Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 10, No. 2/2018

ESSENTIAL NORM ESTIMATES FOR LITTLE HANKEL OPERATORS ON $L^2_a(\mathbb{C}_+)^*$

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Abstract

In this paper, we give estimates for the essential norm of a bounded little Hankel operator defined on the Bergman space of the right half plane. As an application of these estimates, we also give a necessary and sufficient condition for the little Hankel operator to be compact.

MSC: 47B35, 30H20.

keywords: Bergman space; right half plane; essential norm; little Hankel operators; automorphism.

1 Introduction

Let $\mathbb{C}_+ = \{s = x + iy \in \mathbb{C} : \text{Re}s > 0\}$ be the right half plane. Let $d\mu(s) = dxdy$ be the area measure. Let $L^2(\mathbb{C}_+, d\mu)$ be the space of complex-valued, square-integrable, measurable functions on $\mathbb{C}_+$ with respect to the area measure. Let $L^2_a(\mathbb{C}_+)$ be the closed subspace [1] of $L^2(\mathbb{C}_+, d\mu)$ consisting of those functions in $L^2(\mathbb{C}_+, d\mu)$ that are analytic. The space $L^2_a(\mathbb{C}_+)$ is referred to as the Bergman space of the right half plane. The functions

*accepted for publication in revised form on July 12, 2018
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$H(s,w) = \frac{1}{(s+\overline{w})^2}, s \in \mathbb{C}_+, w \in \mathbb{C}_+$ is the reproducing kernel [2] for $L^2_a(\mathbb{C}_+)$. Let $L^\infty(\mathbb{C}_+)$ be the space of complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{C}_+$. For $f \in L^\infty(\mathbb{C}_+), ||f||_\infty = \text{ess} \sup_{s\in\mathbb{C}_+} |f(s)| < \infty$. The space $L^\infty(\mathbb{C}_+)$ is a Banach space with respect to the essential supremum norm. For $\phi \in L^\infty(\mathbb{C}_+)$, we define the multiplication operator $\mathcal{M}_\phi$ from $L^2(\mathbb{C}_+, d\mu)$ into $L^2(\mathbb{C}_+, d\mu)$ by $(\mathcal{M}_\phi f)(s) = \phi(s)f(s)$ and the little Hankel operator $\hbar_\phi$ is a mapping from $L^2_a(\mathbb{C}_+)$ into $\overline{L^2_a(\mathbb{C}_+)}$ defined by $\hbar_\phi f = \overline{P}_+(\phi f)$, where $\overline{P}_+$ is the projection operator from $L^2(\mathbb{C}_+, d\mu)$ onto $\overline{L^2_a(\mathbb{C}_+)} = \{\overline{f} : f \in L^2_a(\mathbb{C}_+)\}$. There are also many equivalent ways of defining little Hankel operators on $L^2_a(\mathbb{C}_+)$. Let $\mathcal{S}_\phi$ be the mapping from $L^2_a(\mathbb{C}_+)$ into $L^2_a(\mathbb{C}_+)$ defined by $\mathcal{S}_\phi f = P_+(\mathcal{J}(\phi f))$, where P_+ denote the orthogonal projection from $L^2(\mathbb{C}_+, d\mu)$ onto $L^2_a(\mathbb{C}_+)$ and $\mathcal{J}$ is the mapping from $L^2(\mathbb{C}_+, d\mu)$ into $L^2(\mathbb{C}_+, d\mu)$ such that $\mathcal{J}f(s) = f(\overline{s})$. Notice that $\mathcal{J}$ is unitary and $\mathcal{J}\mathcal{S}_\phi f = \mathcal{J}(P_+(\mathcal{J}(\phi f))) = \mathcal{J}P_+\mathcal{J}(\phi f) = \overline{P}_+(\phi f) = \hbar_\phi f$ for $f \in L^2_a(\mathbb{C}_+)$. Let $\mathbf{\Gamma}_\phi$ be the mapping from $L^2_a(\mathbb{C}_+)$ into $L^2_a(\mathbb{C}_+)$ defined by $\mathbf{\Gamma}_\phi f = P_+\mathcal{M}_\phi\mathcal{J}f$. Thus $\mathbf{\Gamma}_\phi f = P_+\mathcal{M}_\phi\mathcal{J}f = P_+(\phi(s)f(\overline{s})) = P_+(\mathcal{J}(\phi(\overline{s})f(s))) = \mathcal{S}_{\mathcal{J}\phi}f$ for all $f \in L^2_a(\mathbb{C}_+)$. Hence $\mathbf{\Gamma}_\phi f = \mathcal{S}_{\mathcal{J}\phi}f$. Thus we obtain $\hbar_\phi = \mathcal{J}\mathcal{S}_\phi$ and $\mathbf{\Gamma}_\phi = \mathcal{S}_{\mathcal{J}\phi}$. Since $\mathcal{J}$ is unitary, the three operators $\hbar_\phi, \mathcal{S}_\phi$ and $\mathbf{\Gamma}_\phi$ are referred to as little Hankel operators on $L^2_a(\mathbb{C}_+)$ and a given result on little Hankel operators can be stated using the operators $\hbar_\phi, \mathcal{S}_\phi$ and $\mathbf{\Gamma}_\phi$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $L^2(\mathbb{D}, dA)$ be the space of complex- valued, square-integrable, measurable functions on $\mathbb{D}$ with respect to the normalized area measure $dA(z) = \frac{1}{\pi}dxdy$. Let $L^2_a(\mathbb{D})$ be the space consisting of those functions of $L^2(\mathbb{D}, dA)$ that are analytic. The space $L^2_a(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, dA)$ and is called the Bergman space of the open unit disk $\mathbb{D}$. Let $L^\infty(\mathbb{D})$ be the space of complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{D}$ with the essential supremum norm. For $\phi \in L^\infty(\mathbb{D})$, the multiplication operator M_ϕ from $L^2(\mathbb{D}, dA)$ into $L^2(\mathbb{D}, dA)$ is defined by $M_\phi f = \phi f$ and the little Hankel operator h_ϕ is a mapping from $L^2_a(\mathbb{D})$ into $\overline{L^2_a(\mathbb{D})}$ defined by $h_\phi f = \overline{P}(\phi f)$, where $\overline{P}$ is the projection operator from $L^2(\mathbb{D}, dA)$ onto $\overline{L^2_a(\mathbb{D})} = \{\overline{f} : f \in L^2_a(\mathbb{D})\}$. Let S_ϕ be the mapping from $L^2_a(\mathbb{D})$ into $L^2_a(\mathbb{D})$ defined by $S_\phi f = P(J(\phi f))$, where P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$ and J is the mapping from $L^2(\mathbb{D}, dA)$ into itself such that $Jf(z) = f(\overline{z})$. Notice that J is unitary and $JS_\phi f = J(P(J(\phi f))) = JPJ(\phi f) = \overline{P}(\phi f) = h_\phi f$ for all $f \in L^2_a(\mathbb{D})$. Let Γ_ϕ be the mapping from $L^2_a(\mathbb{D})$ into $L^2_a(\mathbb{D})$ defined by $\Gamma_\phi f = PM_\phi Jf$, where M_ϕ

is the mapping from $L^2(\mathbb{D}, dA)$ into $L^2(\mathbb{D}, dA)$ defined by $M_\phi f = \phi f$. Thus $\Gamma_\phi f = PM_\phi Jf = P(\phi(z)f(\overline{z})) = P(J(\phi(\overline{z})f(z))) = S_{J\phi} f$ for all $f \in L^2_a(\mathbb{D})$. Hence $\Gamma_\phi = S_{J\phi}$. Since J is unitary, the three operators h_ϕ, S_ϕ and Γ_ϕ are referred to as little Hankel operators on $L^2_a(\mathbb{D})$. The sequence of functions $\{e_n(z)\}_{n=0}^{\infty} = \{\sqrt{n+1}z^n\}_{n=0}^{\infty}$ form an orthonormal basis for $L^2_a(\mathbb{D})$. Since point evaluation at $z \in \mathbb{D}$ is a bounded linear functional on the Hilbert space $L^2_a(\mathbb{D})$, the Riesz representation theorem implies that there exists a unique function K_z in $L^2_a(\mathbb{D})$ such that

$$f(z) = \int_{\mathbb{D}} f(w)\overline{K_z(w)}dA(w).$$

for all f in $L^2_a(\mathbb{D})$. Let $K(z,w)$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by

$$K(z,w) = \overline{K_z(w)}.$$

The function $K(z,w)$ is analytic in z and co-analytic in w. Since

$$f(z) = \int_{\mathbb{D}} f(w)K(z,w)dA(w), f \in L^2_a(\mathbb{D}),$$

the function $K(z,w) = \frac{1}{(1-z\overline{w})^2},\ z,w \in \mathbb{D}$ and is the reproducing kernel [7] of $L^2_a(\mathbb{D})$. For $a \in \mathbb{D}$, let $k_a(z) = \frac{K(z,a)}{\sqrt{K(a,a)}} = \frac{(1-|a|^2)}{(1-\overline{a}z)^2}$. The function k_a is called the normalized reproducing kernel for $L^2_a(\mathbb{D})$. It is clear that $||k_a||_2 = 1$. Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$ an automorphism ϕ_a in $Aut(\mathbb{D})$ such that

