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EXISTENCE AND CONTROLLABILITY OF FRACTIONAL NEUTRAL INTEGRO-DIFFERENTIAL SYSTEMS WITH STATE-DEPENDENT DELAY*

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Abstract

In light of ideas for semigroups, fractional calculus and Banach contraction principle, this manuscript is mainly concerned with existence and controllability of fractional neutral integro-differential structures with state-dependent delay in Banach spaces. To obtain our results, our working hypotheses are that the functions determining the equation satisfy certain Lipschitz conditions of local type which is similar to the hypotheses [5]. Examples are presented to demonstrate the application of the results established.

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1 Introduction

The hypothesis of semigroups of bounded linear operator is carefully related to tackling differential and integro-differential conditions in Banach spaces. As of late, this idea has been utilized to a noteworthy kind of nonlinear differential conditions in Banach spaces. For more purposes of enthusiasm on this idea, we insinuate the peruser to Pazy [25]. The concept of controllability is in accordance with the mathematical criteria of the dynamical system. In respect to control theory, a dynamical system is controllable if, with a acceptable selection of inputs, it can be influenced from any initial state to any preferred last state in just only a certain time. The control theory has been sufficiently created amid the most recent three decades (for instance, see [36, 6, 21]). Nevertheless, because of the rise of fractional neutral integro-differential systems (abbreviated, FNIDS) in numerous practical designs one needs more intense improvements.

The breakthrough of fractional calculus come up new request in vital physics, which offers incredible convoluted interest for the physicists and mathematicians in the basic principle of fractional calculus. The fractional differential equations (abbreviated, FDEs) were regarded as to be the significant tool, which could depict dynamical movements of real existence phenomena greater exactly. For living proof, the nonlinear wavering of seismic tremor may be fairly displayed with fractional derivatives. We will locate the different usages of FDE in control speculation, nonlinear wavering of quake, the fluid-dynamic site visitors version, the study of air and in pretty much every field of technology and technological innovation. For critical assurances approximately fractional frameworks, it is easy to make reference to the treatises [7, 20], and the papers [10, 28, 1, 19, 23, 35, 3, 15, 27], and the references cited therein. Fractional equation with delay features occur in a few areas, for example, therapeutic and physical with state-dependent delay (abbreviated, SDD) or non-constant delay. Nowadays, existence results of mild solutions for such issues became very appealing and numerous researchers taking a shot at it, see for example [2, 9, 4, 11, 12, 32, 31].

The presence, controllability and different subjective and quantitative properties of FDEs are the most propelling domain of investigation, for example, see [8, 24, 29, 16]. As of late, Carvalho dos Santos et al. [11] analyzed the existence of solutions for fractional integro-differential equation with SDD in Banach spaces. Sakthivel et al. [29] cooperate with the approximate controllability of fractional neutral stochastic model with infinite delay by put on the payroll expropriate fixed point techniques. In [24, 16], the writers offer adequate circumstances for the stochastic differential models with infinite delay. Lately, Benchohra et al. [4] researched the existence of mild solutions on a compact interval for fractional integro-differential equation with SDD in Banach spaces. However, existence results for FNIDS with SDD in $\mathscr{B}_h$ phase space adages have not yet been totally inspected.

Inspired by the above mentioned papers [5, 4, 16], the principle motivation behind this manuscript is to analyze the existence results for the following model

$$
\begin{aligned}
{}^{C}D_t^{\alpha}\left[u(t) - \mathscr{G}\left(t, u_{\zeta(t,u_t)}, \int_0^t e_1(t,s,u_{\zeta(s,u_s)})ds\right)\right] &= \mathscr{A}u(t) \\
+ \mathscr{F}\left(t, u_{\zeta(t,u_t)}, \int_0^t e_2(t,s,u_{\zeta(s,u_s)})ds\right) &+ \int_0^t e_3(t,s,u_{\zeta(s,u_s)})ds, \\
t \in \mathscr{I} = [0,T], & \qquad (1.1) \\
u_0 = \varsigma(t) \in \mathscr{B}_h, \quad t \in (-\infty, 0], & \qquad (1.2)
\end{aligned}
$$

where ${}^{C}D_t^{\alpha}(0 < \alpha < 1)$ is the Caputo's fractional derivative of order α, and the operator $\mathscr{A}$ is the infinitesimal generator of a strongly continuous semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$ in a Banach space $\mathbb{X}$. $\mathscr{F} : \mathscr{I} \times \mathscr{B}_h \times \mathbb{X} \to \mathbb{X}, e_i : \mathscr{D} \times \mathscr{B}_h \to \mathbb{X}$, $i = 1, 2, 3$; $\mathscr{D} = \{(t,s) \in \mathscr{I} \times \mathscr{I} : 0 \leq s \leq t \leq T\}, \mathscr{G} : \mathscr{I} \times \mathscr{B}_h \times \mathbb{X} \to \mathbb{X}$, $\zeta : \mathscr{I} \times \mathscr{B}_h \to \mathbb{R}$ are appropriate functions and $\mathscr{B}_h$ is a phase space characterized in Preliminaries.

For nearly any continuous function x signalize on $(-\infty, T]$ and any $t \geq 0$, we decide on by u_t the part of $\mathscr{B}_h$ indicate by $u_t(\theta) = u(t+\theta)$ for $\theta \leq 0$. Now $u_t(\cdot)$ speaks to the historical backdrop of the state from every $\theta \in (-\infty, 0]$ in all likelihood the current time t.

Contrary to the existing consequence, this manuscript has a few effective elements: initially, we consist of the integral term in the non-linear term $\mathscr{F}$ and present a suitable thought of mild solution of the version (1.1)-(1.2). At that point, in light of local Lipschitz conditions of the concerned functions, we examine the existence and controllability of mild solutions for FNIDS with SDD of the trouble (1.1)-(1.2) beneath Banach fixed point hypothesis, and the outcomes in [4] may be seen as the unique conditions. And furthermore, we actualize $\mathscr{B}_h$ phase space axioms to look at the model (1.1)-(1.2).

We move forward as follows. Section 2 is focused on a survey of some indispensable viewpoints that will be utilized in this work to accomplish our key results. In Section 3 and 4, we declare and show the existence and controllability results by suggests of fixed point hypothesis correspondingly. As a very last point, examples are given to demonstrate our consequences.

2 Preliminaries

In this section, we present some primary components which are required to confirm the main results.

Let $\mathcal{L}(\mathbb{X}) : \mathbb{X} \to \mathbb{X}$ represents the Banach space of all bounded linear operators, obtain its norm recognized as $\| \cdot \|_{\mathcal{L}(\mathbb{X})}$.

Define the continuous functions $\mathbb{C}(\mathscr{I}, \mathbb{X}) : \mathscr{I} \to \mathbb{X}$, secure its norm recognized as $\| \cdot \|_{\mathbb{C}(\mathscr{I},\mathbb{X})}$. As well, $B_r(u, \mathbb{X})$ represents the closed ball in $\mathbb{X}$ with the middle at u and therefore the the distance r.

Let $\mathscr{A} : D(\mathscr{A}) \to \mathbb{X}$ be the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$. Without loss of simplification, we expect that $0 \in \rho(\mathscr{A})$. Then it is attainable to determine the fractional power $\mathscr{A}^\beta$ for $0 < \beta \leq 1$, as a closed linear operator on its domain $D(\mathscr{A})^\beta$, being dense in $\mathbb{X}$. Moreover, the subspace $D(\mathscr{A})^\beta$ is dense in $\mathbb{X}$ and the expression $\|z\|_\beta = \|\mathscr{A}^\beta z\|$, $z \in D(\mathscr{A}^\beta)$, defines a norm on $D(\mathscr{A}^\beta)$. For $0 < \alpha \leq \beta \leq 1$, $\mathbb{X}_\beta \to \mathbb{X}_\alpha$ and the imbedding is compact whenever the resolvent operator of $\mathscr{A}$ is compact. Also for every $0 < \beta \leq 1$, there exists $\mathcal{M}_\beta > 0$ ensure that

$$\| \mathscr{A}^\beta \mathbb{T}(t) \| \leq \frac{\mathcal{M}_\beta}{t^\beta}, \quad 0 < t \leq T.$$

With this discussion, we recall fundamental properties of fractional powers $\mathscr{A}^\beta$ from Pazy [30, 25].

