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FIXED POINTS FOR NONSPREADING-TYPE MULTI-VALUED MAPPINGS: EXISTENCE AND CONVERGENCE RESULTS*

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Abstract

In this paper, we introduce a new class of nonlinear multi-valued mappings which is called a *nonspreading-type mapping* in Hilbert spaces, and prove some properties and the existence results for the proposed mapping. Furthermore, we prove weak and strong convergence theorems for a finite family of nonspreading-type multi-valued mappings in Hilbert spaces. As applications, we give examples and numerical results to illustrate our iteration and results.

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keywords: Fixed point; nonspreading-type multi-valued mapping; weak convergence; strong convergence; Opial's condition.

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1 Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ and C be a nonempty convex subset of H. Let $\{x_n\}$ be a sequence in H. Now, we denote the weak convergence of $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and the strong convergence of $\{x_n\}$ to a point $x \in H$ by $x_n \to x$.

For a single-valued mapping $T : C \to H$, $I - T$ is said to be *demiclosed* at $y \in C$ if $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $(I - T)x_n \to y$ imply $(I - T)x = y$, where I denotes the identity mapping on H.

One of the fundamental and celebrated results in the theory of nonexpansive single-valued mappings is Browder's demiclosedness principle [5]. The principle is also valid in a space satisfying Opial's condition. It has been known that the demiclosedness principle plays a key role in studying the asymptotic and ergodic behaviour of a nonexpansive single-valued mapping (see for example [11, 16, 24, 31, 43]).

Since 1965, fixed point theorems and the existence of fixed points of single-valued mappings have been intensively studied and considered by many authors (see, for examples, [1, 4, 13, 15, 19, 40]).

Recall that a single-valued mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

On the other hand, in 2008, Kohsaka and Takahashi [21, 22] introduced class of mappings, which is called the class of *nonspreading mappings*.

Let H be a Hilbert space and C be nonempty closed convex subset of H. Then a mapping $T : C \to C$ is said to be *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$$

for all $x, y \in C$. Recently, Iemoto and Takahashi [17] showed that $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Ty, y - Ty \rangle$$

for all $x, y \in C$. Further, Takahashi [38] defined a class of nonlinear mappings, which is said to be *hybrid*, i.e.,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty \rangle$$

for all $x, y \in C$. Recently, Aoyama et al. [2] introduced a new class of nonlinear mappings in a Hilbert space containing the class of nonexpansive mappings, nonspreding mappings and hybrid mappings, i.e.,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\lambda\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$. This mapping is called a *λ-hybrid mapping*. They proved obtained necessary and sufficient conditions for the existence of fixed points of λ-hybrid mappings in Hilber spaces. For some more results on some new nonlinear mappings, refer to [27, 34, 35, 36].

A subset $C \subset H$ is said to be *proximinal* if, for all $x \in H$, there exists $y \in C$ such that

$$\|x - y\| = d(x, C) = \inf\{\|x - z\| : z \in C\}.$$

Let $CB(C)$, $K(C)$ and $P(C)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximinal bounded subset of C, respectively. The *Hausdorff metric* on $CB(C)$ is defined by

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B),\ \sup_{y \in B} d(y, A)\right\}$$

for all $A, B \in CB(C)$, where $d(x, B) = \inf_{b \in B}\|x - b\|$. A multi-valued mapping $T : C \to CB(C)$ is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|$$

for all $x, y \in C$. An element $p \in C$ is called a *fixed point* of a single-valued mapping $T : C \to C$ (resp., a multi-valued mapping $T : C \to CB(C)$) if $p = Tp$ (resp., $p \in Tp$). The fixed points set of T is denoted by $F(T)$. If $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\|$$

for all $x \in C$ and $p \in F(T)$, then T is said to be *quasi-nonexpansive*.

Let $T : C \to CB(H)$ be a multi-valued mapping. Then $I - T$ is said to be *demiclosed* at $y \in C$ if $\{x_n\} \subset C$ is a sequence such that $x_n \rightharpoonup x$ and $x_n - z_n \to y$, where $z_n \in Tx_n$, imply $x - y \in Tx$.

Fixed point theory of multi-valued mappings is much more complicated and difficult than the corresponding theory of single-valued mappings. So some classical fixed point theorems for single-valued mappings have already been extended to multi-valued mappings.

In 1968, Markin [26] firstly established the nonexpansive multivalued convergence results in Hilbert spaces. Banach's Contraction Principle was extended to a multi-valued contraction in 1969 by [28]. In 1974, one breakthrough was achieved by Lim using Edelstein's method of asymptotic centers [23]. In 1999, Kirk and Massa [20] obtained another important result for the existence fixed point theorem of nonexpansive multi-valued mappings. In 1999, Sahu [32] obtained the strong convergence theorems of the nonexpansive type and non-self multi-valued mappings in uniformly convex Banach spaces. In 2001, Xu [44] extended the result of Kirk and Massa [20] to a nonexpansive multi-valued non-self mapping and obtained the fixed point theorem. Some fixed point results for nonexpansive multi-valued mappings can be found in [3, 7, 8, 9, 10, 12, 18, 30, 33, 42] and references therein.

In this paper, we introduce a new multi-valued mapping which is called a *k-nonspreading multi-valued mapping* and prove some properties of this mapping for the existence results. Also, we prove some fixed point theorems and weak convergence theorems for this mapping under some conditions. Finally, we give examples with its numerical results to illustrate our main theorems.

2 Preliminaries and lemmas

In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$ (see, for instance, [39]). Further, we have

$$2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2$$

for all $x, y, z, w \in H$ (see [17] for more details). It is also known that a Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \in H$ with $y \neq x$.