(i) $(\phi_a \circ \phi_a)(z) = z$;

(ii) $\phi_a(0) = a, \phi_a(a) = 0$;

(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.

In fact, $\phi_a(z) = \frac{a-z}{1-\overline{a}z}$ for all a and z in $\mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is $J_{\phi_a}(z) = |k_a(z)|^2 = \frac{(1-|a|^2)}{|1-\overline{a}z|^4}$. Given $a \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define a function $U_a f$ on $\mathbb{D}$ by $U_a f(z) = k_a(z)f(\phi_a(z))$. Let $\mathcal{L}(H)$ be the set of all bounded linear operators from the Hilbert space H into itself. Let $\mathcal{LC}(H)$ be the set of all compact operators in $\mathcal{L}(H)$. The essential norm of an operator $T \in \mathcal{L}(H)$ is the distance of the operator from the space of compact operators, that is

$$||T||_e = \inf\{||T-K|| : K \text{ is compact. }\}$$

In particular, T is compact if and only if $||T||_e = 0$. Essential norm estimates for bounded operators on the Bergman space are studied in [4] and [6].
The layout of this paper is as follows. In §2, we introduce a class of unitary operators defined on $L_a^2(\mathbb{C}_+)$ induced by the automorphisms $t_a(s)$ of $\mathbb{C}_+$. In §3, we introduce the functions $B(s,w), B_{\overline{w}}(s)$ and $b_{\overline{w}}(s), s, w \in \mathbb{C}_+$ and establish relations between them. We also show that the function $B(s,w)$ satisfy an inequality like the Bergman kernel (see [3]) $K(z,w)$ defined for the space $L_a^2(\mathbb{D})$. In §4, we introduce the operators Q_1 and $\mathcal{V}_1$ and show that they are bounded on $L^2(\mathbb{C}_+, d\mu)$. In §5, we establish that if $\phi \in L^2(\mathbb{C}_+, d\mu)$, then the little Hankel operator $\hbar_{\overline{\phi}}$ is bounded if and only if $\mathcal{V}_1\phi$ is bounded on $\mathbb{C}_+$. In §6, we give estimates for the essential norm of bounded little Hankel operators on the Bergman space $L_a^2(\mathbb{C}_+, d\mu)$ in terms of the function $\mathcal{V}_1\phi$ and applications of the result are also obtained.

2 A class of unitary operators on $L_a^2(\mathbb{C}_+)$

In this section, we introduce a class of unitary operators defined on $L_a^2(\mathbb{C}_+)$ induced by the automorphisms $t_a(s)$ of $\mathbb{C}_+$.

Define $M : \mathbb{C}_+ \to \mathbb{D}$ by $Ms = \frac{1-s}{1+s}$. Then M is one-one, onto and $M^{-1} : \mathbb{D} \to \mathbb{C}_+$ is given by $M^{-1}(z) = \frac{1-z}{1+z}$. Thus M is its self-inverse. Let $W : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{C}_+)$ be defined by $Wg(s) = \frac{2}{\sqrt{\pi}} g(Ms) \frac{1}{(1+s)^2}$. The map W is one-one and onto. Hence W^{-1} exists and $W^{-1} : L_a^2(\mathbb{C}_+) \to L_a^2(\mathbb{D})$ is given by $W^{-1}G(z) = 2\sqrt{\pi} G(Mz) \frac{1}{(1+z)^2}$, where $Mz = \frac{1-z}{1+z}$.

Lemma 1. *If $a \in \mathbb{D}$ and $a = c + id$, $c, d \in \mathbb{R}$, then the following hold:*

(i) $t_a(s) = \frac{-ids+(1-c)}{(1+c)s+id}$ *is an automorphism from $\mathbb{C}_+$ onto $\mathbb{C}_+$.*

(ii) $(t_a \circ t_a)(s) = s$.

(iii) $t_a'(s) = -l_a(s)$, *where* $l_a(s) = \frac{1-|a|^2}{((1+c)s+id)^2}$.

Proof. This can be verified by direct calculations. □

For $a \in \mathbb{D}$, define $V_a : L_a^2(\mathbb{C}_+) \to L_a^2(\mathbb{C}_+)$ by $(V_a g)(s) = (g \circ t_a)(s) l_a(s)$. In Proposition 1, we show that V_a is a self-adjoint, unitary operator which is also an involution.

Proposition 1. *For $a \in \mathbb{D}$,*

(i) $V_a l_a = 1$.

(ii) $V_a^{-1} = V_a$ *and* V_a *is an involution, i.e.* $V_a^2 = I_{\mathcal{L}(L_a^2(\mathbb{C}_+))}$, *where* $I_{\mathcal{L}(L_a^2(\mathbb{C}_+))}$ *is the identity operator from* $L_a^2(\mathbb{C}_+)$ *into itself.*

(iii) V_a *is self-adjoint.*

(iv) V_a *is unitary,* $||V_a|| = 1$.

(v) $V_a P_+ = P_+ V_a$.

Proof. One can prove (i), (ii), (iii) and (iv) by direct calculations. Notice that V_a can also be defined from $L^2(\mathbb{C}_+)$ into itself. To prove (v), observe that $V_a(L_a^2(\mathbb{C}_+)) \subset L_a^2(\mathbb{C}_+)$ and $V_a(L_a^2(\mathbb{C}_+))^\perp \subset (L_a^2(\mathbb{C}_+))^\perp$. Now let $f \in L^2(\mathbb{C}_+)$ and $f = f_1 + f_2$, where $f_1 \in L_a^2(\mathbb{C}_+)$ and $f_2 \in (L_a^2(\mathbb{C}_+))^\perp$. Hence,

$$P_+V_af = P_+V_a(f_1 + f_2) = P_+(V_af_1 + V_af_2) = P_+V_af_1 = V_af_1 = V_aP_+f.$$

□

3 The function $B(s, w)$

In this section, we introduce the functions $B(s, w)$ and $b_{\overline{w}}(s), s, w \in \mathbb{C}_+$ and establish relations between them. We also show that the function $B(s, w)$ satisfy an inequality like the Bergman kernel (see [3]) $K(z, w)$ defined for the space $L_a^2(\mathbb{D})$.

Suppose $a \in \mathbb{D}$ and $w = \frac{1-\overline{a}}{1+\overline{a}} = M\overline{a} \in \mathbb{C}_+$. Define $b_{\overline{w}}(s) = \frac{1}{\sqrt{\pi}} \frac{1+w}{1+\overline{w}} \frac{2Rew}{(s+w)^2}$. Let $B(s, w) = B_{\overline{w}}(s) = \frac{1}{\pi} \frac{(1+a)^2}{(1-\overline{a}Ms)^2} \frac{1}{(1+s)^2}$.

Lemma 2. *Let* $s, w \in \mathbb{C}_+$. *The following hold:*

(i) $(b_{\overline{w}}(\overline{w}))^2 = B(\overline{w}, w)$.

(ii) $|b_{\overline{w}}(s)|\ ||B_{\overline{w}}|| = |B_{\overline{w}}(s)|$.