It must be printed that, as soon as the delay is infinite, then we had like to debate the theoretical phase space $\mathscr{B}_h$ in an exceedingly useful method. During this manuscript, we tend to deliberate phase spaces $\mathscr{B}_h$ that square measure same as delineated in [13]. So, we tend to bypass the small print.

We count on that the phase space $(\mathscr{B}_h, \| \cdot \|_{\mathscr{B}_h})$ could be a semi-normed linear area of functions mapping $(-\infty, 0]$ into $\mathbb{X}$, and enjoyable the next elementary adages as a results of Hale and Kato (see case in purpose in [17, 18, 33, 14]).

If u is continuous function from $(-\infty, T], T > 0$ into $\mathbb{X}$, defined on $\mathscr{I}$ and $u_0 \in \mathscr{B}_h$, then for every $t \in \mathscr{I}$ the following situations preserve:

(P_1) u_t is in $\mathscr{B}_h$;

(P_2) $\|u(t)\|_{\mathbb{X}} \leq H\|u_t\|_{\mathscr{B}_h}$;

(P_3) $\|u_t\|_{\mathscr{B}_h} \leq \mathscr{D}_1(t)\sup\{\|u(s)\|_{\mathbb{X}} : 0 \leq s \leq t\} + \mathscr{D}_2(t)\|u_0\|_{\mathscr{B}_h}$, where $H > 0$ is a constant and $\mathscr{D}_1(\cdot) : [0, +\infty) \to [0, +\infty)$ is continuous, $\mathscr{D}_2(\cdot) : [0, +\infty) \to [0, +\infty)$ is locally bounded, and $\mathscr{D}_1, \mathscr{D}_2$ are independent of $u(\cdot)$.
For our convenience, denote $\mathscr{D}_1^* = \sup_{s\in\mathscr{I}} \mathscr{D}_1(s), \quad \mathscr{D}_2^* = \sup_{s\in\mathscr{I}} \mathscr{D}_2(s)$.

Recognize the space

$$\mathscr{B}_T = \{u : (-\infty, T] \to \mathbb{X} : u \in \mathscr{I} \text{ is continuous and } u_0 \in \mathscr{B}_h\}.$$

The function $\|\cdot\|_{\mathscr{B}_T}$ is defined as

$$\|u\|_{\mathscr{B}_T} = \|\varsigma\|_{\mathscr{B}_h} + \sup\{\|u(t)\|_{\mathbb{X}} : t \in \mathscr{I}\}, \ u \in \mathscr{B}_T.$$

To stay away from the reiterations of a few definitions utilized as a part of this paper we refer the readers: such as for the definition of the fractional integral, Riemann-Liouville fractional integral operator, the generalized Mittag-Leffler special function, Wright-type function and the Caputo's derivative one can see the papers [31, 30] and the monographs [20, 26, 37].

Assume that the subsequent system

$$^{C}D_t^{\alpha}u(t) = \mathscr{A}u(t) + \mathscr{F}(t), \tag{2.1}$$

$$u(0) = u_0, \tag{2.2}$$

where $^{C}D_t^{\alpha}$ and $\mathscr{A}$ are much the same as defined in (1.1)-(1.2).

By thinking the proofs as in [30, Lemma 6 and Lemma 9], we directly define the mild solution for the model (2.1)-(2.2).

Definition 1. *u is the function from $\mathscr{I}$ into $\mathbb{X}$. Assume that u is a mild solution of model (2.1)-(2.2) if $u \in \mathbb{C}(\mathscr{I}, \mathbb{X})$ make happen the supporters integral equation:*

$$u(t) \ = \mathbb{T}_{\alpha}(t)u_0 + \int_0^t \mathbb{S}_{\alpha}(t-s)\mathscr{F}(s)ds, \quad t \in \mathscr{I},$$

where

$$\mathbb{S}_{\alpha}(t) = \alpha\int_0^{\infty} \theta\xi_{\alpha}(\theta)t^{\alpha-1}\mathbb{T}(t^{\alpha}\theta)d\theta \quad \textit{and} \quad \mathbb{T}_{\alpha}(t) = \int_0^{\infty} \xi_{\alpha}(\theta)\mathbb{T}(t^{\alpha}\theta)d\theta.$$

Remark 1. *On the results received in the papers [31, 30, 38, 33], we clearly see that our definition of mild solution fulfills the given model (1.1)-(1.2).*

Definition 2. *[38, Definition 3.1] Let u be a function from $(-\infty, T]$ into $\mathbb{X}$ said to be mild solution of the model (1.1)-(1.2) if $u_0 = \varsigma \in \mathscr{B}_h$, and for each $s \in [0,t)$ the function $\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{G}\left(t, u_{\zeta(t,u_t)}, \int_0^t e_1(t,s,u_{\zeta(s,u_s)})ds\right)$ is integrable and the subsequent integral equation*

$$\begin{aligned} u(t) =& \mathbb{T}_\alpha(t)[\varsigma(0) - \mathscr{G}(0,\varsigma(0),0)] + \mathscr{G}\left(t, u_{\zeta(t,u_t)}, \int_0^t e_1(t,s,u_{\zeta(s,u_s)})ds\right) \\ &+ \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{G}\left(s, u_{\zeta(s,u_s)}, \int_0^s e_1(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right) ds \\ &+ \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{F}\left(s, u_{\zeta(s,u_s)}, \int_0^s e_2(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right) ds \\ &+ \int_0^t \mathbb{S}_\alpha(t-s)\left(\int_0^s e_3(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right) ds, \quad t \in \mathscr{I} \end{aligned} \tag{2.4}$$

is satisfied.

3 Existence Results

In this section, we consider $\varsigma \in \mathscr{B}_h$ a fixed function, $\mathscr{I} = [0,T]$. To simplify the writing of the text, in what follows, we assume that $0 \le \zeta(t,\psi) \le t$ for all $\psi \in \mathscr{B}_h$.

Presently, we listing the subsequent hypotheses:

(H1) The continuous function $\mathscr{G} : \mathscr{I} \times \mathscr{B}_h \times \mathbb{X} \to \mathbb{X}$ and we can find constants $\beta \in (0,1), \overline{\nu}_1, \overline{\nu}_2 > 0$ and $\overline{\nu}_1^* > 0$ in ways that $\mathscr{G}$ is $\mathbb{X}_\beta$ -valued and fulfills the subsequent assumptions:

$$\| \mathscr{A}^\beta \mathscr{G}(t,\psi_1,x) - \mathscr{A}^\beta \mathscr{G}(t,\psi_2,\overline{x}) \|_{\mathbb{X}} \le \overline{\nu}_1 \| \psi_1 - \psi_2 \|_{\mathscr{B}_h} + \overline{\nu}_2 \|x - \overline{x}\|_{\mathbb{X}},$$
$t \in \mathscr{I},\ \psi_1, \psi_2 \in \mathscr{B}_h,\ x, \overline{x} \in \mathbb{X},$

$$\| \mathscr{A}^\beta \mathscr{G}(t,\psi,0) \|_{\mathbb{X}} \le \overline{\nu}_1 \| \psi \|_{\mathscr{B}_h} + \overline{\nu}_1^*, \quad t \in \mathscr{I},\ \psi \in \mathscr{B}_h,$$

where

$$\overline{\nu}_1^* = \max_{t \in \mathscr{I}} \|\mathscr{A}^\beta \mathscr{G}(t,0,0)\|_{\mathbb{X}}.$$

(H2) $\mathscr{F} : \mathscr{I} \times \mathscr{B}_h \times \mathbb{X} \to \mathbb{X}$ is continuous and we can find constants $\nu_1, \nu_2 > 0$, and $\nu_1^* > 0$ in a way that

$$\|\mathscr{F}(t,\psi_1,x) - \mathscr{F}(t,\psi_2,\overline{x})\|_{\mathbb{X}} \le \nu_1 \|\psi_1 - \psi_2\|_{\mathscr{B}_h} + \nu_2 \|x - \overline{x}\|_{\mathbb{X}},$$
$t \in \mathscr{I},\ (\psi_1, \psi_2) \in \mathscr{B}_h^2, x, \overline{x} \in \mathbb{X},$

and

$$\nu_1^* = \max_{t \in \mathscr{I}} \|\mathscr{F}(t, 0, 0)\|_{\mathbb{X}}.$$

(H3) (i) The continuous functions $e_i : \mathscr{D} \times \mathscr{B}_h \to \mathbb{X}$ and we can observe the positive constants ξ_i, ξ_i^* to verify that

$$\big\|e_i(t, s, \psi_1) - e_i(t, s, \psi_2)\big\|_{\mathbb{X}} \leq \xi_i \|\psi_1 - \psi_2\|_{\mathscr{B}_h},$$
$$(t, s) \in \mathscr{D}, \quad (\psi_1, \psi_2) \in \mathscr{B}_h^2,\ i = 1, 2, 3;$$

and

$$\xi_i^* = \max_{t \in \mathscr{I}} \|e_i(t, s, 0)\|_{\mathbb{X}}, \quad i = 1, 2, 3.$$