Let C be a closed and convex subset of H. For all point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C ([14]).

The following results will be used for the proof of our main results in the sequel.

Condition(A) Let H be a Hilbert space and C be a subset of H. A multi-valued mapping $T : C \to CB(C)$ is said to satisfy *Condition* (A) if

$$\|x - p\| = d(x, Tp)$$

for all $x \in H$ and $p \in F(T)$.

Remark 1 We see that T satisfies Condition (A) if and only if $Tp = \{p\}$ for all $p \in F(T)$. It is known that the best approximation operator P_T, which is defined by

$$P_T x = \{y \in Tx : \|y - x\| = d(x, Tx)\},$$

also satisfies Condition (A).

Lemma 1 *Let H be a real Hilbert space. Then the following results hold:*

(1) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$ *for all* $x, y \in H$ *and* $t \in [0, 1]$.

(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ *for all* $x, y \in H$.

(3) *If* $\{x_n\}_{n=1}^{\infty}$ *is a sequence in* H *which converges weakly to* $z \in H$*, then we have*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2$$

for all $y \in H$.

Lemma 2 [14, 25] *Let C be a closed and convex subset of a real Hilbert space H and P_C be the metric projection from H onto C. For any $x \in H$ and $z \in C$, $z = P_C x$ if and only if*

$$\langle x - z, y - z\rangle \leq 0$$

for all $y \in C$.

Lemma 3 [41] *Let C be a nonempty closed convex subset of a Hilbert space H, $\{x_n\}$ be a bounded sequence in H and μ be a Banach limit. If $g : C \to R$ is defined by*

$$g(z) = \mu_n \|x_n - z\|^2$$

for all $z \in C$, then there exists a unique $z_0 \in C$ such that

$$g(z_0) = \min\{g(z) : z \in C\}.$$

Lemma 4 [6] *Let H be a real Hilbert space and let $x_i \in H$ for each $i = 1, 2, \cdots, m$. For each $i = 1, 2, \cdots, m$, if $\alpha_i \in (0, 1)$ with $\sum_{i=1}^m \alpha_i = 1$, the following identity holds:*

$$\left\| \sum_{i=1}^{m} \alpha_i x_i \right\|^2 = \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \sum_{1 \le i < j \le m} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 5 [37] *Let X be a Banach space which satisfies Opial's condition and $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

3 Main results

Let H be a real Hilbert space and C be a nonempty convex subset of H. A multi-valued mapping $T : C \to CB(C)$ is a *k-nonspreading* if there exists $k > 0$ such that

$$H(Tx, Ty)^2 \le k\big(d(Tx, y)^2 + d(x, Ty)^2\big)$$

for all $x, y \in C$. We say that a multi-valued mapping $T : C \to CB(C)$ is a *nonspreading-type* if $k = \frac{1}{2}$, i.e.,

$$2H(Tx, Ty)^2 \le d(Tx, y)^2 + d(x, Ty)^2 \tag{1}$$

for all $x, y \in C$.

It is easy to see that, if T is nonspreading-type, then T is nonspreading in the case of single-valued mappings (see [21, 22]). Moreover, if T is nonspreading-type and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. Indeed, for all $x \in C$ and $p \in F(T)$, we have

$$\begin{aligned} 2H(Tx, Tp)^2 &\le d(Tx, p)^2 + d(x, Tp)^2 \\ &\le H(Tx, Tp)^2 + \|x - p\|^2. \end{aligned}$$

Thus it follows that

$$H(Tx, Tp) \le \|x - p\|. \tag{2}$$

Now, we give an example of a nonspreading-type multi-valued mapping which is not a nonexpansive multi-valued mapping.

Example 1 Consider $C = [-3, 3]$ with the usual norm. Define a multi-valued mapping $T : C \to CB(C)$ by

$$Tx = \begin{cases} \{0\}, & x \in [-2, 2]; \\ \big[-\frac{|x|}{|x|+1}, \frac{|x|}{|x|+1}\big], & x \notin [-2, 2]. \end{cases}$$

To see that T is nonspreading-type, we observe the following cases:

Case 1: if $x, y \in [-2, 2]$, then $H(Tx, Tx) = 0$.

Case 2: if $x \in [-2, 2]$ and $y \notin [-2, 2]$, then $Tx = \{0\}$ and $Ty = \big[-\frac{|y|}{|y|+1}, \frac{|y|}{|y|+1}\big]$. This implies that

$$2H(Tx, Ty)^2 = 2\Big(\frac{|y|}{|y|+1}\Big)^2 < 2 < y^2 \leq d(Tx, y)^2 + d(x, Ty)^2.$$

Case 3: if $x, y \notin [-2, 2]$, then $Tx = \big[-\frac{|x|}{|x|+1}, \frac{|x|}{|x|+1}\big]$ and $Ty = \big[-\frac{|y|}{|y|+1}, \frac{|y|}{|y|+1}\big]$. This implies that

$$2H(Tx, Ty)^2 = 2\Big(\frac{|x|}{|x|+1} - \frac{|y|}{|y|+1}\Big)^2 < 2 < d(Tx, y)^2 + d(x, Ty)^2.$$

But T is not nonexpansive since for $x = 2$ and $y = \frac{5}{2}$, we have $Tx = \{0\}$ and $Ty = \big[-\frac{5}{7}, \frac{5}{7}\big]$. This implies that

$$H(Tx, Ty) = \frac{5}{7} > \frac{1}{2} = \big|2 - \frac{5}{2}\big| = \|x - y\|.$$

Let C be a nonempty set in a Hilbert space H. We define $T(C) = \cup_{x\in C} Tx$ and $(ST)x = S(Tx)$ for all $x \in C$.