Proof. Let $w \in \mathbb{C}_+$ and $w = M\overline{a} = \frac{1-\overline{a}}{1+\overline{a}}$. Since

$$\begin{aligned}
b_{\overline{w}}(s) &= \frac{1}{\sqrt{\pi}}\frac{1+w}{1+\overline{w}}\frac{2Rew}{[s+w]^2} = \frac{2}{\sqrt{\pi}}\frac{\mathrm{Re}w}{(1+w)(1+\overline{w})}\frac{(1+w)^2}{[s+w]^2} \\
&= \frac{2}{\sqrt{\pi}}\frac{4\mathrm{Re}w}{|1+w|^2}\frac{(1+w)^2}{4}\frac{1}{[s+w]^2} = \frac{2}{\sqrt{\pi}}\frac{\frac{|1+w|^2-|1-w|^2}{|1+w|^2}}{\left[\frac{2}{(1+w)}\right]^2}\frac{1}{[s+w]^2} \\
&= \frac{2}{\sqrt{\pi}}\frac{1-\left|\frac{1-w}{1+w}\right|^2}{(1+\frac{1-w}{1+w})^2}\frac{1}{[s+w]^2} = \frac{2}{\sqrt{\pi}}\frac{1-|a|^2}{(1+\overline{a})^2}\frac{1}{[s+w]^2}, \text{where } \frac{1-\overline{a}}{1+\overline{a}} = w, \\
&= \frac{2}{\sqrt{\pi}}\frac{1-|a|^2}{(1+\overline{a})^2}\frac{1}{\left[s+\frac{1-\overline{a}}{1+\overline{a}}\right]^2} = \frac{2}{\sqrt{\pi}}\frac{1-|a|^2}{(1+\overline{a})^2\left[s+\frac{1-\overline{a}}{1+\overline{a}}\right]^2} \\
&= \frac{2}{\sqrt{\pi}}\frac{1-|a|^2}{[1-\overline{a}+s(1+\overline{a})]^2} = \frac{2}{\sqrt{\pi}}\frac{1-|a|^2}{[1+s-\overline{a}+\overline{a}s]^2} \\
&= \frac{(-1)}{\sqrt{\pi}}\frac{(1-|a|^2)(1+s)^2}{[1+s-\overline{a}+\overline{a}s]^2}\frac{(-2)}{(1+s)^2} = \frac{(-1)}{\sqrt{\pi}}\frac{1-|a|^2}{\left[1-\overline{a}\frac{1-s}{1+s}\right]^2}\frac{(-2)}{(1+s)^2} \\
&= \frac{2}{\sqrt{\pi}}\frac{1-|a|^2}{[1-\overline{a}(Ms)]^2}\frac{1}{(1+s)^2},
\end{aligned}$$

we obtain

$$\begin{aligned}
b_{\overline{w}}(\overline{w}) &= \frac{2}{\sqrt{\pi}}\frac{(1-|a|^2)}{(1-\overline{a}M\overline{w})^2}\frac{1}{(1+\overline{w})^2} = \frac{2}{\sqrt{\pi}}\frac{(1-|a|^2)}{(1-|a|^2)^2}\frac{1}{\left(1+\frac{1-a}{1+a}\right)^2} \\
&= \frac{2}{\sqrt{\pi}}\frac{1}{(1-|a|^2)}\frac{(1+a)^2}{4} = \frac{1}{2\sqrt{\pi}}\frac{(1+a)^2}{(1-|a|^2)}.
\end{aligned}$$

Thus

$$\begin{aligned}
b_{\overline{w}}(s)b_{\overline{w}}(\overline{w}) &= \frac{2}{\sqrt{\pi}}\frac{(1-|a|^2)}{(1-\overline{a}Ms)^2}\frac{1}{(1+s)^2}\frac{1}{2\sqrt{\pi}}\frac{(1+a)^2}{(1-|a|^2)} \\
&= \frac{1}{\pi}\frac{1}{(1-\overline{a}Ms)^2}\frac{(1+a)^2}{(1+s)^2} = \frac{(-1)}{2\pi}\frac{(1+a)^2}{(1-\overline{a}Ms)^2}\frac{(-2)}{(1+s)^2} \\
&= \frac{(-1)}{2\pi}\frac{(1+a)^2}{(1-\overline{a}Ms)^2}M' = B(s,w).
\end{aligned}$$

Thus $b_{\overline{w}}(s) = \frac{B(s,w)}{b_{\overline{w}}(\overline{w})}$ and $(b_{\overline{w}}(\overline{w}))^2 = B(\overline{w}, w)$. This proves (i). To prove (ii), notice that

$$\begin{aligned}||B_{\overline{w}}||^2 &= \langle B_{\overline{w}}, B_{\overline{w}}\rangle = \int_{\mathbb{C}_+} |B_{\overline{w}}(s)|^2 d\mu(s) = \int_{\mathbb{C}_+} |B(s,w)|^2 d\mu(s) \\ &= \int_{\mathbb{C}_+} |b_{\overline{w}}(\overline{w})|^2 |b_{\overline{w}}(s)|^2 d\mu(s) = |b_{\overline{w}}(\overline{w})|^2 \int_{\mathbb{C}_+} |b_{\overline{w}}(s)|^2 d\mu(s) \\ &= |b_{\overline{w}}(\overline{w})|^2 ||b_{\overline{w}}||_2^2 = |b_{\overline{w}}(\overline{w})|^2,\end{aligned}$$

since $||b_{\overline{w}}||_2 = 1$. Thus $||B_{\overline{w}}|| = |b_{\overline{w}}(\overline{w})|$ and hence $|b_{\overline{w}}(s)|\ ||B_{\overline{w}}|| = |B_{\overline{w}}(s)|$. □

Lemma 3. *Suppose* $-\frac{1}{2} < q < p-1$. *Then there exists a positive constant* C *such that*

$$\int_{\mathbb{C}_+} |B(\overline{s}, \overline{w})|^p\ |B(\overline{w}, w)|^{-q}\ d\mu(\overline{w}) \le C |B(\overline{s}, s)|^{p-q-1}$$

for all $s \in \mathbb{C}_+$.

Proof. Since $B(s,w) = \frac{1}{\pi}\frac{(1+a)^2}{(1-\overline{a}Ms)^2}\frac{1}{(1+s)^2}$ and $Ma = \overline{w}$, we obtain

$$\int_{\mathbb{C}_+} |B(\overline{s}, \overline{w})|^p\ |B(\overline{w}, w)|^{-q}\ d\mu(\overline{w})$$

$$= \int_{\mathbb{C}_+} \left|\frac{1}{\pi}\frac{(1+\overline{a})^2}{(1-aM\overline{s})^2}\frac{1}{(1+\overline{s})^2}\right|^p \left|\frac{1}{4\pi}\frac{(1+a)^4}{(1-|a|^2)^2}\right|^{-q} d\mu(Ma)$$

$$= \int_{\mathbb{D}} \left|\frac{1}{\pi}\frac{(1+\overline{a})^2}{(1-a\overline{z})^2}\frac{1}{(1+M\overline{z})^2}\right|^p \left|\frac{1}{4\pi}\frac{(1+a)^4}{(1-|a|^2)^2}\right|^{-q} \left|\frac{(-2)}{(1+a)^2}\right|^2 dA(a)$$

$$= \int_{\mathbb{D}} \left|\frac{1}{\pi}\frac{(1+\overline{a})^2}{(1-a\overline{z})^2}\frac{1}{\left(1+\frac{1-\overline{z}}{1+\overline{z}}\right)^2}\right|^p \left|\frac{1}{4\pi}\frac{(1+a)^4}{(1-|a|^2)^2}\right|^{-q} \left|\frac{(-2)}{(1+a)^2}\right|^2 dA(a)$$

$$= \int_{\mathbb{D}} \left|\frac{1}{\pi}\frac{(1+\overline{a})^2}{(1-a\overline{z})^2}\frac{(1+\overline{z})^2}{(1+\overline{z}+1-\overline{z})^2}\right|^p \left|\frac{1}{4\pi}\frac{(1+a)^4}{(1-|a|^2)^2}\right|^{-q} \left|\frac{(-2)}{(1+a)^2}\right|^2 dA(a)$$