(H4) For every $r > 0$, there exist constants $L_{\mathscr{G}}(r) > 0$, $L_{\mathscr{F}}(r) > 0$ and $L_{e_i}(r) > 0$, for $i = 1, 2, 3$ such that;

(i)

$$\|\mathscr{A}^\beta \mathscr{G}(t, x_{t_2}, x) - \mathscr{A}^\beta \mathscr{G}(t, x_{t_1}, y)\|_{\mathbb{X}} \leq L_{\mathscr{G}}(r)(|t_2 - t_1| + \|x - y\|_{\mathbb{X}}),$$
$$x, y \in \mathbb{X}, \quad t, t_1, t_2 \in \mathscr{I},$$

(ii)

$$\|\mathscr{F}(t, x_{t_2}, x) - \mathscr{F}(t, x_{t_1}, y)\|_{\mathbb{X}} \leq L_{\mathscr{F}}(r)(|t_2 - t_1| + \|x - y\|_{\mathbb{X}}),$$
$$x, y \in \mathbb{X}, \quad t, t_1, t_2 \in \mathscr{I},$$

(iii) $\big\|e_i(t, s, y_{t_2}) - e_i(t, s, y_{t_1})\big\|_{\mathbb{X}} \leq L_{e_i}(r)|t_2 - t_1|, \quad t, t_1, t_2 \in \mathscr{I}.$

(H5) The condition for the function $\zeta : \mathscr{I} \times \mathscr{B}_h \to [0, \infty)$ satisfies:

(i) For every $\psi \in \mathscr{B}_h$, the function $t \mapsto \zeta(t, \psi)$ is continuous.

(ii) There exists a constant $L_\zeta > 0$ such that

$$|\zeta(t, \varsigma_2) - \zeta(t, \varsigma_1)| \leq L_\zeta \|\varsigma_2 - \varsigma_1\|_{\mathscr{B}_h},\ \varsigma_1, \varsigma_2 \in \mathscr{B}_h, \quad \text{for all} \quad t \in \mathscr{I}.$$

(H6) The following inequalities holds:

(i) Let

$$\mathcal{M}\mathcal{M}_0\big[\overline{\nu}_1\|\varsigma\|_{\mathscr{B}_h}+\overline{\nu}_1^*\big]+\Big[(\overline{\nu}_1+\overline{\nu}_2T\xi_1)\left(\mathcal{M}_0+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)$$
$$+\frac{\mathcal{M}T^{\alpha}}{\Gamma(\alpha+1)}(\nu_1+\nu_2T\xi_2)+\frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\xi_3\Big](\mathscr{D}_1^*r+c_n)+\mathcal{M}_0\overline{\nu}_1^*+\mathcal{M}_0\overline{\nu}_2T\xi_1^*$$
$$+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}(\overline{\nu}_2T\xi_1^*+\overline{\nu}_1^*)+\frac{\mathcal{M}T^{\alpha}}{\Gamma(\alpha+1)}(\nu_2T\xi_2^*+\nu_1^*)$$
$$+\frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\xi_3^*\leq r,$$

where $\|\mathscr{A}^{-\beta}\|=\mathcal{M}_0$ and for some $r>0$.

(ii) Let

$$\Lambda^*=\mathscr{D}_1^*\Bigg[\Big(\overline{\nu}_1+\overline{\nu}_2T\xi_1+2L_{\mathscr{G}}(r)L_\zeta(1+TL_{e_1}(r))\Big)\cdot$$
$$\left(\mathcal{M}_0+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)+\frac{\mathcal{M}T^{\alpha}}{\Gamma(\alpha+1)}\cdot$$
$$\cdot\Big(\nu_1+\nu_2T\xi_2+2L_{\mathscr{F}}(r)L_\zeta(1+TL_{e_2}(r))\Big)$$
$$+\frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}(\xi_3+2L_\zeta L_{e_3}(r))\Bigg]<1$$

be ensure that $1>\Lambda^*\geq 0$.

Theorem 1. *Suppose that the conditions (H1)-(H6) hold. At that point the model (1.1)-(1.2) has unique mild solution in* $(-\infty,T]$.

Proof. We can transmute the model (1.1)-(1.2) into a fixed-point system. Perceive the operator $\Upsilon:\mathscr{B}_T\to\mathscr{B}_T$ specified by

$$(\Upsilon u)(t)=\begin{cases}\mathbb{T}_\alpha(t)[\varsigma(0)-\mathscr{G}(0,\varsigma(0),0)]+\mathscr{G}\left(t,u_{\zeta(t,u_t)},\int_0^t e_1(t,s,u_{\zeta(s,u_s)})ds\right)\\ +\int_0^t\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{G}\left(s,u_{\zeta(s,u_s)},\int_0^s e_1(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right)ds\\ +\int_0^t\mathbb{S}_\alpha(t-s)\mathscr{F}\left(s,u_{\zeta(s,u_s)},\int_0^s e_2(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right)ds\\ +\int_0^t\mathbb{S}_\alpha(t-s)\left(\int_0^s e_3(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right)ds,\ t\in\mathscr{I}.\end{cases}$$

It is observable that the fixed points of the operator Υ are mild solutions of the structure (1.1)-(1.2). The function $z(\cdot) : (-\infty, T] \to \mathbb{X}$ is defined by

$$z(s) = \begin{cases} \varsigma(s), & s \leq 0; \\ \mathbb{T}_\alpha(s)\varsigma(0), & s \in \mathscr{I}, \end{cases}$$

then $z_0 = \varsigma$ and $x(0) = 0$ with for each function $x \in \mathbb{C}(\mathscr{I}, \mathbb{R})$. We also defined as $\widetilde{x}$ is characterized by

$$\widetilde{x}(s) = \begin{cases} 0, & s \leq 0; \\ x(s), & s \in \mathscr{I}. \end{cases}$$

If $u(\cdot)$ fulfills (2.4), we can part it as $u(s) = z(s) + \widetilde{x}(s)$, that is $u(s) = z(s) + x(s), s \in \mathscr{I}$, which suggests $u_s = z_s + x_s$, for every $s \in \mathscr{I}$ and also the function $x(\cdot)$ satisfies

$$x(t) = \begin{cases} -\mathbb{T}_\alpha(t)\mathscr{G}(0,\varsigma,0) + \mathscr{G}\left(t, x_{\zeta(t,x_t+z_t)} + z_{\zeta(t,x_t+z_t)}, \right. \\ \left. \int_0^t e_1(t,s,x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)})ds\right) \\ + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{G}\left(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)}, \right. \\ \left. \int_0^s e_1(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\right) ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{F}\left(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)}, \right. \\ \left. \int_0^s e_2(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + x_{\zeta(\tau,x_\tau+z_\tau)})d\tau\right) ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\left(\int_0^s e_3(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\right) ds. \end{cases}$$

Let $\mathscr{B}_T^0 = \{x \in \mathscr{B}_T\colon\ x_0 = 0 \in \mathscr{B}_h\}$. Let $\|\cdot\|_{\mathscr{B}_T^0}$ be the seminorm in $\mathscr{B}_T^0$ described by

$$\|x\|_{\mathscr{B}_T^0} = \sup_{t\in\mathscr{I}} \|x(t)\|_{\mathbb{X}} + \|x_0\|_{\mathscr{B}_h} = \sup_{t\in\mathscr{I}} \|x(t)\|_{\mathbb{X}}, \quad x \in \mathscr{B}_T^0,$$

as a result $(\mathscr{B}_T^0, \|\cdot\|_{\mathscr{B}_T^0})$ is a Banach space. We delimit the operator $\overline{\Upsilon}$:

$\mathscr{B}_T^0 \to \mathscr{B}_T^0$ by

$$(\overline{\Upsilon}z)(t) = \begin{cases} -\mathbb{T}_\alpha(t)\mathscr{G}(0,\varsigma,0) + \mathscr{G}\left(t, x_{\zeta(t,x_t+z_t)} + z_{\zeta(t,x_t+z_t)},\right. \\ \left.\int_0^t e_1(t,s,x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)})ds\right) \\ + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{G}\left(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)},\right. \\ \left.\int_0^s e_1(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\right) ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{F}\left(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)},\right. \\ \left.\int_0^s e_2(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\right) ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\left(\int_0^s e_3(s,\tau,x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)})d\tau\right) ds. \end{cases}$$

It is vindicated that the operator Υ has a fixed point if and only if $\overline{\Upsilon}$ has a fixed point.