Now, we present the following properties of a nonspreading-type multi-valued mapping.

Lemma 6 *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to CB(C)$ be a nonspreading-type multi-valued mapping with $F(T) \neq \emptyset$. Then $F(T)$ is closed.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x$ as $n \to \infty$. Then we have

$$\begin{aligned} d(x, Tx) &\leq \|x - x_n\| + d(x_n, Tx) \\ &\leq \|x - x_n\| + H(Tx_n, Tx) \\ &\leq 2\|x - x_n\|. \end{aligned}$$

It follows that $d(x, Tx) = 0$ and hence $x \in F(T)$. This completes the proof.

Lemma 7 *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to CB(C)$ be a nonspreading-type multi-valued mapping with $F(T) \neq \emptyset$. If T satisfies Condition (A), then $F(T)$ is convex.*

Proof. Let $p = tp_1 + (1-t)p_2$, where $p_1, p_2 \in F(T)$ and $t \in (0,1)$. Let $z \in Tp$. It follows from (2) that

$$\begin{aligned}
\|p-z\|^2 &= \|t(z-p_1) + (1-t)(z-p_2)\|^2 \\
&= t\|z-p_1\|^2 + (1-t)\|z-p_2\|^2 - t(1-t)\|p_1-p_2\|^2 \\
&= td(z,Tp_1)^2 + (1-t)d(z,Tp_2)^2 - t(1-t)\|p_1-p_2\|^2 \\
&\leq tH(Tp,Tp_1)^2 + (1-t)H(Tp,Tp_2)^2 - t(1-t)\|p_1-p_2\|^2 \\
&\leq t\|p-p_1\|^2 + (1-t)\|p-p_2\|^2 - t(1-t)\|p_1-p_2\|^2 \\
&= t(1-t)^2\|p_1-p_2\|^2 + (1-t)t^2\|p_1-p_2\|^2 - t(1-t)\|p_1-p_2\|^2 \\
&= 0
\end{aligned}$$

and hence $p = z$. Therefore, $p \in F(T)$. This completes the proof.

Lemma 8 *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a nonspreading-type multi-valued mapping. If $x, y \in C$ and $a \in Tx$, then there exists $b \in Ty$ such that*

$$\|a-b\|^2 \leq H(Tx,Ty)^2 \leq \|x-y\|^2 + 2\langle x-a, y-b\rangle.$$

Proof. Let $x, y \in C$ and $a \in Tx$. From Nadler's theorem (see [28]), there exists $b \in Ty$ such that

$$\|a-b\|^2 \leq H(Tx,Ty)^2.$$

It follows that

$$\begin{aligned}
2H(Tx,Ty)^2 &\leq d(Tx,y)^2 + d(x,Ty)^2 \\
&\leq \|a-y\|^2 + \|x-b\|^2 \\
&\leq \|a-x\|^2 + 2\langle a-x, x-y\rangle + \|x-y\|^2 + \|x-a\|^2 \\
&\quad +2\langle x-a, a-b\rangle + \|a-b\|^2 \\
&= 2\|a-x\|^2 + \|x-y\|^2 + \|a-b\|^2 \\
&\quad +2\langle a-x, x-a-(y-b)\rangle \\
&\leq 2\|a-x\|^2 + \|x-y\|^2 + H(Tx,Ty)^2 \\
&\quad +2\langle a-x, x-a-(y-b)\rangle.
\end{aligned}$$

This implies that

$$H(Tx,Ty)^2 \leq \|x-y\|^2 + 2\langle x-a, y-b\rangle.$$

This completes the proof.

Lemma 9 *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a nonspreading-type multi-valued mapping. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Then $p \in Tp$.*

Proof. Let $\{x_n\}$ be a sequence in C which converges weakly to p and $\{x_n - y_n\}$ converges strongly to 0 for some $y_n \in Tx_n$.

Now, we show that $p \in F(T)$. By Lemma 8, there exists $z_n \in Tp$ such that

$$\|y_n - z_n\|^2 \le \|x_n - p\|^2 + 2\langle x_n - y_n, p - z_n\rangle.$$

Since Tp is compact and $z_n \in Tp$, there exists $\{z_{n_i}\} \subset \{z_n\}$ such that $z_{n_i} \to z \in Tp$. Since $\{x_n\}$ converges weakly, it is bounded. For each $x \in H$, define a function $f : H \to [0, \infty)$ by

$$f(x) := \limsup_{i\to\infty} \|x_{n_i} - x\|^2.$$

Then, by Lemma 1(3), we obtain

$$f(x) = \limsup_{i\to\infty} \|x_{n_i} - p\|^2 + \|p - x\|^2$$

for all $x \in H$. Thus $f(x) = f(p) + \|p - x\|^2$ for all $x \in H$. It follows that

$$f(z) = f(p) + \|p - z\|^2. \tag{3}$$

We observe that

$$f(z) = \limsup_{i\to\infty} \|x_{n_i} - z\|^2 = \limsup_{i\to\infty} \|x_{n_i} - y_{n_i} + y_{n_i} - z\|^2 \le \limsup_{i\to\infty} \|y_{n_i} - z\|^2.$$

This implies that

$$\begin{aligned} f(z) &\le \limsup_{i\to\infty} \|y_{n_i} - z\|^2 \\ &= \limsup_{i\to\infty} \left(\|y_{n_i} - z_{n_i} + z_{n_i} - z\|\right)^2 \\ &\le \limsup_{i\to\infty} \left(\|x_{n_i} - p\|^2 + 2\langle x_{n_i} - y_{n_i}, p - z_{n_i}\rangle\right) \\ &= \limsup_{i\to\infty} \|x_{n_i} - p\|^2 \\ &= f(p). \end{aligned} \tag{4}$$

Hence it follows from (3) and (4) that $\|p - z\| = 0$. This completes the proof.