$$= \int_{\mathbb{D}} \left|\frac{1}{4\pi}\frac{(1+\overline{a})^2(1+\overline{z})^2}{(1-a\overline{z})^2}\right|^p \left|\frac{1}{4\pi}\frac{(1+a)^4}{(1-|a|^2)^2}\right|^{-q} \left|\frac{(-2)}{(1+a)^2}\right|^2 dA(a)$$

$$= \left(\tfrac{1}{4\pi}\right)^{p-q} \ 2^2 \ |1+\overline{z}|^{2p} \int_{\mathbb{D}} \left|\frac{1}{(1-\overline{a}z)^2}\right|^p \left|\frac{1}{(1-|a|^2)^2}\right|^{-q} |1+a|^{-4q-4}|1+\overline{a}|^{2p} dA(a)$$

$$= \left(\tfrac{1}{4\pi}\right)^{p-q} \ 2^2 \ |1+\overline{z}|^{2p} \int_{\mathbb{D}} |K(z,a)|^p \ |K(a,a)|^{-q} \ |1+a|^{-4q-4}|1+\overline{a}|^{2p} dA(a)$$

$$\leq \tfrac{4}{(4\pi)^{p-q}} \ 2^{2p} \ 2^{2p} \ \ 2^{-4q-4} \int_{\mathbb{D}} |K(z,a)|^p \ |K(a,a)|^{-q} dA(a)$$

$$\leq \tfrac{1}{4} \left(\tfrac{4}{\pi}\right)^{p-q} \int_{\mathbb{D}} |K(z,a)|^p \ |K(a,a)|^{-q} dA(a).$$

From [3], we obtain

$$\int_{\mathbb{C}_+} |B(\overline{s},\overline{w})|^p |B(\overline{w},w)|^{-q} d\mu(\overline{w}) \leq C\frac{1}{4}\left(\frac{4}{\pi}\right)^{p-q} K(z,z)^{p-q-1}$$

for some constant C. Let $C_1 = C\frac{1}{4}\left(\frac{4}{\pi}\right)^{p-q}$. Then

$$\int_{\mathbb{C}_+} |B(\overline{s},\overline{w})|^p |B(\overline{w},w)|^{-q} d\mu(\overline{w}) \leq C_1 K(z,z)^{p-q-1} = C_1 K_z(z)^{p-q-1}$$

$$= C_1 \langle K_z, K_z\rangle^{p-q-1} = C_1 ||K_z||^{2(p-q-1)} \left\langle \frac{K_z}{||K_z||}, \frac{K_z}{||K_z||}\right\rangle^{p-q-1}$$

$$= C_2 \langle k_z, k_z\rangle^{p-q-1} \text{ where } C_2 = C_1 ||K_z||^{2(p-q-1)}.$$

Thus, if $z = M\overline{s}$, then

$$\int_{\mathbb{C}_+} |B(\overline{s},\overline{w})|^p |B(\overline{w},w)|^{-q} d\mu(\overline{w}) \leq C_2 \left(\frac{||K_z||^2}{||B_{\overline{s}}||^2} |B(\overline{s},s)|\right)^{p-q-1}$$

$$= C_3 |B(\overline{s},s)|^{p-q-1},$$

where $C_3 = C_2 \frac{||K_z||^{2(p-q-1)}}{||B_{\overline{s}}||^{2(p-q-1)}}$. This complete the proof.

□

Lemma 4. *Let $s, w \in \mathbb{C}_+$, and $w = M\overline{a}$. Then $|B(\overline{s},\overline{w})| = |B(\overline{w},\overline{s})|$.*

Proof. Let $s, w \in \mathbb{C}_+$ and $w = M\overline{a}$. Since $B(s,w) = \frac{1}{\pi}\frac{1}{(1-\overline{a}Ms)^2}\frac{(1+a)^2}{(1+s)^2}$, we obtain

$$\begin{aligned}
B(\overline{s},\overline{w}) &= \frac{1}{\pi}\frac{1}{(1-aM\overline{s})^2}\frac{(1+Mw)^2}{(1+\overline{s})^2} = \frac{1}{\pi}\frac{\frac{(1+w+1-w)^2}{(1+w)^2}}{\left(1-a\frac{1-\overline{s}}{1+\overline{s}}\right)^2}\frac{1}{(1+\overline{s})^2} \\
&= \frac{4}{\pi}\frac{1}{(1+w)^2}\frac{(1+\overline{s})^2}{(1+\overline{s}-a+a\overline{s})^2}\frac{1}{(1+\overline{s})^2} = \frac{4}{\pi}\frac{1}{(1+w)^2}\frac{1}{(1-a+\overline{s}(1+a))^2} \\
&= \frac{4}{\pi}\frac{1}{(1+w)^2}\frac{1}{(1+a)^2\left(\frac{1-a}{1+a}+\overline{s}\right)^2} = \frac{4}{\pi}\frac{1}{(1+w)^2}\frac{1}{(1+a)^2(\overline{s}+\overline{w})^2} \\
&= \frac{4}{\pi}\frac{1}{(1+w)^2}\frac{1}{(1+M\overline{w})^2(\overline{s}+\overline{w})^2} = \frac{4}{\pi}\frac{(1+\overline{w})^2}{4(1+w)^2}\frac{1}{(\overline{s}+\overline{w})^2} \\
&= \frac{1}{\pi}\left(\frac{1+\overline{w}}{1+w}\right)^2\frac{1}{(\overline{s}+\overline{w})^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
B(\overline{w},\overline{s}) &= \frac{4}{\pi}\frac{1}{(1+s)^2}\frac{1}{(1+M\overline{s})^2(\overline{s}+\overline{w})^2} = \frac{4}{\pi}\frac{1}{(1+s)^2}\frac{1}{(\overline{s}+\overline{w})^2}\frac{1}{\left(1+\frac{1-\overline{s}}{1+\overline{s}}\right)^2} \\
&= \frac{4}{\pi}\frac{1}{(1+s)^2}\frac{1}{(\overline{s}+\overline{w})^2}\frac{(1+\overline{s})^2}{4} = \frac{1}{\pi}\frac{(1+\overline{s})^2}{(1+s)^2}\frac{1}{(\overline{s}+\overline{w})^2}.
\end{aligned}$$

Thus $|B(\overline{s},\overline{w})| = |B(\overline{w},\overline{s})|$ for all $s, w \in \mathbb{C}_+$.

□

4 Integral operator

In this section, we introduce the operators Q_1 and $\mathcal{V}_1$ and prove that these operators are bounded on $L^2(\mathbb{C}_+, d\mu)$. For $\phi \in L^2(\mathbb{C}_+, d\mu)$ and $s \in \mathbb{C}_+$, we define

$$Q_1\phi(s) = 3\int_{\mathbb{C}_+}\frac{|B(\overline{s},\overline{w})|^2}{|B(\overline{w},w)|}\phi(w)d\mu(w)$$

and

$$\mathcal{V}_1\phi(w) = 3\int_{\mathbb{C}_+}\frac{|B(\overline{s},\overline{w})|^2}{|B(\overline{w},w)|}\phi(s)d\mu(s).$$

Proposition 2. *The operators Q_1 and $\mathcal{V}_1$ are bounded on $L^2(\mathbb{C}_+, d\mu)$.*

