})(\acute{\mathcal{X}} - \tilde{\mathcal{X}}) + (\acute{\mathcal{X}} - \tilde{\mathcal{X}})(A - \mathcal{S}\tilde{\mathcal{X}}) \\
= \; & \mathcal{D}\acute{\mathcal{X}} - \tilde{\mathcal{X}}\mathcal{S}\acute{\mathcal{X}} - (\mathcal{D} - \tilde{\mathcal{X}}\mathcal{S})\tilde{\mathcal{X}} + \acute{\mathcal{X}}A - \acute{\mathcal{X}}\mathcal{S}\tilde{\mathcal{X}} - \tilde{\mathcal{X}}(A - \mathcal{S}\tilde{\mathcal{X}}) \\
= \; & \mathcal{D}\acute{\mathcal{X}} - \tilde{\mathcal{X}}\mathcal{S}\acute{\mathcal{X}} + \acute{\mathcal{X}}A - \acute{\mathcal{X}}\mathcal{S}\tilde{\mathcal{X}} + \mathcal{Q} + \tilde{\mathcal{X}}\mathcal{S}\tilde{\mathcal{X}} \\
= \; & \acute{\mathcal{X}}\mathcal{S}\acute{\mathcal{X}} - \tilde{\mathcal{X}}\mathcal{S}\acute{\mathcal{X}} - \acute{\mathcal{X}}\mathcal{S}\tilde{\mathcal{X}} + \tilde{\mathcal{X}}\mathcal{S}\tilde{\mathcal{X}} \\
= \; & (\acute{\mathcal{X}} - \tilde{\mathcal{X}})\mathcal{S}(\acute{\mathcal{X}} - \tilde{\mathcal{X}}) \leq 0\,.
\end{aligned}$$

Since $(A - \mathcal{S}\tilde{\mathcal{X}})$ is asymptotically stable, we can apply Lemma 1.3 [14] and therefore conclude that the solution $(\acute{\mathcal{X}} - \tilde{\mathcal{X}})$ to the above matrix equation is unique nonnegative. Thus $\acute{\mathcal{X}} \geq \tilde{\mathcal{X}}$ which is a contradiction. We conclude $\acute{\mathcal{X}} = \tilde{\mathcal{X}}$.

The proof is complete. □

4 Numerical examples

We consider a two-player game and we apply the iterative methods on two numerical examples. The matrix coefficients A, B_i, Q_i and R_{ii} for $i = 1, 2$ are defined using the Matlab description. The numerical experiments are constructed following the approach applied in [14].

Example 1. (based on Example 1 from [2]) The matrix coefficients are:
A=abs(randn(n))/99; s=max(abs(eig(A)))+4.5; $\gamma = -5.0$;
for i=1:n, A(i,i)=-(A(i,i))-s; end
B_1 = abs(randn(n,1))/2;
B_2 = eye(n,n); B_2(n,n)=n/3;

Table 1. Example 1. Comparison between iterations with tol=1.0e-9.

	NI (5)			ALIDI (7)-(10)		
n	maxIt	avIt	CPU	maxIt	avIt	CPU
8	4	3.1	0.31s	6	5.01	0.125s
16	4	3.27	0.43s	10	5.5	0.19s
24	5	3.57	0.73s	17	6.67	0.37s
32	5	3.67	1.16s	14	8.06	0.73s

Q_1=zeros(n,n); Q_1(1,1)=n/2; Q_1(n,n)=1.5;
$Q_2 = 2 * Q_1$;
R_{11}=-1;
R_{22} = -eye(n,n); R_{22}(1,1)=-50; R_{22}(n,n)=-30;

We are executing Example 1 for different values of n, and 100 runs are completed for each value of n. We take $X_1^{(0)} = X_2^{(0)} = 0$ and thus $\mathcal{R}(X_1^{(0)}, X_2^{(0)}) = -Q_i \leq 0$, (i.e. the matrix is nonpositive). We take $n \times n$ matrices

$$\hat{X}_1 = \begin{pmatrix} 0.3 & 0.01 & \dots & 0.01 \\ 0.01 & 0.3 & \dots & 0.01 \\ \vdots & \vdots & \vdots & \vdots \\ 0.01 & 0.01 & \dots & 0.3 \end{pmatrix} \text{ and } \hat{X}_1 = \begin{pmatrix} 0.5 & 0.01 & \dots & 0.01 \\ 0.01 & 0.5 & \dots & 0.01 \\ \vdots & \vdots & \vdots & \vdots \\ 0.01 & 0.01 & \dots & 0.5 \end{pmatrix}.$$

The conditions of Lemma 3, and Lemma 4, and Theorem 2 are fulfilled for this choice of $\hat{X}_1, \hat{X}_2$, i.e. $X_1^{(0)} \leq \hat{X}_1, X_2^{(0)} \leq \hat{X}_2, \mathcal{R}(X_1^{(0)}, X_2^{(0)}) \leq 0$, and $\mathcal{R}(\hat{X}_1, \hat{X}_2) \geq 0, i = 1, 2$, and $-A + S_1\hat{X}_1 + S_2\hat{X}_2$ is a nonsingular M-matrix. The computed solution $\tilde{X}_1, \tilde{X}_2$ satisfies the inequality $\tilde{X}_1 \leq \hat{X}_1, \tilde{X}_2 \leq \hat{X}_2$, and the fact that $-A + S_1\tilde{X}_1 + S_2\tilde{X}_2$ is an M-matrix.

Table 1 explains the computational results for different values of n.

Example 2. The matrix coefficients are:
A=abs(randn(n))/10; s=max(abs(eig(A)))+4.5; $\gamma = -5.0$;
for i=1:n, A(i,i)=-(A(i,i))-s; end
B_1 = abs(randn(n,1))/2;
B_2 = eye(n,n); B_2(n,n)=abs(randn);
Q_1=zeros(n,n); Q_1(1,1)=n/2; Q_1(n,n)=1.5;
$Q_2 = 2 * Q_1$;
R_{11}=-1;
R_{22} = -eye(n,n); R_{22}(1,1)=-80; R_{22}(n,n)=-90;

Table 2 presents the computational results for different values of n.

Table 2. Example 2. Comparison between iterations with tol=1.0e-9.

	NI (5)			LINI (6)			ALIDI (7)-(10)		
n	maxIt	avIt	CPU	maxIt	avIt	CPU	maxIt	avIt	CPU
15	4	3.2	0.39s	7	5.16	0.17s	8	5.11	0.10s
25	4	3.38	0.73s	28	7.14	0.56s	12	6.9	0.29s
40	6	3.69	1.66s	14	8.9	1.16s	10	8.85	0.89s
55	6	3.66	3.25s	17	10.74	2.23s	22	10.8	2.03s

The Newton method converges to the solution quadratically and it provides the lower number of iteration steps while another iterations have the linear rate of convergence. Iteration (6) is the linearized modification of the Newton method. The numbers of iterations are bigger than the Newton method, however the CPU times (for different values of n) for (6) are less than the corresponding CPU times for the Newton method. In Table 2 are presented results from experiments with the introduced here linearized Sylvester iteration (7)-(10). This iteration is faster than the remain iterations.

5 Conclusion

We have made numerical experiments for computing the stabilizing solution to a nonsymmetric Riccati equation (4) and we have compared the numerical results. The numerical experiments confirm the effectiveness of the proposed ALIDI iteration. In addition, the ALIDI method can be naturally divided and applied on a two processors computer. Thus, the ALIDI iteration is an effective alternative method to the Newton method.

Acknowledgement. The paper was supported by the projects RD-08-107/06.02.2017 from the Shumen University, Bulgaria.

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