Now, by using lemmas, we give our main results in this paper.

Theorem 1 *Let C be a nonempty closed convex subset of a real Hilbert space H and T be a multi-valued mapping of C into $CB(C)$. Suppose that there exists an element $z_0 \in C$ and $z_n \in Tz_{n-1}$ for all $n \geq 1$ such that $\{z_n\}$ is bounded and, for all $y \in C$, there exists $a \in Ty$ such that*

$$\mu_n \|z_n - a\|^2 \leq \mu_n \|z_n - y\|^2$$

for a Banach limit μ. Then T has a fixed point in C.

Proof. Using a Banach limit μ on ℓ^∞, we can define a function $g : C \to R$ as follows:

$$g(y) := \mu_n \|z_n - y\|^2$$

for all $y \in C$. From Lemma 3, there exists a unique $y_0 \in C$ such that

$$g(y_0) = \min\{g(y) : y \in C\}.$$

So, there exist $a_0 \in Ty_0$ such that

$$g(a_0) = \mu_n \|z_n - a_0\|^2 \leq \mu_n \|z_n - y_0\|^2 = g(y_0).$$

Since $a_0 \in C$ and $y_0 \in C$ is a unique element such that

$$g(y_0) = \min\{g(y) : y \in C\},$$

we have $y_0 = a_0 \in Ty_0$. This completes the proof.

Theorem 2 *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to K(C)$ be a nonspreading-type multi-valued mapping. Then the following are equivalent:*

(1) *There exists $z_0 \in C$ and $z_n \in Tz_{n-1}$ for all $n \geq 1$ such that $\{z_n\}$ is bounded.*

(2) *$F(T)$ is nonempty.*

Proof. It is obvious that (2) implies (1). Now, we show that (1) implies (2). Assume that there exists $z_0 \in C$ and $z_n \in Tz_{n-1}$ for all $n \geq 1$ such that $\{z_n\}$ is bounded. Let $y \in C$. From Lemma 8, there exists $b \in Ty$ such that

$$\begin{aligned} & \|z_{n+1} - b\|^2 \leq \|z_n - y\|^2 + 2\langle z_n - z_{n+1}, y - b\rangle \\ \iff \ & \|z_{n+1} - b\|^2 \leq \|z_n - y\|^2 + \big(\|z_n - b\|^2 + \|z_{n+1} - y\|^2 - \|z_n - y\|^2 \\ & \qquad - \|z_{n+1} - b\|^2\big) \\ \iff \ & 2\|z_{n+1} - b\|^2 - \|z_n - b\|^2 \leq \|z_{n+1} - y\|^2. \end{aligned}$$

Let μ be a Banach limit on ℓ^∞. For any $n \in N$, we have

$$2\mu_n\|z_{n+1} - b\|^2 - \mu_n\|z_n - b\|^2 \leq \mu_n\|z_{n+1} - y\|^2.$$

This implies that

$$\mu_n\|z_n - b\|^2 \leq \mu_n\|z_n - y\|^2.$$

By Theorem 1, T has a fixed point in C. This completes the proof.

Next, we prove weak and strong convergence theorems for nonspreading-type multi-valued mappings in a Hilbert space H.

Theorem 3 *Let C be a nonempty closed convex subset of a real Hilbert space H. For each $i = 1, 2, \cdots, m$, let $T_i : C \to CB(C)$ be a family of nonspreading-type multi-valued mappings such that $\cap_{i=1}^m F(T_i) \neq \emptyset$. Let $\alpha_{i,n} \in (0, 1)$ for each $i = 0, 1, 2, \cdots, m$ and $\sum_{i=0}^m \alpha_{i,n} = 1$ for each $n \geq 1$. Define a sequence $\{x_n\}$ by $x_1 \in C$ arbitrary and*

$$x_{n+1} \in \alpha_{0,n}x_n + \sum_{i=1}^m \alpha_{i,n}T_ix_n \tag{5}$$

for each $n \geq 1$. Assume that the following conditions hold:

(a) *T_i satisfies Condition (A) for each $i = 1, 2, \cdots, m$;*

(b) *$\liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} > 0$ for each $i = 1, 2, \cdots, m$.*

Then, for each $i = 1, 2, \cdots, m$, $\lim_{n\to\infty} d(x_n, T_ix_n) = 0$.

Proof. Let $p \in \bigcap_{i=1}^m F(T_i)$. Since T_i satisfies Condition (A) for all $i = 1, 2, \cdots, m$, for some $y_n^i \in T_ix_n$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_{0,n}(x_n - p) + \sum_{i=1}^m \alpha_{i,n}(y_n^i - p)\| \\
&\leq \alpha_{0,n}\|x_n - p\| + \sum_{i=1}^m \alpha_{i,n}\|y_n^i - p\| \\
&= \alpha_{0,n}\|x_n - p\| + \sum_{i=1}^m \alpha_{i,n}d(y_n^i, T_ip) \\
&\leq \alpha_{0,n}\|x_n - p\| + \sum_{i=1}^m \alpha_{i,n}H(T_ix_n, T_ip) \\
&\leq \|x_n - p\|. \qquad (6)
\end{aligned}$$