Proof. Notice that the boundedness of Q_1 follows from the boundedness of $\mathcal{V}_1$. Thus we only show that Q_1 is a bounded operator on $L^2(\mathbb{C}_+, d\mu)$. Take $t > 0$ and let $h(s) = |B(\overline{s}, s)|^t$. Then by Lemma 3 and Lemma 4, we obtain

$$\begin{aligned}
\int_{\mathbb{C}_+} \frac{|B(\overline{s}, \overline{w})|^2}{|B(\overline{w}, w)|} h(s) d\mu(s) &= |B(\overline{w}, w)|^{-1} \int_{\mathbb{C}_+} |B(\overline{s}, \overline{w})|^2 \; |B(\overline{s}, s)|^t d\mu(s) \\
&= |B(\overline{w}, w)|^{-1} \int_{\mathbb{C}_+} |B(\overline{w}, \overline{s})|^2 \; |B(\overline{s}, s)|^t d\mu(s) \\
&\leq |B(\overline{w}, w)|^{-1} \; C|B(\overline{w}, w)|^{2+t-1} \\
&= C|B(\overline{w}, w)|^t = Ch(w); \qquad (1)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{C}_+} \frac{|B(\overline{s}, \overline{w})|^2}{|B(\overline{w}, w)|} h(w) d\mu(w) &= \int_{\mathbb{C}_+} |B(\overline{s}, \overline{w})|^2 |B(\overline{w}, w)|^t \; |B(\overline{w}, w)|^{-1} d\mu(w) \\
&= \int_{\mathbb{C}_+} |B(\overline{s}, \overline{w})|^2 |B(\overline{w}, w)|^{t-1} \; d\mu(w) \\
&\leq CB(\overline{s}, s)^{2+t-1-1} = CB(\overline{s}, s)^t = Ch(s), \qquad (2)
\end{aligned}$$

for some constant $C > 0$. From Schur's theorem [7], it follows that Q_1 is a bounded operator on $L^2(\mathbb{C}_+, d\mu)$. Moreover (1) and (2) also yield the boundedness of $\mathcal{V}_1$. □

The boundedness of Q_1 or $\mathcal{V}_1$ on $L^2(\mathbb{C}_+, d\mu)$ enables us to use Fubini's theorem [5]. Let $\phi, g \in L^2(\mathbb{C}_+, d\mu)$. Then

$$\begin{aligned}
\langle \mathcal{V}_1 \phi, g \rangle &= \int_{\mathbb{C}_+} \left(3 \int_{\mathbb{C}_+} \frac{|B(\overline{s}, \overline{w})|^2}{|B(\overline{w}, w)|} \phi(s) d\mu(s) \right) \overline{g(w)} d\mu(w) \\
&= \int_{\mathbb{C}_+} \overline{\left(3 \int_{\mathbb{C}_+} \frac{|B(\overline{w}, \overline{s})|^2}{|B(\overline{w}, w)|} g(w) d\mu(w) \right)} \phi(s) d\mu(s) \\
&= \langle \phi, Q_1 g \rangle, \qquad (3)
\end{aligned}$$

where the second equality of (3) follows from Fubini's theorem because

$$\begin{aligned}
&3 \int_{\mathbb{C}_+} \int_{\mathbb{C}_+} \left| \frac{B(\overline{w}, \overline{s})^2}{B(\overline{w}, w)} \phi(s) g(w) \right| d\mu(s) d\mu(w) \\
&\leq ||Q_1|| \; ||g|| \; ||\phi|| < \infty.
\end{aligned}$$

Therefore, the adjoint operator of $\mathcal{V}_1$ on $L^2(\mathbb{C}_+, d\mu)$ is equal to Q_1.

Lemma 5. *For* $\phi \in L^2(\mathbb{C}_+, d\mu)$,

$$\int_{\mathbb{C}_+} f(w)\overline{\phi(w)}d\mu(w) = \int_{\mathbb{C}_+} f(w)\overline{\mathcal{V}_1\phi(w)}d\mu(w)$$

for all $f \in L^2_a(\mathbb{C}_+)$.

Proof. As $Q_1 f = f$ for $f \in L^2_a(\mathbb{C}_+)$, we have

$$\int_{\mathbb{C}_+} f(w)\overline{\phi(w)}d\mu(w) = \langle Q_1 f, \phi\rangle = \langle f, \mathcal{V}_1\phi\rangle = \int_{\mathbb{C}_+} f(w)\overline{\mathcal{V}_1\phi(w)}d\mu(w).$$

□

5 Little Hankel operators

In this section, we establish that if $\phi \in L^2(\mathbb{C}_+, d\mu)$, then the little Hankel operator $\hbar_{\overline{\phi}}$ is bounded if and only if $(\mathcal{V}_1\phi)(w)$ is bounded in $\mathbb{C}_+$. Let $H^\infty(\mathbb{C}_+)$ be the space of bounded analytic functions on $\mathbb{C}_+$. It is not difficult to verify that $H^\infty(\mathbb{C}_+) = WH^\infty(\mathbb{D})$ and $H^\infty(\mathbb{C}_+)$ is dense in $L^2_a(\mathbb{C}_+)$.

Proposition 3. *If* $\phi \in L^2(\mathbb{C}_+, d\mu)$, *then* $\hbar_{\overline{\phi}} = \hbar_{\overline{P_+\phi}}$ *in the sense that* $\hbar_{\overline{\phi}} g = \hbar_{\overline{P_+\phi}} g$ *for all* $g \in H^\infty(\mathbb{C}_+)$.

Proof. Let $h \in L^2_a(\mathbb{C}_+)$ and $g \in H^\infty(\mathbb{C}_+)$. Then

$$\begin{aligned}\langle \hbar_{\overline{\phi}} g, \overline{h}\rangle &= \langle \overline{P}_+(\overline{\phi} g), \overline{h}\rangle = \langle \overline{\phi} g, \overline{h}\rangle = \langle gh, \phi\rangle \\ &= \langle gh, P_+\phi\rangle = \langle \overline{P_+\phi} g, \overline{h}\rangle = \langle \overline{P_+\phi} g, \overline{P}_+\overline{h}\rangle \\ &= \langle \overline{P}_+(\overline{P_+\phi} g), \overline{h}\rangle = \langle \hbar_{\overline{P_+\phi}} g, \overline{h}\rangle.\end{aligned}$$

Hence $\hbar_{\overline{\phi}} g = \hbar_{\overline{P_+\phi}} g$ for all $g \in H^\infty(\mathbb{C}_+)$. □

Lemma 6. *Let* $G(s) \in L^\infty(\mathbb{C}_+)$. *Then the little Hankel operator* $\mathbf{\Gamma}_G$ *determined on* $L^2_a(\mathbb{C}_+)$ *by* G *is equivalent to the little Hankel operator* Γ_ϕ *determined on* $L^2_a(\mathbb{D})$ *by the function* $\phi(z) = \left(\frac{1+\overline{z}}{1+z}\right)^2 G(Mz)$.

Proof. Notice that the sequence of vectors $\left\{\sqrt{n+1}z^n\right\}_{n=0}^\infty$ forms an orthonormal basis for $L^2_a(\mathbb{D})$. Then

$$\begin{aligned}\mathbf{\Gamma}_G(W(\sqrt{n+1}z^n)) &= P_+\left(G\mathcal{J}\left(\frac{2}{\sqrt{\pi}}\left(\frac{1-s}{1+s}\right)^n \frac{1}{(1+s)^2}\sqrt{n+1}\right)\right) \\ &= WPW^{-1}\left(G(s)\frac{2}{\sqrt{\pi}}\left(\frac{1-\overline{s}}{1+\overline{s}}\right)^n \frac{1}{(1+\overline{s})^2}\sqrt{n+1}\right) \\ &= W\Gamma_{\left(\frac{1+\overline{z}}{1+z}\right)^2 G(Mz)}\left(\sqrt{n+1}z^n\right) \text{ for all } n \geq 0.\end{aligned}$$

Thus $\mathbf{\Gamma}_G$ is unitarily equivalent to Γ_ϕ where $\phi(z) = \left(\frac{1+\overline{z}}{1+z}\right)^2 G(Mz)$. The result follows. □

Proposition 4. *If* $\phi \in L^\infty(\mathbb{C}_+)$, *then* $\hbar_{\overline{\phi}} W = W h_{\overline{\phi} \circ M}$.