Remark 2. *Assume that $B_r = \{x \in \mathbb{X} : \|x\| \le r\}$ for a few $r > 0$. By the conditions $(P_1) - (P_3)$, we can get the following estimates:*

(i)

$$\begin{aligned} &\|x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)}\|_{\mathscr{B}_h} \\ &\le \|x_{\zeta(s,x_s+y_s)}\|_{\mathscr{B}_h} + \|z_{\zeta(s,x_s+z_s)}\|_{\mathscr{B}_h} \\ &\le \mathscr{D}_1^* \sup_{0\le\tau\le\zeta(s,x_s+z_s)} \|x(\tau)\|_{\mathbb{X}} + \mathscr{D}_2^*\|x_0\|_{\mathscr{B}_h} + \mathscr{D}_1^* \sup_{0\le\tau\le\zeta(s,x_s+z_s)} \|z(\tau)\| + \mathscr{D}_2^*\|z_0\|_{\mathscr{B}_h} \\ &\le \mathscr{D}_1^* \sup_{0\le\tau\le s} \|x(\tau)\|_{\mathbb{X}} + \mathscr{D}_1^*\|\mathbb{T}_\alpha(t)\|_{\mathcal{L}(\mathbb{X})}\|\varsigma(0)\| + \mathscr{D}_2^*\|\varsigma\|_{\mathscr{B}_h} \\ &\le \mathscr{D}_1^* \sup_{0\le\tau\le s} \|x(\tau)\|_{\mathbb{X}} + (\mathscr{D}_1^*\mathcal{M}H + \mathscr{D}_2^*)\|\varsigma\|_{\mathscr{B}_h} \\ &\le \mathscr{D}_1^* r + c_n, \end{aligned}$$

where $c_n = (\mathscr{D}_1^*\mathcal{M}H + \mathscr{D}_2^*)\|\varsigma\|_{\mathscr{B}_h}$. From suppositions (H1)-(H5) in concert with the earlier mentioned discussion, we sustain

(ii)

$$
\begin{aligned}
&\left\|\mathscr{G}\left(t, x_{\zeta(t,x_t+z_t)}+z_{\zeta(t,x_t+z_t)}, \int_0^t e_1(t,s,x_{\zeta(s,x_s+y_s)}+z_{\zeta(s,x_s+z_s)})ds\right)\right\|_{\mathbb{X}} \\
&\leq \|\mathscr{A}^{-\beta}\|\left[\left\|\mathscr{A}^{\beta}\mathscr{G}\left(t, x_{\zeta(t,x_t+z_t)}+z_{\zeta(t,x_t+z_t)}, \int_0^t e_1(t,s,x_{\zeta(s,x_s+y_s)}+z_{\zeta(s,x_s+z_s)})ds\right)\right.\right. \\
&\left.\left.-\mathscr{A}^{\beta}\mathscr{G}(t,0,0)\right\|_{\mathbb{X}}+\|\mathscr{A}^{\beta}\mathscr{G}(t,0,0)\|_{\mathbb{X}}\right] \\
&\leq \mathcal{M}_0\left[\overline{\nu}_1\|x_{\zeta(t,x_t+z_t)}+z_{\zeta(t,x_t+z_t)}\|_{\mathscr{B}_h}\right. \\
&\left.+\overline{\nu}_2\left\|\int_0^t e_1(t,s,x_{\zeta(s,x_s+y_s)}+z_{\zeta(s,x_s+z_s)})ds\right\|_{\mathbb{X}}+\overline{\nu}_1^*\right] \\
&\leq \mathcal{M}_0\overline{\nu}_1(\mathscr{D}_1^* r+c_n)+\mathcal{M}_0\overline{\nu}_2\int_0^t\left[\left\|e_1(t,s,x_{\zeta(s,x_s+y_s)}+z_{\zeta(s,x_s+z_s)})-e_1(t,s,0)\right\|_{\mathbb{X}}\right. \\
&\left.+\|e_1(t,s,0)\|_{\mathbb{X}}\right]ds+\mathcal{M}_0\overline{\nu}_1^* \\
&\leq \mathcal{M}_0\overline{\nu}_1(\mathscr{D}_1^* r+c_n)+\mathcal{M}_0\overline{\nu}_1^*+\mathcal{M}_0\overline{\nu}_2 T\big[\xi_1\|x_{\zeta(t,x_t+z_t)}+z_{\zeta(t,x_t+z_t)}\|_{\mathscr{B}_h}+\xi_1^*\big] \\
&\leq \mathcal{M}_0\overline{\nu}_1(\mathscr{D}_1^* r+c_n)+\mathcal{M}_0\overline{\nu}_1^*+\mathcal{M}_0\overline{\nu}_2 T\xi_1(\mathscr{D}_1^* r+c_n)+\mathcal{M}_0\overline{\nu}_2 T\xi_1^*,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\mathscr{G}\left(t, x_{\zeta(t,x_t+z_t)}+z_{\zeta(t,x_t+z_t)}, \int_0^t e_1(t,s,x_{\zeta(s,x_s+y_s)}+z_{\zeta(s,x_s+z_s)})ds\right)\right. \\
&\qquad\left.-\mathscr{G}\left(t, \overline{x}_{\zeta(t,\overline{x}_t+z_t)}+z_{\zeta(t,\overline{x}_t+z_t)}, \int_0^t e_1(t,s,\overline{x}_{\zeta(s,\overline{x}_s+z_s)}+z_{\zeta(s,\overline{x}_s+z_s)})\right)\right\|_{\mathbb{X}} \\
&\leq \left\|\mathscr{G}\left(t, x_{\zeta(t,x_t+z_t)}+z_{\zeta(t,x_t+z_t)}, \int_0^t e_1(t,s,x_{\zeta(s,x_s+y_s)}+z_{\zeta(s,x_s+z_s)})ds\right)\right. \\
&\qquad-\mathscr{G}\left(t, \overline{x}_{\zeta(t,x_t+z_t)}+z_{\zeta(t,x_t+z_t)}, \int_0^t e_1(t,s,\overline{x}_{\zeta(s,x_s+z_s)}+z_{\zeta(s,x_s+z_s)})ds\right) \\
&\qquad+\mathscr{G}\left(t, \overline{x}_{\zeta(t,x_t+z_t)}+z_{\zeta(t,x_t+z_t)}, \int_0^t e_1(t,s,\overline{x}_{\zeta(s,x_s+z_s)}+z_{\zeta(s,x_s+z_s)})ds\right) \\
&\qquad\left.-\mathscr{G}\left(t, \overline{x}_{\zeta(t,\overline{x}_t+z_t)}+z_{\zeta(t,\overline{x}_t+z_t)}, \int_0^t e_1(t,s,\overline{x}_{\zeta(s,\overline{x}_s+z_s)}+z_{\zeta(s,\overline{x}_s+z_s)})\right)\right\|_{\mathbb{X}}
\end{aligned}
$$