Hence $\lim_{n\to\infty}\|x_n - p\|$ exists. This implies that $\{x_n\}$ is bounded. From Lemma 4, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_{0,n}(x_n-p)+\sum_{i=1}^{m}\alpha_{i,n}(y_n^i-p)\|^2,\ y_n^i\in T_ix_n \\
&= \alpha_{0,n}\|x_n-p\|^2+\sum_{i=1}^{m}\alpha_{i,n}\|y_n^i-p\|^2-\sum_{i=1}^{m}\alpha_{0,n}\alpha_{i,n}\|x_n-y_n^i\|^2 \\
&= \alpha_{0,n}\|x_n-p\|^2+\sum_{i=1}^{m}\alpha_{i,n}d(y_n^i,T_ip)^2-\sum_{i=1}^{m}\alpha_{0,n}\alpha_{i,n}\|x_n-y_n^i\|^2 \\
&\le \alpha_{0,n}\|x_n-p\|^2+\sum_{i=1}^{m}\alpha_{i,n}H(T_ix_n^i,T_ip)^2-\sum_{i=1}^{m}\alpha_{0,n}\alpha_{i,n}\|x_n-y_n^i\|^2 \\
&\le \|x_n-p\|^2-\sum_{i=1}^{m}\alpha_{0,n}\alpha_{i,n}\|x_n-y_n^i\|^2.
\end{aligned}$$

It follows that

$$\sum_{i=1}^{m}\alpha_{0,n}\alpha_{i,n}\|x_n-y_n^i\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2.$$

Since $\liminf_{n\to\infty}\alpha_{0,n}\alpha_{i,n}>0$ for each $i=1,2,\cdots,m$, we obtain

$$\lim_{n\to\infty}\|x_n-y_n^i\|=0, \tag{7}$$

which implies that

$$\lim_{n\to\infty}d(x_n,T_ix_n)\le\lim_{n\to\infty}\|x_n-y_n^i\|=0 \tag{8}$$

for each $i=1,2,\cdots,m$. This completes the proof.

Theorem 4 *Let C be a nonempty closed convex subset of a real Hilbert space H. For each $i=1,2,\cdots,m$, let $T_i: C\to CB(C)$ be a finite family of nonspreading-type multi-valued mappings such that $\bigcap_{i=1}^m F(T_i)\neq\emptyset$. Let $\alpha_{i,n}\in(0,1)$ for each $i=0,1,2,\cdots,m$ and $\sum_{i=0}^m\alpha_{i,n}=1$ for each $n\ge1$. Assume that the following conditions hold:*

(a) *T_i satisfies Condition (A) for each $i=1,2,\cdots,m$;*
(b) *$\liminf_{n\to\infty}\alpha_{0,n}\alpha_{i,n}>0$ for each $i=1,2,\cdots,m$.*

Then the sequence $\{x_n\}$ defined by (5) converges weakly to a common fixed point of the family $\{T_i: i=1,2,\cdots,m\}$.

Proof. By Theorem 3, we know that $\{x_n\}$ is bounded. Thus there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q \in C$. From (7), $\lim_{n\to\infty} \|x_n - y_n^i\| = 0$ for each $i = 1, 2, \cdots, m$. By Lemma 9, we obtain $q \in \cap_{i=1}^m F(T_i)$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$. Using Lemma 9, we get $p \in \bigcap_{i=1}^m F(T_i)$. Thus, applying Lemma 5, we obtain $p = q$. This completes the proof.

Theorem 5 *Under the hypotheses of Theorem 3, assume that one of T_i is completely continuous. Then the iterative sequence $\{x_n\}$ defined by* (5) *converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \cdots, m\}$.*

Proof. Suppose that T_{i_0} is completely continuous for some $i_0 \in \{1, 2, \cdots, m\}$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} d(T_{i_0}x_{n_k}, p) = 0$ for some $p \in C$. It follows from (8) that

$$\|x_{n_k} - p\| \leq d(x_{n_k}, T_{i_0}x_{n_k}) + d(T_{i_0}x_{n_k}, p) \to 0 \tag{9}$$

as $k \to \infty$. From Lemma 8, for any $y_{n_k^i} \in T_i x_{n_k}$, there exists $b_{n_k}^i \in T_i p$ such that

$$\begin{aligned} H(T_i x_{n_k}, T_i p)^2 &\leq \|x_{n_k} - p\|^2 + 2\langle x_{n_k} - y_{n_k}^i, p - b_{n_k}^i\rangle \\ &\leq \|x_{n_k} - p\|^2 + 2\|x_{n_k} - y_{n_k}^i\|\|p - b_{n_k}^i\|. \end{aligned}$$

Thus it follows from (7) that

$$\lim_{k\to\infty} H(T_i x_{n_k}, T_i p) = 0 \tag{10}$$

for each $i \in \{1, 2, \cdots, m\}$. For each $i \in \{1, 2, \cdots, m\}$, we have

$$d(p, T_i p) \leq \|p - x_{n_k}\| + d(x_{n_k}, T_i x_{n_k}) + H(T_i x_{n_k}, T_i p). \tag{11}$$

From (8), (9) and (10), we obtain $d(p, T_i p) = 0$ for each $i \in \{1, 2, \cdots, m\}$. Since $T_i p$ is closed, we have $p \in \cap_{i=1}^m F(T_i)$. By Theorem 3, it follows that $\lim_{n\to\infty} \|x_n - p\|$ exists. This implies that $\lim_{n\to\infty} \|x_n - p\| = 0$. This completes the proof.