Proof. For $\phi \in L^\infty(\mathbb{C}_+)$, notice that $\hbar_\phi = \mathcal{J}\mathcal{S}_\phi$ and $\mathbf{\Gamma}_\phi = \mathcal{S}_{\mathcal{J}\phi}$, where $\mathcal{J}$ is the mapping from $L^2(\mathbb{C}_+, d\mu)$ into itself defined by $\mathcal{J}f(s) = f(\overline{s})$. Then from Lemma (6), we obtain

$$W^{-1}\mathcal{J}\hbar_{\mathcal{J}\overline{\phi}} W = Jh_{J\left(\left(\frac{1+\overline{z}}{1+z}\right)^2 (\overline{\phi} \circ M)(z)\right)}.$$

Hence

$$(W^{-1}\mathcal{J}W)(W^{-1}\hbar_{\mathcal{J}\overline{\phi}} W) = Jh_{J\left(\left(\frac{1+\overline{z}}{1+z}\right)^2 (\overline{\phi} \circ M)(z)\right)}.$$

Thus

$$J[J(W^{-1}\hbar_{\mathcal{J}\overline{\phi}} W)] = J\left(Jh_{J\left(\left(\frac{1+\overline{z}}{1+z}\right)^2 (\overline{\phi} \circ M)\right)(z)}\right).$$

Therefore

$$W^{-1}\hbar_{\mathcal{J}\overline{\phi}} W = h_{J\left(\left(\frac{1+\overline{z}}{1+z}\right)^2 (\overline{\phi} \circ M)(z)\right)}.$$

Hence

$$\hbar_{\mathcal{J}\overline{\phi}} W = W h_{J\left(u(\overline{\phi} \circ M)\right)}, \tag{4}$$

where $u(z) = \left(\frac{1+\overline{z}}{1+z}\right)^2 = J(M' \circ M)(z)M'(z)$. Now from (4), it follows that

$$\hbar_{\overline{\phi}} W = W h_{J\left(u(\mathcal{J}\overline{\phi} \circ M)\right)}. \tag{5}$$

Now

$$Ju = J(J(M' \circ M)M') = (M' \circ M)JM'.$$

Hence

$$(Ju \circ M) = (M' \circ M \circ M)(JM' \circ M) = M'(J(M' \circ M)).$$

Thus

$$\begin{aligned}(Ju)(Ju \circ M) &= (M' \circ M)(JM')M'(J(M' \circ M)) \\ &= (M' \circ M)M'J[(M' \circ M)M'] = 1. \end{aligned} \tag{6}$$

Further notice that

$$W^{-1}\overline{\phi} = (-1)\sqrt{\pi}(\overline{\phi} \circ M)M'.$$

Hence

$$J(W^{-1}\overline{\phi}) = (-1)\sqrt{\pi}(J\overline{\phi} \circ M)(JM').$$

This implies

$$WJW^{-1}\overline{\phi} = (-1)\sqrt{\pi}\frac{(-1)}{\sqrt{\pi}}(J\overline{\phi})(JM' \circ M)M' = (J\overline{\phi})(J(M' \circ M))M'.$$

Thus

$$\mathcal{J}\overline{\phi} = WJW^{-1}\overline{\phi} = u(J\overline{\phi}).$$

Hence

$$(\mathcal{J}\overline{\phi}) \circ M = (u \circ M)(J\overline{\phi} \circ M) = (u \circ M)J(\overline{\phi} \circ M).$$

Therefore

$$\begin{aligned} J((\mathcal{J}\overline{\phi}) \circ M) &= (J(u \circ M))(JJ(\overline{\phi} \circ M)) \\ &= (J(u \circ M))(\overline{\phi} \circ M) \\ &= ((Ju) \circ M)(\overline{\phi} \circ M). \end{aligned}$$

Form (5), we obtain

$$\begin{aligned} \hbar_{\overline{\phi}}W &= Wh_{J\left(u(\mathcal{J}\overline{\phi}\circ M)\right)} = Wh_{(Ju)(J(\mathcal{J}\overline{\phi}\circ M))} \\ &= Wh_{Ju[(Ju\circ M)(\overline{\phi}\circ M)]} = Wh_{[(Ju)(Ju\circ M)](\overline{\phi}\circ M)}. \end{aligned}$$

From (6), it follows that $\hbar_{\overline{\phi}}W = Wh_{\overline{\phi}\circ M}$. □

For $\phi \in L^2(\mathbb{C}_+, d\mu)$, it is not difficult to show that $(\mathcal{V}_1\phi)(w) = 3\langle \overline{b_{\overline{w}}}, \hbar_{\overline{\phi}} b_{\overline{w}}\rangle$. For $z \in \mathbb{D}, f \in L^2(\mathbb{D}, dA)$, define

$$(Vf)(z) = 3(1-|z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^4} dA(w).$$

Proposition 5. *Let* $\phi \in L^2(\mathbb{C}_+, d\mu)$*, then* $(\mathcal{V}_1\phi)(w) = V(\phi \circ M)(a)$*, for all* $a \in \mathbb{D}$*.*

Proof. Let $\phi \in L^2(\mathbb{C}_+, d\mu)$ and $w = M\overline{a}, a \in \mathbb{D}, w \in \mathbb{C}_+$. Then

$$\begin{aligned} \mathcal{V}_1\phi(w) &= 3\langle \overline{b_{\overline{w}}}, \hbar_{\overline{\phi}} b_{\overline{w}}\rangle = 3\langle \overline{Wk_a}, \hbar_{\overline{\phi}} Wk_a\rangle = 3\langle W\overline{k_a}, \hbar_{\overline{\phi}} Wk_a\rangle \\ &= 3\langle \overline{k_a}, W^{-1}\hbar_{\overline{\phi}} Wk_a\rangle = 3\langle \overline{k_a}, h_{\overline{\phi}\circ M} k_a\rangle = 3\langle \overline{k_a}, h_{\overline{\phi\circ M}} k_a\rangle \\ &= V(\phi \circ M)(a), \end{aligned}$$

for all $a \in \mathbb{D}$. □

Proposition 6. *For* $\phi \in L^2(\mathbb{C}_+, d\mu)$,

(i) $\mathcal{V}_1 P_+ = \mathcal{V}_1$.

(ii) $P_+\mathcal{V}_1 = P_+$.

(iii) $\mathcal{V}_1^2 = \mathcal{V}_1$.

Proof. From Proposition 3, we obtain

$$\mathcal{V}_1 P_+\phi = 3\langle \bar{b}_{\overline{w}}, \hbar_{\overline{P_+\phi}} b_{\overline{w}}\rangle = 3\langle \bar{b}_{\overline{w}}, \hbar_{\overline{\phi}} b_{\overline{w}}\rangle = \mathcal{V}_1\phi,$$

for $\phi \in L^2(\mathbb{C}_+, d\mu)$. This proves (i). To prove (ii), let $\phi, g \in L^2(\mathbb{C}_+, d\mu)$ and $g = g_1 + g_2$ where $g_1 \in L_a^2(\mathbb{C}_+)$ and $g_2 \in (L_a^2(\mathbb{C}_+))^\perp$. Then

$$\begin{aligned}\langle P_+\mathcal{V}_1\phi, g\rangle = \langle \mathcal{V}_1\phi, P_+ g\rangle = \langle \mathcal{V}_1\phi, g_1\rangle &= \int_{\mathbb{C}_+} (\mathcal{V}_1\phi)(w)\overline{g_1(w)}\ d\mu(w)\\ &= \pi\int_{\mathbb{D}} [(\mathcal{V}_1\phi)\circ M](z)\overline{(g_1\circ M)(z)}|M'(z)|^2\ dA(z)\\ &= \pi\int_{\mathbb{D}} [V(\phi\circ M)](z)\overline{(g_1\circ M)(z)}|M'(z)|^2\ dA(z).\end{aligned}$$