$$\leq \mathcal{M}_0 \Big\| \mathscr{A}^\beta \mathscr{G}\left(t, x_{\zeta(t,x_t+z_t)} + z_{\zeta(t,x_t+z_t)}, \int_0^t e_1(t,s,x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)})ds\right)$$
$$- \mathscr{A}^\beta \mathscr{G}\left(t, \overline{x}_{\zeta(t,x_t+z_t)} + z_{\zeta(t,x_t+z_t)}, \int_0^t e_1(t,s,\overline{x}_{\zeta(s,\overline{x}_s+z_s)} + z_{\zeta(s,x_s+z_s)})ds\right)\Big\|_{\mathbb{X}}$$
$$+ \mathcal{M}_0 \Big\| \mathscr{A}^\beta \mathscr{G}\left(t, \overline{x}_{\zeta(t,x_t+z_t)} + z_{\zeta(t,x_t+z_t)}, \int_0^t e_1(t,s,\overline{x}_{\zeta(s,\overline{x}_s+z_s)} + z_{\zeta(s,x_s+z_s)})ds\right)$$
$$- \mathscr{A}^\beta \mathscr{G}\left(t, \overline{x}_{\zeta(t,\overline{x}_t+z_t)} + z_{\zeta(t,\overline{x}_t+z_t)}, \int_0^t e_1(t,s,\overline{x}_{\zeta(s,\overline{x}_s+z_s)} + z_{\zeta(s,\overline{x}_s+z_s)})\right)\Big\|_{\mathbb{X}}$$
$$\leq \mathcal{M}_0 \Big[\overline{\nu}_1 \|x_{\zeta(t,x_t+z_t)} - \overline{x}_{\zeta(t,x_t+z_t)}\|_{\mathscr{B}_h} + \overline{\nu}_2 T \xi_1 \|x_{\zeta(t,x_t+z_t)} - \overline{x}_{\zeta(t,x_t+z_t)}\|_{\mathscr{B}_h}$$
$$+ L_{\mathscr{G}}(r)[2L_\zeta \|x_t - \overline{x}_t\|_{\mathscr{B}_h} + 2TL_{e_1}(r)L_\zeta \|x_t - \overline{x}_t\|_{\mathscr{B}_h}]\Big]$$
$$\leq \mathcal{M}_0 \mathscr{D}_1^* \Big[\overline{\nu}_1 + \overline{\nu}_2 T \xi_1 + 2L_{\mathscr{G}}(r) L_\zeta (1 + TL_{e_1}(r))\Big] \|x - \overline{x}\|_{\mathscr{B}_T^0},$$

since

$$\begin{aligned}\|x_{\zeta(s,x_s+y_s)} - \overline{x}_{\zeta(s,x_s+z_s)}\|_{\mathscr{B}_h} &\leq \mathscr{D}_1^* \max_{0\leq\tau\leq\zeta(s,x_s+z_s)} \|x(\tau) - \overline{x}(\tau)\| + \mathscr{D}_2^* \|x_0 - \overline{x}_0\|_{\mathscr{B}_h} \\ &\leq \mathscr{D}_1^* \max_{0\leq\tau\leq s} \|x(s) - \overline{x}(s)\|_{\mathbb{X}} \\ &\leq \mathscr{D}_1^* \|x - \overline{x}\|_{\mathscr{B}_T^0}.\end{aligned}$$

(iii) By employing the results of [29, Lemma 2.2 (iii), pp. 295] and Definition 2.2, we receive

$$\| \int_0^t \mathscr{A} \mathbb{S}_\alpha(t-s) \mathscr{G}\left(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)},\right.$$
$$\left.\int_0^s e_1(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\right) ds\|_{\mathbb{X}}$$
$$\leq \int_0^t \|\mathscr{A}^{1-\beta} \mathbb{S}_\alpha(t-s)\|_{\mathcal{L}(\mathbb{X})} \times \Big[\|\mathscr{A}^\beta \mathscr{G}\left(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)},\right.$$
$$\left.\int_0^s e_1(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\right) - \mathscr{A}^\beta \mathscr{G}(s,0,0)\|_{\mathbb{X}} + \|\mathscr{A}^\beta \mathscr{G}(s,0,0)\|_{\mathbb{X}}\Big] ds$$

$$\leq \int_0^t \left\| \left\{ \alpha \int_0^\infty \theta \xi_\alpha(\theta)(t-s)^{\alpha-1} \mathscr{A}^{1-\beta} \mathbb{T}((t-s)^\alpha \theta) d\theta \right\} \right\|_{\mathcal{L}(\mathbb{X})}$$
$$\left[\overline{\nu}_1 \| x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)} \|_{\mathscr{B}_h} + \overline{\nu}_2 \left\| \int_0^s e_1(s, \right.\right.$$
$$\left.\left. \tau, x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)}) d\tau \right\|_{\mathbb{X}} + \overline{\nu}_1^* \right] ds$$
$$\leq \int_0^t \alpha \mathcal{M}_{1-\beta}(t-s)^{\alpha\beta-1} \left[\int_0^\infty \theta^\beta \xi_\alpha(\theta) d\theta \right] \left[\overline{\nu}_1 \| x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)} \|_{\mathscr{B}_h} \right.$$
$$\left. + \overline{\nu}_2 \left\| \int_0^s e_1(s, \tau, x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)}) d\tau \right\|_{\mathbb{X}} + \overline{\nu}_1^* \right] ds. \tag{3.1}$$

On the other hand, from [38, Lemma 3.2], we see that

$$\int_0^\infty \theta^{-q} \xi_\alpha(\theta) d\theta = \frac{\Gamma(1+\frac{q}{\alpha})}{\Gamma(1+q)}, \quad q \in [0,1],$$

and

$$\int_0^\infty \theta^\beta \xi_\alpha(\theta) d\theta = \int_0^\infty \frac{1}{\theta^{\beta\alpha}} \xi_\alpha(\theta) d\theta = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}. \tag{3.2}$$

Then utilizing the result (3.2) in (3.1), we get

$$\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)} \left[\overline{\nu}_1(\mathscr{D}_1^* r + c_n) + \overline{\nu}_2 \int_0^t \left[\| e_1(t,s,x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)}) \right.\right.$$
$$\left.\left. - e_1(t,s,0) \|_{\mathbb{X}} + \| e_1(t,s,0) \|_{\mathbb{X}} \right] ds + \overline{\nu}_1^* \right]$$
$$\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)} \left[\overline{\nu}_1(\mathscr{D}_1^* r + c_n) + \overline{\nu}_2 T \left[\xi_1 \| x_{\zeta(t,x_t+z_t)} + \right.\right.$$
$$\left.\left. z_{\zeta(t,x_t+z_t)} \|_{\mathscr{B}_h} + \xi_1^* \right] + \overline{\nu}_1^* \right]$$
$$\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)} \left[\overline{\nu}_1(\mathscr{D}_1^* r + c_n) + \overline{\nu}_2 T \left[\xi_1(\mathscr{D}_1^* r + c_n) + \xi_1^* \right] + \overline{\nu}_1^* \right],$$

and

$$\Big\| \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{G}\left(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)},\right.$$
$$\left.\int_0^s e_1(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\right) ds$$
$$-\int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{G}\left(s, \overline{x}_{\zeta(s,\overline{x}_s+z_s)} + z_{\zeta(s,\overline{x}_s+z_s)},\right.$$
$$\left.\int_0^s e_1(s,\tau,\overline{x}_{\zeta(\tau,\overline{x}_\tau+z_\tau)} + z_{\zeta(t,\overline{x}_t+z_t)})d\tau\right) ds\Big\|_{\mathbb{X}}$$
$$\le \int_0^t \|\mathscr{A}^{1-\beta}\mathbb{S}_\alpha(t-s)\|_{\mathcal{L}(\mathbb{X})}$$
$$\times\left[\Big\|\mathscr{A}^\beta\mathscr{G}\left(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)}, \int_0^s e_1(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\right)\right.$$
$$-\mathscr{A}^\beta\mathscr{G}\left(s, \overline{x}_{\zeta(s,x_s+z_s)} + z_{\zeta(s,x_s+z_s)}, \int_0^s e_1(s,\tau,\overline{x}_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\right)$$
$$+\mathscr{A}^\beta\mathscr{G}\left(s, \overline{x}_{\zeta(s,x_s+z_s)} + z_{\zeta(s,x_s+z_s)}, \int_0^s e_1(s,\tau,\overline{x}_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\right)$$
$$-\mathscr{A}^\beta\mathscr{G}\left(s, \overline{x}_{\zeta(s,\overline{x}_s+z_s)} + z_{\zeta(s,\overline{x}_s+z_s)}, \int_0^s e_1\left(s,\tau,\overline{x}_{\zeta(\tau,\overline{x}_\tau+z_\tau)}+\right.\right.$$
$$\left.\left.\left. z_{\zeta(\tau,\overline{x}_\tau+z_\tau)}\right)d\tau\right)\Big\|_{\mathbb{X}}\right]ds$$
$$\le \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\mathscr{D}_1^*\Big[\overline{\nu}_1 + \overline{\nu}_2 T\xi_1 + 2L_{\mathscr{G}}(r)L_\zeta(1+TL_{e_1}(r))\Big]\|z-\overline{z}\|_{\mathscr{B}_T^0}.$$