Theorem 6 *Under the hypotheses of Theorem 3, assume that one of T_i is hemicompact. Then the iterative sequence $\{x_n\}$ defined by* (5) *converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \cdots, m\}$.*

Proof. Suppose that T_{i_0} is hemicompact for some $i_0 \in \{1, 2, \cdots, m\}$. From (8), we have

$$\lim_{n\to\infty} d(x_n, T_{i_0}x_n) = 0.$$

Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in C$. From Lemma 8, for any $y_{n_k^i} \in T_i x_{n_k}$, there exists $b^i_{n_k} \in T_i p$ such that

$$\begin{aligned} H(T_i x_{n_k}, T_i p)^2 &\leq \|x_{n_k} - p\|^2 + 2\langle x_{n_k} - y^i_{n_k}, p - b^i_{n_k}\rangle \\ &\leq \|x_{n_k} - p\|^2 + 2\|x_{n_k} - y^i_{n_k}\|\|p - b^i_{n_k}\|. \end{aligned}$$

Thus it follows from (7) that

$$\lim_{k\to\infty} H(T_i x_{n_k}, T_i p) = 0 \tag{12}$$

for each $i \in \{1, 2, \cdots, m\}$. For each $i \in \{1, 2, \cdots, m\}$, we have

$$d(p, T_i p) \leq \|p - x_{n_k}\| + d(x_{n_k}, T_i x_{n_k}) + H(T_i x_{n_k}, T_i p). \tag{13}$$

Since $x_{n_k} \to p$, by (8), (12) and (13), we obtain $d(p, T_i p) = 0$ for each $i \in \{1, 2, \cdots, m\}$. Since $T_i p$ is closed, we have $p \in \bigcap_{i=1}^m F(T_i)$. By Theorem 3, it follows that $\lim_{n\to\infty} \|x_n - p\|$ exists. This implies that $\lim_{n\to\infty} \|x_n - p\| = 0$. This completes the proof.

If, for each $i \in \{1, 2, \cdots, m\}$, $T_i p = \{p\}$ for all $p \in F(T_i)$, then T_i satisfies Condition (A) for each $i \in \{1, 2, \cdots, m\}$. Then we obtain the following results:

Corollary 1 *Let C be a nonempty closed convex subset of a real Hilbert space H. For each $i = 1, 2, \cdots, m$, let $T_i : C \to CB(C)$ be a finite family of nonspreading-type multi-valued mappings such that $\bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let $\alpha_{i,n} \in (0, 1)$ for each $i = 0, 1, 2, \cdots, m$ and $\sum_{i=0}^m \alpha_{i,n} = 1$ for each $n \geq 1$. Define a sequence $\{x_n\}$ by $x_1 \in C$ arbitrary and*

$$x_{n+1} \in \alpha_{0,n} x_n + \sum_{i=1}^m \alpha_{i,n} T_i x_n \tag{14}$$

for each $n \geq 1$. Assume that the following conditions hold:

(a) *for each $i \in \{1, 2, \cdots, m\}$, $T_i p = \{p\}$ for all $p \in F(T_i)$;*

(b) $\liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} > 0$ *for each $i = 1, 2, \cdots, m$.*

Then, for each $i = 1, 2, \cdots, m$, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$.

Corollary 2 *Let C be a nonempty closed convex subset of a real Hilbert space H. For each $i = 1, 2, \cdots, m$, let $T_i : C \to CB(C)$ be a finite family of nonspreading-type multi-valued mappings such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\alpha_{i,n} \in (0,1)$ for each $i = 0, 1, 2, \cdots, m$ and $\sum_{i=0}^{m} \alpha_{i,n} = 1$ for each $n \geq 1$. Assume that the following conditions hold:*

(a) *for each $i \in \{1, 2, \cdots, m\}$, $T_i p = \{p\}$ for all $p \in F(T_i)$;*

(b) $\liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} > 0$ *for each $i = 1, 2, \cdots, m$.*

Then the sequence $\{x_n\}$ defined by (14) converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, \cdots, m\}$.

Corollary 3 *Under the hypotheses of Corollary 1, assume that one of T_i is completely continuous. Then the iterative sequence $\{x_n\}$ defined by (14) converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \cdots, m\}$.*

Corollary 4 *Under the hypotheses of Corollary 1, assume that one of T_i is hemicompact. Then the iterative sequence $\{x_n\}$ defined by (14) converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \cdots, m\}$.*

Since P_{T_i} satisfies Condition (A) for each $i \in \{1, 2, \cdots, m\}$, we also obtain the following results:

Corollary 5 *Let C be a nonempty closed convex subset of a real Hilbert space H. For each $i = 1, 2, \cdots, m$, let $T_i : C \to CB(C)$ be a finite family of multi-valued mappings such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\alpha_{i,n} \in (0,1)$ for each $i = 0, 1, 2, \cdots, m$ and $\sum_{i=0}^{m} \alpha_{i,m} = 1$ for each $n \geq 1$. Define a sequence $\{x_n\}$ by $x_1 \in C$ arbitrary and*

$$x_{n+1} \in \alpha_{0,n} x_n + \sum_{i=1}^{m} \alpha_{i,n} P_{T_i} x_n \tag{15}$$

for each $n \geq 1$. Assume that the following conditions hold:

(a) *for each $i \in \{1, 2, \cdots, m\}$, P_{T_i} is a nonspreading-type multi-valued mapping;*

(b) $\liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} > 0$ *for each $i = 1, 2, \cdots, m$.*

Then, for each $i = 1, 2, \cdots, m$, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$.

*Proof.*By the same proof as in Theorem 3, we have $x_n \to y_n^i \in P_{T_i} x_n$. This implies that

$$d(x_n, T_i x_n) \leq d(x_n, P_{T_i} x_n) \leq \|x_n - y_n^i\| \to 0 \tag{16}$$

as $n \to \infty$ for each $i \in \{1, 2, \cdots, m\}$.