Under the complex integral pairing with respect to dA, we have $V = P_2^*$ where $P_2 h(z) = 3\int_{\mathbb{D}} \frac{(1-|u|^2)^2}{(1-z\overline{u})^4} h(u) dA(u)$ is a projection from $L^1(\mathbb{D}, dA)$ onto $L_a^1(\mathbb{D})$. From Fubini's theorem [5] and the fact that both P and P_2 reproduce analytic functions it follows that $PV = P$, where P is the Bergman projection from $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$. Thus for $\phi, g \in L^2(\mathbb{C}_+, d\mu)$,

$$\begin{aligned}\langle P_+\mathcal{V}_1\phi, g\rangle &= \pi\int_{\mathbb{D}} [V(\phi\circ M)](z)\overline{(g_1\circ M)(z)}\ |M'(z)|^2\ dA(z)\\ &= \pi\int_{\mathbb{D}} V[(\phi\circ M)M'](z)\overline{(g_1\circ M)(z)M'(z)}\ dA(z)\\ &= \int_{\mathbb{D}} V[(-1)\sqrt{\pi}(\phi\circ M)M'](z)\overline{(-1)\sqrt{\pi}(g_1\circ M)(z)M'(z)}\ dA(z)\\ &= \int_{\mathbb{D}} V(W^{-1}\phi)(z)\overline{(W^{-1}g_1)(z)}\ dA(z)\\ &= \langle VW^{-1}\phi, W^{-1}g_1\rangle = \langle VW^{-1}\phi, W^{-1}P_+g_1\rangle = \langle VW^{-1}\phi, PW^{-1}g_1\rangle\\ &= \langle PVW^{-1}\phi, W^{-1}g_1\rangle = \langle PW^{-1}\phi, W^{-1}g_1\rangle = \langle WPW^{-1}\phi, g_1\rangle\\ &= \langle P_+\phi, g_1\rangle = \langle P_+^2\phi, g_1\rangle = \langle P_+\phi, P_+g_1\rangle\\ &= \langle P_+\phi, P_+g\rangle = \langle P_+^2\phi, g\rangle = \langle P_+\phi, g\rangle.\end{aligned}$$

Thus $P_+\mathcal{V}_1\phi = P_+\phi$ for all $\phi \in L^2(\mathbb{C}_+, d\mu)$ and therefore $P_+\mathcal{V}_1 = P_+$. This proves (ii). To prove (iii), notice that

$$\begin{aligned}(\mathcal{V}_1^2\phi)(w) &= \mathcal{V}_1(\mathcal{V}_1\phi)(w) = 3\langle \bar{b}_{\overline{w}}, \hbar_{\overline{\mathcal{V}_1\phi}} b_{\overline{w}}\rangle = 3\langle \bar{b}_{\overline{w}}, \hbar_{\overline{P_+\mathcal{V}_1\phi}} b_{\overline{w}}\rangle \\ &= 3\langle \bar{b}_{\overline{w}}, \hbar_{\overline{P_+\phi}} b_{\overline{w}}\rangle = 3\langle \bar{b}_{\overline{w}}, \hbar_{\overline{\phi}} b_{\overline{w}}\rangle = (\mathcal{V}_1\phi)(w)\end{aligned}$$

for all $w \in \mathbb{C}_+$ and $\phi \in L^2(\mathbb{C}_+, d\mu)$. Hence $\mathcal{V}_1^2 = \mathcal{V}_1$. □

Proposition 7. *Let $a \in \mathbb{D}, \overline{f} \in \overline{L_a^2(\mathbb{D})}$ and $f = W^{-1}g, g \in L_a^2(\mathbb{C}_+)$. Then*

$$h^*_{\overline{\phi}\circ M}\overline{f}(a) = c_a\langle \hbar^*_{\overline{\phi}}\overline{g}, B_{\overline{w}}\rangle,$$

for all $g \in L_a^2(\mathbb{C}_+)$ and for some constant c_a.

Proof. Let $a \in \mathbb{D}, \overline{f} \in \overline{L_a^2(\mathbb{D})}$ and $f = W^{-1}g, g \in L_a^2(\mathbb{C}_+)$. Then by Lemma 2, there exists a constant $\alpha, |\alpha| = 1$ such that

$$\begin{aligned}h^*_{\overline{\phi}}\overline{f}(a) &= \langle h^*_{\overline{\phi}}\overline{f}, K_a\rangle = \langle \overline{f}, h_{\overline{\phi}}K_a\rangle = \langle W\overline{f}, Wh_{\overline{\phi}}K_a\rangle \\ &= ||K_a||\langle W\overline{f}, Wh_{\overline{\phi}}k_a\rangle = ||K_a||\langle \overline{g}, Wh_{\overline{\phi}}W^{-1}b_{\overline{w}}\rangle = ||K_a||\langle \overline{g}, \hbar_{\overline{\phi}\circ M}b_{\overline{w}}\rangle \\ &= ||K_a||\langle \hbar^*_{\overline{\phi}\circ M}\overline{g}, b_{\overline{w}}\rangle = \alpha||K_a||\left\langle \hbar^*_{\overline{\phi}\circ M}\overline{g}, \frac{B_{\overline{w}}}{||B_{\overline{w}}||}\right\rangle = \frac{\alpha||K_a||}{||B_{\overline{w}}||}\langle \hbar^*_{\overline{\phi}\circ M}\overline{g}, B_{\overline{w}}\rangle \\ &= c_a\langle \hbar^*_{\overline{\phi}\circ M}\overline{g}, B_{\overline{w}}\rangle,\end{aligned}$$

where $c_a = \frac{\alpha||K_a||}{||B_{\overline{w}}||}$. Thus,

$$\hbar^*_{\overline{\phi}\circ M}\overline{f}(a) = c_a\langle \hbar^*_{\overline{\phi}}\overline{g}, B_{\overline{w}}\rangle.$$

□

Theorem 1. *Suppose $\phi \in L^2(\mathbb{C}_+, d\mu)$. Then $\hbar_{\overline{\phi}}$ is bounded if and only if $(\mathcal{V}_1\phi)(w)$ is bounded in $\mathbb{C}_+$ and there is a constant $C > 0$ such that $C^{-1}||\mathcal{V}_1\phi||_\infty \leq ||\hbar_{\overline{\phi}}|| \leq C||\mathcal{V}_1\phi||_\infty$.*

Proof. Notice that $b_{\overline{w}} \in L^2(\mathbb{C}_+, d\mu)$ and $||b_{\overline{w}}||_2 = 1$. Hence $|(\mathcal{V}_1\phi)(w)| = 3|\langle \bar{b}_{\overline{w}}, \hbar_{\overline{\phi}}b_{\overline{w}}\rangle| \leq 3||b_{\overline{w}}||_2\ ||\hbar_{\overline{\phi}}||\ ||b_{\overline{w}}||_2 = 3||b_{\overline{w}}||_2^2\ ||\hbar_{\overline{\phi}}|| = 3||\hbar_{\overline{\phi}}||$. Further, $\hbar_{\overline{\phi}} = \hbar_{\overline{P_+\phi}} = \hbar_{\overline{P_+\mathcal{V}_1\phi}} = \hbar_{\overline{\mathcal{V}_1\phi}}$. Thus $\mathcal{V}_1\phi \in L^\infty(\mathbb{C}_+)$ implies that $\hbar_{\overline{\phi}}$ is bounded with $||\hbar_{\overline{\phi}}|| \leq ||\mathcal{V}_1\phi||_\infty$. The result follows since $\hbar_{\overline{\phi}} = \hbar_{\overline{\mathcal{V}_1\phi}}$ for all $\phi \in L^2(\mathbb{C}_+, d\mu)$. □

Theorem 2. *Suppose $\phi \in L^2(\mathbb{C}_+, d\mu)$ such that $\hbar_{\overline{\phi}}$ is bounded. Then*

$$\frac{1}{3}\lim_{Rew\to 0}\sup|\mathcal{V}_1\phi(w)| \leq ||\hbar_{\overline{\phi}}||_e.$$

Proof. Consider a compact operator T from $L_a^2(\mathbb{C}_+, d\mu)$ to $\overline{L_a^2(\mathbb{C}_+, d\mu)}$ arbitarily. Since $b_{\overline{w}} \to 0$ weakly in $L_a^2(\mathbb{C}_+)$ as $Rew \to 0$, we obtain $||Tb_{\overline{w}}|| \longrightarrow 0$ as $Rew \to 0$. Hence,

$$||\hbar_{\overline{\phi}} - T|| \geq \lim_{Rew\to 0} \sup ||(\hbar_{\overline{\phi}} - T)b_{\overline{w}}|| \geq \lim_{Rew\to 0} \sup ||\hbar_{\overline{\phi}} b_{\overline{w}}||. \tag{7}$$