(iv)

$$\begin{aligned}\|-\mathbb{T}_\alpha(t)\mathscr{G}(0,\varsigma,0)\|_{\mathbb{X}} &= \Big\|\Big\{\int_0^\infty \xi_\alpha(\theta)\mathbb{T}(t^\alpha\theta)d\theta\Big\}\mathscr{G}(0,\varsigma,0)\Big\|_{\mathbb{X}}\\ &= \mathcal{M}\|(\mathscr{A})\|^{-\beta}\|(\mathscr{A})^\beta\mathscr{G}(0,\varsigma,0)\|_{\mathbb{X}}\\ &\le \mathcal{M}\mathcal{M}_0\Big[\overline{\nu}_1\|\varsigma\|_{\mathscr{B}_h} + \overline{\nu}_1^*\Big].\end{aligned}$$

(v)

$$\Big\|\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{F}\left(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)},\right.$$
$$\left.\int_0^s e_2(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\right) ds\Big\|_{\mathbb{X}}$$
$$\le \int_0^t \Big\|\Big\{\alpha\int_0^\infty \theta\xi_\alpha(\theta)(t-s)^{\alpha-1}\mathbb{T}((t-s)^\alpha\theta)d\theta\Big\}\Big\|_{\mathcal{L}(\mathbb{X})}$$

$$\times \Big[\|\mathscr{F}\Big(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)}, \int_0^s e_2(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\Big)$$
$$- \mathscr{F}(s,0,0)\|_{\mathbb{X}} + \|\mathscr{F}(s,0,0)\|_{\mathbb{X}}\Big]ds$$
$$\leq \frac{\mathcal{M}T^\alpha}{\alpha\Gamma(\alpha)}\Big[\nu_1\|x_{\zeta(t,x_t+z_t)} + z_{\zeta(t,x_t+z_t)}\|_{\mathscr{B}_h}$$
$$+ \nu_2\Big\|\int_0^t e_2(t,s,x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)})ds\Big\|_{\mathbb{X}} + \nu_1^*\Big]$$
$$\leq \frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}\Bigg[\nu_1(\mathscr{D}_1^* r + c_n) + \nu_2\int_0^t\Big[\big\|e_2(t,s,x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)}) - e_2(t,s,0)\big\|_{\mathbb{X}}$$
$$+ \|e_2(t,s,0)\|_{\mathbb{X}}\Big]ds + \nu_1^*\Bigg]$$
$$\leq \frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}\Big[\nu_1(\mathscr{D}_1^* r + c_n) + \nu_2 T\big[\xi_2\|x_{\zeta(t,x_t+z_t)} + z_{\zeta(t,x_t+z_t)}\|_{\mathscr{B}_h} + \xi_2^*\big] + \nu_1^*\Big]$$
$$\leq \frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}\Big[\nu_1(\mathscr{D}_1^* r + c_n) + \nu_2 T\big[\xi_2(\mathscr{D}_1^* r + c_n) + \xi_2^*\big] + \nu_1^*\Big],$$

and

$$\Big\|\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{F}\Big(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)}, \int_0^s e_2(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\Big)ds$$
$$- \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{F}\Big(s, \overline{x}_{\zeta(s,\overline{x}_s+z_s)} + z_{\zeta(s,\overline{x}_s+z_s)}, \int_0^s e_2(s,\tau,\overline{x}_{\zeta(\tau,\overline{x}_\tau+z_\tau)} + z_{\zeta(\tau,\overline{x}_\tau+z_\tau)})d\tau\Big)ds\Big\|_{\mathbb{X}}$$
$$\leq \int_0^t \Big\|\Big\{\alpha\int_0^\infty \theta\xi_\alpha(\theta)(t-s)^{\alpha-1}\mathbb{T}((t-s)^\alpha\theta)d\theta\Big\}\Big\|_{\mathcal{L}(\mathbb{X})}$$
$$\times\Big\|\mathscr{F}\Big(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)}, \int_0^s e_2(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\Big)$$
$$- \mathscr{F}\Big(s, \overline{x}_{\zeta(s,x_s+z_s)} + z_{\zeta(s,x_s+z_s)}, \int_0^s e_2(s,\tau,\overline{x}_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\Big)$$
$$+ \mathscr{F}\Big(s, \overline{x}_{\zeta(s,x_s+z_s)} + z_{\zeta(s,x_s+z_s)}, \int_0^s e_2(s,\tau,\overline{x}_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\Big)$$
$$- \mathscr{F}\Big(s, \overline{x}_{\zeta(s,\overline{x}_s+z_s)} + z_{\zeta(s,\overline{x}_s+z_s)}, \int_0^s e_2(s,\tau,\overline{x}_{\zeta(\tau,\overline{x}_\tau+z_\tau)} + z_{\zeta(\tau,\overline{x}_\tau+z_\tau)})d\tau\Big)\Big\|_{\mathbb{X}}ds$$

$$\leq \frac{\mathcal{M}T^{\alpha}}{\Gamma(\alpha+1)}\Big[\nu_1\|x_{\zeta(t,x_t+z_t)}-\overline{x}_{\zeta(t,x_t+z_t)}\|_{\mathscr{B}_h}+\nu_2$$

$$\cdot\Big\|\int_0^t e_2(t,s,x_{\zeta(s,x_s+y_s)}+z_{\zeta(s,x_s+z_s)})ds-\int_0^t e_2(t,s,\overline{x}_{\zeta(s,x_s+z_s)}+z_{\zeta(s,x_s+z_s)})ds\Big\|_{\mathbb{X}}$$

$$+L_{\mathscr{F}}(r)\Big(2|\zeta(t,x_t+z_t)-\zeta(t,\overline{x}_t+z_t)|+2TL_{e_2}(r)|\zeta(t,x_t+z_t)-\zeta(t,\overline{x}_t+z_t)|\Big)\Big]$$

$$\leq \frac{\mathcal{M}T^{\alpha}}{\Gamma(\alpha+1)}\mathscr{D}_1^*\Big[\nu_1+\nu_2T\xi_2+2L_{\mathscr{F}}(r)L_{\zeta}(1+TL_{e_2}(r))\Big]\|x-\overline{x}\|_{\mathscr{B}_T^0}.$$

(vi)

$$\Big\|\int_0^t \mathbb{S}_{\alpha}(t-s)\int_0^s e_3(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)}+z_{\zeta(\tau,x_\tau+z_\tau)})d\tau ds\Big\|_{\mathbb{X}}$$

$$\leq \int_0^t \Big\|\Big\{\alpha\int_0^\infty \theta\xi_{\alpha}(\theta)(t-s)^{\alpha-1}\mathbb{T}((t-s)^{\alpha}\theta)d\theta\Big\}\Big\|_{\mathcal{L}(\mathbb{X})}\cdot$$

$$\cdot\|e_3(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)}+z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\|_{\mathbb{X}}ds$$

$$\leq \frac{\mathcal{M}T^{\alpha}}{\Gamma(\alpha+1)}\int_0^t\Big[\big\|e_3(t,s,x_{\zeta(s,x_s+y_s)}+z_{\zeta(s,x_s+z_s)})-e_3(t,s,0)\big\|_{\mathbb{X}}$$

$$+\|e_3(t,s,0)\|_{\mathbb{X}}\Big]ds$$

$$\leq \frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\Big[\xi_3\|x_{\zeta(t,x_t+z_t)}+z_{\zeta(t,x_t+z_t)}\|_{\mathscr{B}_h}+\xi_3^*\Big]$$

$$\leq \frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\Big[\xi_3(\mathscr{D}_1^*r+c_n)+\xi_3^*\Big],$$

and

$$\Big\|\int_0^t \mathbb{S}_{\alpha}(t-s)\int_0^s e_3(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)}+z_{\zeta(\tau,x_\tau+z_\tau)})d\tau ds$$