Corollary 6 *Let C be a nonempty closed convex subset of a real Hilbert space H. For each $i = 1, 2, \cdots, m$, let $T_i : C \to CB(C)$ be a finite family of multi-valued mappings such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and $I - T_i$ is demiclosed at 0. Let $\alpha_{i,n} \in (0, 1)$ for each $i = 0, 1, 2, \cdots, m$ and $\sum_{i=0}^{m} \alpha_{i,m} = 1$ for each $n \geq 1$. Assume that the following conditions hold:*

(a) *for each $i \in \{1, 2, \cdots, m\}$, P_{T_i} is a nonspreading-type multi-valued mapping;*

(b) $\liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} > 0$ *for each $i = 1, 2, \cdots, m$.*

Then the sequence $\{x_n\}$ defined by (15) *converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, \cdots, m\}$.*

Proof. From $I - T_i$ is demiclosed at 0 and (16) for each $i \in \{1, 2, \cdots, m\}$, we obtain the result.

Corollary 7 *Under the hypotheses of Corollary 5, assume that one of T_i is completely continuous. Then the iterative sequence $\{x_n\}$ defined by* (15) *converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \cdots, m\}$.*

Proof. Suppose that T_{i_0} is completely continuous for some $i_0 \in \{1, 2, \cdots, m\}$. Since $\{x_n\}$ is bounded, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} d(T_{i_0}x_{n_k}, p) = 0$ for some $p \in C$. It follows from (16) that

$$\|x_{n_k} - p\| \leq d(x_{n_k}, T_{i_0}x_{n_k}) + d(T_{i_0}x_{n_k}, p) \to 0 \tag{17}$$

as $k \to \infty$. From Lemma 8, for any $y_{n_k^i} \in P_{T_i}x_{n_k}$, there exists $b^i_{n_k} \in P_{T_i}p$ such that

$$\begin{aligned} H(P_{T_i}x_{n_k}, P_{T_i}p)^2 &\leq \|x_{n_k} - p\|^2 + 2\langle x_{n_k} - y^i_{n_k}, p - b^i_{n_k}\rangle \\ &\leq \|x_{n_k} - p\|^2 + 2\|x_{n_k} - y^i_{n_k}\|\|p - b^i_{n_k}\|. \end{aligned}$$

Thus it follows from (16) that

$$\lim_{k\to\infty} H(P_{T_i}x_{n_k}, P_{T_i}p) = 0 \tag{18}$$

for each $i \in \{1, 2, \cdots, m\}$. For each $i \in \{1, 2, \cdots, m\}$, we have

$$d(p, T_ip) \leq d(p, P_{T_i}p) \leq \|p - x_{n_k}\| + d(x_{n_k}, P_{T_i}x_{n_k}) + H(P_{T_i}x_{n_k}, P_{T_i}p). \tag{19}$$

From (16), (17) and (18), we obtain $d(p, T_ip) = 0$ for each $i \in \{1, 2, \cdots, m\}$. Since T_ip is closed, we have $p \in \bigcap_{i=1}^{m} F(T_i)$. By Theorem 3, it follows that $\lim_{n\to\infty} \|x_n - p\|$ exists. This implies that $\lim_{n\to\infty} \|x_n - p\| = 0$. This completes the proof.

Corollary 8 *Under the hypotheses of Corollary 5, assume that one of T_i is hemicompact. Then the iterative sequence $\{x_n\}$ defined by* (15) *converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \cdots, m\}$.*

Proof. Suppose that T_{i_0} is hemicompact for some $i_0 \in \{1, 2, \cdots, m\}$. From (16), we have

$$\lim_{n\to\infty} d(x_n, P_{T_{i_0}} x_n) = 0.$$

Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in C$. From Lemma 8, for any $y_{n_k^i} \in P_{T_i} x_{n_k}$, there exists $b^i_{n_k} \in P_{T_i} p$ such that

$$\begin{aligned} H(P_{T_i} x_{n_k}, P_{T_i} p)^2 &\leq \|x_{n_k} - p\|^2 + 2\langle x_{n_k} - y^i_{n_k}, p - b^i_{n_k}\rangle \\ &\leq \|x_{n_k} - p\|^2 + 2\|x_{n_k} - y^i_{n_k}\|\|p - b^i_{n_k}\|. \end{aligned}$$

Thus it follows from (16) that

$$\lim_{k\to\infty} H(P_{T_i} x_{n_k}, P_{T_i} p) = 0 \tag{20}$$

for each $i \in \{1, 2, \cdots, m\}$. For each $i \in \{1, 2, \cdots, m\}$, we have

$$d(p, T_i p) \leq d(p, P_{T_i} p) \leq \|p - x_{n_k}\| + d(x_{n_k}, P_{T_i} x_{n_k}) + H(P_{T_i} x_{n_k}, P_{T_i} p). \tag{21}$$

Since $x_{n_k} \to p$, by (16) and (20), we obtain $d(p, T_i p) = 0$ for each $i \in \{1, 2, \cdots, m\}$. Since $T_i p$ is closed, we have $p \in \cap_{i=1}^m F(T_i)$. By Theorem 3, it follows that $\lim_{n\to\infty} \|x_n - p\|$ exists. This implies that $\lim_{n\to\infty} \|x_n - p\| = 0$. This completes the proof.

4 Examples and Numerical Results

In this section, we give examples and numerical results to illustrate Theorem 4.