Since (7) holds for every compact operator T, it follows that,

$$||\hbar_{\overline{\phi}}||_e \geq \lim_{Rew\to 0} \sup ||\hbar_{\overline{\phi}} b_{\overline{w}}||. \tag{8}$$

On the other hand,

$$|(\mathcal{V}_1\phi)(w)| = 3|\langle b_{\overline{w}}, \hbar_{\overline{\phi}} b_{\overline{w}}\rangle| \leq 3||\hbar_{\overline{\phi}} b_{\overline{w}}||. \tag{9}$$

From (8) and (9), the theorem follows. □

6 Main Result

In this section, we give estimates for the essential norm of bounded little Hankel operators on the Bergman space $L_a^2(\mathbb{C}_+, d\mu)$ in terms of the function $\mathcal{V}_1\phi$ and applications of the result are also derived. Assume that $\hbar_\phi$ is bounded operator from $L_a^2(\mathbb{C}_+, d\mu)$ to $\overline{L_a^2(\mathbb{C}_+, d\mu)}$. The following holds:

Theorem 3. *Suppose $\phi \in L^2(\mathbb{C}_+, d\mu)$ and $\hbar_{\overline{\phi}}$ is bounded. Then*

$$||\hbar_{\overline{\phi}}||_e \leq C \lim_{Rew\to 0} \sup |\mathcal{V}_1\phi(w)|.$$

Proof. For $f \in \overline{L_a^2(\mathbb{D})}$ and $0 < r < 1$, define

$$F_r f(z) = \int_{\mathbb{D}} \left(\int_{r\mathbb{D}} \frac{1}{(1 - z\overline{u})^2} \frac{1}{(1 - v\overline{u})^2} V\phi(u) dA(u) \right) f(v) dA(v).$$

Then PF_r is a compact operator from $\overline{L_a^2(\mathbb{D})}$ to $L_a^2(\mathbb{D})$ because

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \left| \int_{r\mathbb{D}} \frac{1}{(1 - z\overline{u})^2} \frac{1}{(1 - v\overline{u})^2} V\phi(u) dA(u) \right|^2 dA(z) dA(v)$$

$$\leq ||V\phi||_\infty^2 \int_{r\mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|1 - z\overline{u}|^4} \frac{1}{|1 - v\overline{u}|^4} dA(z) dA(v) dA(u)$$

$$= ||V\phi||_\infty^2 \int_{r\mathbb{D}} \frac{1}{(1-|u|^2)^2} dA(u) < \infty$$

and the result follows from Theorem 3.5 in [7]. Thus, we have

$$||h_{\overline{\phi}}||_e = ||h_{\overline{\phi}}^*||_e = ||h_{\overline{\phi}}^* - PF_r||_e \leq ||h_{\overline{\phi}}^* - PF_r||.$$

Moreover, $Ph_{\overline{\phi}}^* = h_{\overline{\phi}}^*$ yields,

$$||h_{\overline{\phi}}^* - PF_r|| = \sup_{f \in \overline{L_a^2(\mathbb{D})}} \frac{||Ph_{\overline{\phi}}^* f - PF_r f||}{||f||}$$

$$\leq \sup_{f \in \overline{L_a^2(\mathbb{D})}} \frac{||h_{\overline{\phi}}^* f - F_r f||}{||f||}.$$

Define

$$K_r(z,v) = \int_{\mathbb{D}/r\mathbb{D}} \frac{1}{(1-z\overline{u})^2} \frac{1}{(1-v\overline{u})^2} V\phi(u) dA(u),$$

and $K_r^+(z,v) = |K_r(z,v)|$. Let G_r(respectively G_r^+) be the integral operator on $L^2(\mathbb{D}, dA)$ with kernel K_r(respectively K_r^+). Then $G_r f = h_{\overline{\phi}}^* f - F_r f$ for any $f \in \overline{L_a^2(\mathbb{D})}$. For details see [7]. Thus

$$||h_{\overline{\phi}}||_e \leq ||G_r^+||.$$

Using Schur's theorem, we will obtain the operator norm of G_r^+ on $L^2(\mathbb{D}, dA)$. Take $t > 0$ and $h(z) = \frac{1}{(1-|z|^2)^{2t}}$. Then we have,

$$\int_{\mathbb{D}} K_r^+(z,v) h(v) dA(v)$$

$$= \int_{\mathbb{D}} \left| \int_{\mathbb{D}/r\mathbb{D}} \frac{1}{(1-z\overline{u})^2} \frac{1}{(1-v\overline{u})^2} V\phi(u) dA(u) \right| \frac{1}{(1-|v|^2)^{2t}} dA(v)$$

$$\leq \left(\sup_{r<|u|<1} |V\phi(u)| \right) \int_{\mathbb{D}} \int_{\mathbb{D}/r\mathbb{D}} \left| \frac{1}{(1-z\overline{u})^2} \frac{1}{(1-v\overline{u})^2} \right| \frac{1}{(1-|v|^2)^{2t}} dA(u) dA(v).$$

Since

$$\int_{\mathbb{D}} \int_{\mathbb{D}/r\mathbb{D}} \left| \frac{1}{(1-z\overline{u})^2} \frac{1}{(1-v\overline{u})^2} \right| \frac{1}{(1-|v|^2)^{2t}} dA(u) dA(v)$$

$$\leq \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|1-v\overline{u}|^2} \frac{1}{(1-|v|^2)^{2t}} \frac{1}{|1-z\overline{u}|^2} dA(v) dA(u)$$

$$\leq Ch(z),$$

we obtain from [3] that,

$$\int_{\mathbb{D}} K_r^+(z,v)h(v)dA(v) \leq C\left(\sup_{r<|u|<1}|V\phi(u)|\right)h(z).$$

Thus using Schur's theorem [7], we have

$$||G_r^+|| \leq C \sup_{r<|u|<1}|V\phi(u)|.$$

Thus

$$||h_{\overline{\phi}}||_e \leq C \sup_{r<|u|<1}|V\phi(u)|,$$

for any $0 < r < 1$. Letting $r \to 1$, we obtain

$$||h_{\overline{\phi}}||_e \leq C \lim_{u\to\partial\mathbb{D}} \sup |V\phi(u)|.$$

Hence

$$\begin{aligned} ||\hbar_{\overline{\phi}}||_e &= \inf\{||\hbar_{\overline{\phi}} - T|| : T \text{ is compact}\} \\ &= \inf\{||W^{-1}\hbar_{\overline{\phi}}W - W^{-1}TW|| : T \text{ is compact}\} \\ &= \{||h_{\overline{\phi}\circ M}W - L|| : L \text{ is compact in } \mathcal{L}(L_a^2(\mathbb{D}))\} \\ &= ||h_{\overline{\phi}\circ M}||_e \\ &\leq C \lim_{a\to\partial\mathbb{D}} \sup |V(\phi\circ M)(a)| = C \lim_{Rew\to 0} \sup |(\mathcal{V}_1\phi)(w)|. \end{aligned}$$

□

Corollary 1. *Let $\phi \in L^2(\mathbb{C}_+, d\mu)$. Then $\hbar_{\overline{\phi}}$ is a compact operator from $L_a^2(\mathbb{C}_+, d\mu)$ to $\overline{L_a^2(\mathbb{C}_+, d\mu)}$ if and only if $\mathcal{V}_1\phi(w) \to 0$ as $Rew \to 0$.*

Proof. Suppose $\hbar_{\overline{\phi}}$ is compact. Since $\hbar_{\overline{\phi}}$ is bounded and $||\hbar_{\overline{\phi}}||_e = 0$. It thus follows from Theorem 2, that $\lim \sup_{Rew\to 0} |\mathcal{V}_1\phi(w)| = 0$. That is, $\mathcal{V}_1\phi(w) \to 0$ as $Rew \to 0$. On the other hand, $\mathcal{V}_1\phi(w) \to 0$ and since $\mathcal{V}_1\phi$ is a continuous functions, we obtain that $\mathcal{V}_1\phi$ is bounded. Therefore, $\hbar_{\overline{\phi}}$ is bounded. Hence, from Theorem 3, we obtain $||\hbar_{\overline{\phi}}||_e = 0$. Thus $\hbar_{\overline{\phi}}$ is compact. □

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