$$-\int_0^t \mathbb{S}_{\alpha}(t-s)\int_0^s e_3(s,\tau,\overline{x}_{\zeta(\tau,\overline{x}_\tau+z_\tau)}+z_{\zeta(\tau,\overline{x}_\tau+z_\tau)})d\tau ds\Big\|_{\mathbb{X}}$$

$$\leq \int_0^t \Big\|\Big\{\alpha\int_0^\infty \theta\xi_{\alpha}(\theta)(t-s)^{\alpha-1}\mathbb{T}((t-s)^{\alpha}\theta)d\theta\Big\}\Big\|_{\mathcal{L}(\mathbb{X})}$$

$$\cdot\int_0^s\Big\|e_3(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)}+z_{\zeta(\tau,x_\tau+z_\tau)})$$

$$
\begin{aligned}
&- e_3(s,\tau,\overline{x}_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)}) + e_3(s,\tau,\overline{x}_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)}) \\
&- e_3(s,\tau,\overline{x}_{\zeta(\tau,\overline{x}_\tau+z_\tau)} + z_{\zeta(\tau,\overline{x}_\tau+z_\tau)})\Big\|_{\mathbb{X}} d\tau ds \\
&\leq \frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\mathscr{D}_1^*[\xi_3 + 2L_{e_3}(r)L_\zeta]\|x-\overline{x}\|_{\mathscr{B}_T^0}.
\end{aligned}
$$

At first, we show that $\overline{\Upsilon} : B_r(0,\mathscr{B}_T^0) \to B_r(0,\mathscr{B}_T^0)$. For any $x(\cdot) \in \mathscr{B}_T^0$, by take on Remark 3.1, we sustain

$$
\begin{aligned}
&\|(\overline{\Upsilon}x)(t)\|_{\mathbb{X}} \leq \mathcal{M}\mathcal{M}_0\big[\overline{\nu}_1\|\varsigma\|_{\mathscr{B}_h} + \overline{\nu}_1^*\big] \\
&+ \Big[(\overline{\nu}_1 + \overline{\nu}_2T\xi_1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right) + \frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}(\nu_1+\nu_2T\xi_2) \\
&+ \frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\xi_3\Big](\mathscr{D}_1^*r + c_n) + \mathcal{M}_0\overline{\nu}_1^* + \mathcal{M}_0\overline{\nu}_2T\xi_1^* + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}(\overline{\nu}_2T\xi_1^* + \overline{\nu}_1^*) \\
&+ \frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}(\nu_2T\xi_2^* + \nu_1^*) + \frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\xi_3^* \\
&\leq r.
\end{aligned}
$$

As a consequence, $\overline{\Upsilon} : B_r(0,\mathscr{B}_T^0) \to B_r(0,\mathscr{B}_T^0)$. At long last, we demonstrate that $\overline{\Upsilon}$ is a contraction on $B_r(0,\mathscr{B}_T^0)$. If $x,\overline{x} \in B_r(0,\mathscr{B}_T^0)$, from Remark 3.1, we support

$$
\begin{aligned}
&\|(\overline{\Upsilon}x)(t) - (\overline{\Upsilon}\overline{x})(t)\|_{\mathbb{X}} \leq \mathscr{D}_1^*\Bigg[\Big(\overline{\nu}_1 + \overline{\nu}_2T\xi_1 + 2L_{\mathscr{G}}(r)L_\zeta(1+TL_{e_1}(r))\Big) \\
&\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right) + \frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}\Big(\nu_1 + \nu_2T\xi_2 + 2L_{\mathscr{F}}(r)L_\zeta(1+TL_{e_2}(r))\Big) \\
&+ \frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}[\xi_3 + 2L_\zeta L_{e_3}(r)]\Bigg]\|x-\overline{x}\|_{\mathscr{B}_T^0} \leq \Lambda^*\|x-\overline{x}\|_{\mathscr{B}_T^0}.
\end{aligned}
$$

From the assumption (H6) and in the perspective of the contraction mapping principle, we understand that $\overline{\Upsilon}$ includes a unique fixed point $u \in \mathscr{B}_T^0$ which is a mild solution of the model (1.1)-(1.2) on $(-\infty, T]$. The proof is now completed. □

4 Controllability Results

In this section, we present and prove that the controllability of the following model

$$
\begin{aligned}
{}^{C}D_t^{\alpha}&\left[u(t) - \mathscr{G}\left(t, u_{\zeta(t,u_t)}, \int_0^t e_1(t,s,u_{\zeta(s,u_s)})ds\right)\right] \\
&= \mathscr{A}u(t) + \mathscr{F}\left(t, u_{\zeta(t,u_t)}, \int_0^t e_2(t,s,u_{\zeta(s,u_s)})ds\right) \\
&\quad + \int_0^t e_3(t,s,u_{\zeta(s,u_s)})ds + \mathcal{C}v(t), \quad t \in \mathscr{I}, \qquad (4.1)\\
u_0 &= \varsigma(t) \in \mathscr{B}_h, \quad t \in (-\infty, 0], \qquad (4.2)
\end{aligned}
$$

where $\mathcal{C}$ is a bounded linear operator from a Banach space U into $\mathbb{X}$; the control function $v(\cdot) \in L^2(\mathscr{I}, U)$, a Banach space of admissible control functions. The rest of the functions are same as defined in (1.1)-(1.2).

Definition 3. *[34, Definition 4.1] A function* $u : (-\infty, T] \to \mathbb{X}$ *is called a mild solution of the model (4.1)-(4.2) if* $v \in L^2(\mathscr{I}, U), u_0 = \varsigma \in \mathscr{B}_h$*, and for each* $s \in [0, t)$ *the function* $\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{G}\left(t, u_{\zeta(t,u_t)}, \int_0^t e_1(t,s,u_{\zeta(s,u_s)})ds\right)$ *is integrable and the following integral equation*

$$
u(t) = \begin{cases}
\mathbb{T}_\alpha(t)[\varsigma(0) - \mathscr{G}(0, \varsigma(0), 0)] + \mathscr{G}\left(t, u_{\zeta(t,u_t)}, \int_0^t e_1(t,s,u_{\zeta(s,u_s)})ds\right) \\
+ \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{G}\left(s, u_{\zeta(s,u_s)}, \int_0^s e_1(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right) ds \\
+ \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{F}\left(s, u_{\zeta(s,u_s)}, \int_0^s e_2(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right) ds \\
+ \int_0^t \mathbb{S}_\alpha(t-s) \int_0^s e_3(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau ds \\
+ \int_0^t \mathbb{S}_\alpha(t-s)\mathcal{C}v(s)ds, \; t \in \mathscr{I}.
\end{cases}
$$

is satisfied.

Definition 4. *The system (4.1)-(4.2) is said to be controllable on* $\mathscr{I}$*, iff for every* $u_0 = \varsigma \in \mathscr{B}_h, u_1 \in \mathbb{X}$*, there exists a control* $v \in L^2(\mathscr{I}, U)$ *such that the mild solution* $u(\cdot)$ *of (4.1)-(4.2) fulfills* $u(T) = u_1$*.*

For the study of the structure (4.1)-(4.2), we report the further right after hypotheses:

(H7) The linear operator $\mathcal{C} : L^2(\mathscr{I}, U) \to L^1(\mathscr{I}, U)$ is bounded, $W : L^2(\mathscr{I}, U) \to \mathbb{X}$ defined by

$$Wv = \int_0^T (T-s)\mathbb{S}_\alpha(T-s)Bv(s)ds$$

has an inverse operator W^{-1} which takes values in $L^2(\mathscr{I}, U)/$ Ker W, where the kernel space of W is defined by Ker $W = \{u \in L^2(\mathscr{I}, U) : Wu = 0\}$ and there exist two positive constants $\widetilde{M}_1, \widetilde{M}_2 > 0$ such that $\|\mathcal{C}\| \le \widetilde{M}_1$ and $\|W^{-1}\| \le \widetilde{M}_2$.