Example 2 Let $H = R$ and $C = [-3, 0]$. Let

$$T_1 x = \begin{cases} \{-3\}, & x \in [-3, -1]; \\ \big[-3, \log(x+4) - 3\big], & x \in (-1, 0], \end{cases}$$

and

$$T_2 x = \begin{cases} \{-3\}, & x \in [-3, -1]; \\ \big[-3, -2 - \frac{|x|}{|x|+1}\big], & x \in (-1, 0]. \end{cases}$$

Choose $\alpha_{0,n} = \frac{18n-1}{20n}$ and $\alpha_{1,n} = \alpha_{2,n} = \frac{2n-1}{40n}$. We know that T_1 and T_2 are nonspreading-type multi-valued mappings. It is easy to check that F_1 and F_2 satisfy all the conditions in Theorem 4, T_1, T_2 satisfy Condition (A) such that $F(T_1) \cap F(T_2) = \{-3\}$. Thus we compute the sequence $\{x_n\}$ by the following iteration:

$$x_{n+1} = \Big(\frac{18n-1}{20n}\Big)x_n + \Big(\frac{2n-1}{40n}\Big)y_n + \Big(\frac{2n-1}{40n}\Big)z_n,$$

where

$$y_n \in \begin{cases} \{-3\}, & x_n \in [-3,-1]; \\ \big[-3, \log(x_n+4)-3\big], & x_n \in (-1,0], \end{cases}$$

and

$$z_n \in \begin{cases} \{-3\}, & x_n \in [-3,-1]; \\ \big[-3, -2-\frac{|x_n|}{|x_n|+1}\big], & x_n \in (-1,0]. \end{cases}$$

Choose $x_1 = 0$ and take randomly y_n and z_n in the above intervals, we have the following:

n	y_n	z_n	x_n	$\|x_{n+1} - x_n\|$
1	-2.69764E+00	-2.46309E+00	0.00000E+00	1.88163E-01
2	-2.98925E+00	-2.45848E+00	-1.29018E-01	1.66372E-01
3	-2.44224E+00	-2.43879E+00	-3.17181E-01	1.82720E-01
4	-2.48261E+00	-2.93727E+00	-4.83553E-01	1.65138E-01
5	-2.55020E+00	-2.74820E+00	-6.66273E-01	1.63008E-01
6	-2.94919E+00	-2.57249E+00	-8.31410E-01	1.52187E-01
7	-2.89553E+00	-2.67715E+00	-9.94418E-01	1.59423E-01
8	-3.00000E+00	-3.00000E+00	-1.14660E+00	1.45475E-01
9	-3.00000E+00	-3.00000E+00	-1.30603E+00	1.32592E-01
10	-3.00000E+00	-3.00000E+00	-1.45150E+00	1.20754E-01
⋮	⋮	⋮	⋮	⋮
500	-3.00000E+00	-3.00000E+00	-2.99388E+00	1.24072E-05

Table 1: Numerical results of Example 2 being randomized in the first time.

n	y_n	z_n	x_n	$\|x_{n+1} - x_n\|$
1	-2.94261E+00	-2.44324E+00	0.00000E+00	1.87934E-01
2	-2.83431E+00	-2.62610E+00	-1.34646E-01	1.94127E-01
3	-2.65057E+00	-2.91171E+00	-3.22581E-01	1.65696E-01
4	-2.54458E+00	-2.57143E+00	-5.16708E-01	1.78607E-01
5	-2.97085E+00	-2.66629E+00	-6.82404E-01	1.70002E-01
6	-2.88750E+00	-2.85675E+00	-8.61011E-01	1.68106E-01
7	-3.00000E+00	-3.00000E+00	-1.03101E+00	1.53844E-01
8	-3.00000E+00	-3.00000E+00	-1.19912E+00	1.40521E-01
9	-3.00000E+00	-3.00000E+00	-1.35296E+00	1.28184E-01
10	-3.00000E+00	-3.00000E+00	-1.49348E+00	1.16826E-01
⋮	⋮	⋮	⋮	⋮
500	-3.00000E+00	-3.00000E+00	-2.99388E+00	1.24072E-05

Table 2: Numerical results of Example 2 being randomized in the second time.

From Table 1 and Table 2, we see that -3 is the common fixed point of T_1 and T_2 in Example 2.

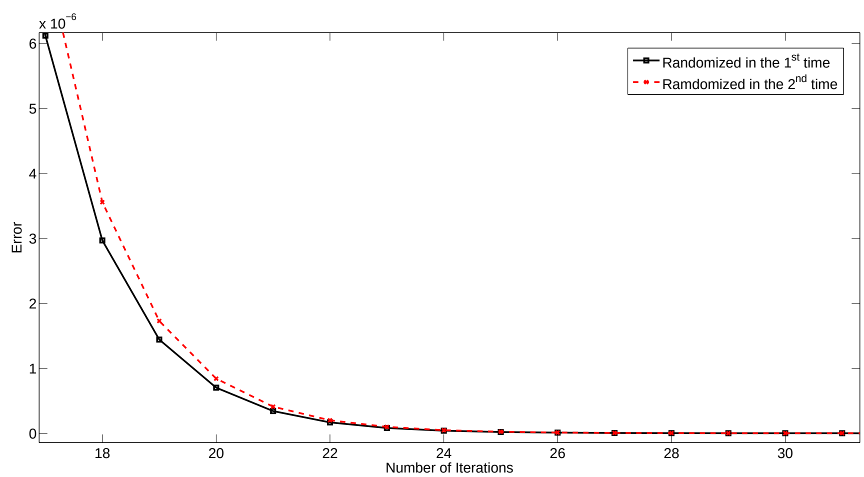

Figure 1: Error plots for all sequences $\{x_n\}$ in Table 1 and Table 2.

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