(H6)* The following inequalities holds:
(i) Let

$$\begin{aligned}&\mathcal{M}\mathcal{M}_0\big[\overline{\nu}_1\|\varsigma\|_{\mathscr{B}_h} + \overline{\nu}_1^*\big] + \Big[\mathcal{M}_0\overline{\nu}_1 + \mathcal{M}_0\overline{\nu}_2 T\xi_1 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\cdot \\ &\cdot(\overline{\nu}_1 + \overline{\nu}_2 T\xi_1) + \frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}(\nu_1 + \nu_2 T\xi_2) + \frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\xi_3\Big](\mathscr{D}_1^* r + c_n) \\ &+ \mathcal{M}_0\overline{\nu}_1^* + \mathcal{M}_0\overline{\nu}_2 T\xi_1^* + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}(\overline{\nu}_2 T\xi_1^* + \overline{\nu}_1^*) \\ &+ \frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}(\nu_2 T\xi_2^* + \nu_1^*) + \frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\xi_3^* + \frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}B_0 \le r,\end{aligned}$$

for some $r > 0$.
(ii) Let

$$\begin{aligned}\Lambda^{**} = {}&\mathscr{D}_1^*\left(1 + \frac{\mathcal{M}\widetilde{M}_1\widetilde{\mathcal{M}}_2 T^\alpha}{\Gamma(\alpha+1)}\right)\Bigg[(\overline{\nu}_1 + \overline{\nu}_2 T\xi_1 + 2L_{\mathscr{G}}(r)L_\zeta(1 + TL_{e_1}(r))) \\ &\times\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right) + \frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}(\nu_1 + \nu_2 T\xi_2 \\ &+ 2L_{\mathscr{F}}(r)L_\zeta(1 + TL_{e_2}(r))) + \frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}[\xi_3 + 2L_\zeta L_{e_3}(r)]\Bigg] < 1\end{aligned}$$

be such that $0 \le \Lambda^{**} < 1$.

Theorem 2. *Expect that the hypotheses (H1)-(H5), (H7) and (H6)* are fulfilled, then the model (4.1)-(4.2) is controllable in $\mathscr{I}$.*

Proof. Using the theory, for an arbitrary function $u(\cdot)$, choose the control function as follows:

$$
\begin{aligned}
v_u(t) = W^{-1}\Bigg[& u_1 - \mathbb{T}_\alpha(T)[\varsigma(0) - \mathscr{G}(0,\varsigma(0),0)] \\
& - \mathscr{G}\left(T, u_{\varrho(T,u_T)}, \int_0^T e_1(T,s,u_{\zeta(s,u_s)})ds\right) \\
& - \int_0^T \mathscr{A}\mathbb{S}_\alpha(T-s)\mathscr{G}\left(s, u_{\zeta(s,u_s)}, \int_0^s e_1(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right) ds \\
& - \int_0^T \mathbb{S}_\alpha(T-s)\mathscr{F}\left(s, u_{\zeta(s,u_s)}, \int_0^s e_2(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right) ds \\
& - \int_0^T \mathbb{S}_\alpha(T-s)\left(\int_0^s e_3(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right) ds\Bigg](t).
\end{aligned} \tag{4.3}
$$

Presently, we determine the operator $\Upsilon_1 : \mathscr{B}_T \to \mathscr{B}_T$ by

$$
(\Upsilon_1 u)(t) = \begin{cases}
\mathbb{T}_\alpha(t)[\varsigma(0) - \mathscr{G}(0,\varsigma(0),0)] + \mathscr{G}\left(t, u_{\zeta(t,u_t)}, \int_0^t e_1(t,s,u_{\zeta(s,u_s)})ds\right) \\
+ \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{G}\left(s, u_{\zeta(s,u_s)}, \int_0^s e_1(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right) ds \\
+ \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{F}\left(s, u_{\zeta(s,u_s)}, \int_0^s e_2(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right) ds \\
+ \int_0^t \mathbb{S}_\alpha(t-s)\left(\int_0^s e_3(s,\tau,u_{\zeta(\tau,u_\tau)})d\tau\right) ds \\
+ \int_0^t \mathbb{S}_\alpha(t-s)\mathcal{C}v_x(s)ds,
\end{cases}
$$

for $t \in \mathscr{I}$.

Notice that the control (4.3) transfers the framework (4.1)-(1.2) from the initial state ς to the last state u_T gave that the operator Υ_1 has a fixed point. To affirm the exact controllability results, it is sufficient to exhibit that the operator Υ_1 has a fixed point on $\mathscr{B}_T$.

By employing same techniques as in Theorem 3.1, We delimit the operator $\overline{\Upsilon}_1 : \mathscr{B}_T^0 \to \mathscr{B}_T^0$ by

$$(\overline{\Upsilon}_1 x)(t) = \begin{cases} -\mathbb{T}_\alpha(t)\mathscr{G}(0,\varsigma,0) + \mathscr{G}\big(t, x_{\zeta(t,x_t+z_t)} + z_{\zeta(t,x_t+z_t)}, \\ \displaystyle\int_0^t e_1(t,s,x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)})ds\Big) \\ \displaystyle + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{G}\big(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)}, \\ \displaystyle\int_0^s e_1(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\Big)\,ds \\ \displaystyle + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{F}\big(s, x_{\zeta(s,x_s+y_s)} + z_{\zeta(s,x_s+z_s)}, \\ \displaystyle\int_0^s e_2(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\Big)\,ds \\ \displaystyle + \int_0^t \mathbb{S}_\alpha(t-s)\Big[\mathcal{C}v_{x+z}(s) + \int_0^s e_3(s,\tau,x_{\zeta(\tau,x_\tau+z_\tau)} \\ + z_{\zeta(\tau,x_\tau+z_\tau)})d\tau\Big]\,ds. \end{cases}$$

By utilizing (4.3), we define the control function v_{x+z} straightly. It is vindicated that the operator Υ_1 has a fixed point if and only if $\overline{\Upsilon}_1$ has a fixed point.

For the convenience of concerns, we consider

$$\begin{aligned} \|\mathcal{C}v_{x+z}(s)\| \le \widetilde{\mathcal{M}}_1\widetilde{\mathcal{M}}_2\Bigg[&\|u_1\| + \mathcal{M}\big[\|\varsigma(0)\| + \mathcal{M}_0(\overline{\nu}_1\|\varsigma\|_{\mathscr{B}_h} + \overline{\nu}_1^*)\big] \\ + \Big[(\overline{\nu}_1 + \overline{\nu}_2 T\xi_1)&\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right) \\ + \frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}(\nu_1 + \nu_2 T\xi_2) &+ \frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\xi_3\Big](\mathscr{D}_1^* r + c_n) + \mathcal{M}_0(\overline{\nu}_1^* + \overline{\nu}_2 T\xi_1^*) \\ + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}(\overline{\nu}_2 T\xi_1^* + \overline{\nu}_1^*) &+ \frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}(\nu_2 T\xi_2^* + \nu_1^* + T\xi_3^*)\Bigg] = B_0, \end{aligned}$$

and

$$
\begin{aligned}
&\|\mathcal{C}v_{x+z}(s)-\mathcal{C}v_{\overline{x}+z}(s)\| \leq \widetilde{\mathcal{M}}_1\widetilde{\mathcal{M}}_2\mathscr{D}_1^*\\
&\Bigg[\Big(\overline{\nu}_1+\overline{\nu}_2T\xi_1+2L_{\mathscr{G}}(r)L_\zeta(1+TL_{e_1}(r))\Big)\left(\mathcal{M}_0+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)\\
&+\frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}\Big(\nu_1+\nu_2T\xi_2+2L_{\mathscr{F}}(r)L_\zeta(1+TL_{e_2}(r))\Big)\\
&+\frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\Big(\xi_3+2L_\zeta L_{e_3}(r)\Big)\Bigg]\|x-\overline{x}\|_{\mathscr{B}_T^0}.
\end{aligned}
$$

Further, thinking as in Theorem 3.1 along with Remark 3.1, we receive

$$
\begin{aligned}
&\|(\overline{\Upsilon}_1x)(t)\|_{\mathbb{X}} \leq \mathcal{M}\mathcal{M}_0\big[\overline{\nu}_1\|\varsigma\|_{\mathscr{B}_h}+\overline{\nu}_1^*\big]\\
&+\Big[\mathcal{M}_0\overline{\nu}_1+\mathcal{M}_0\overline{\nu}_2T\xi_1+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}(\overline{\nu}_1+\overline{\nu}_2T\xi_1)\\
&+\frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}(\nu_1+\nu_2T\xi_2)+\frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\xi_3\Big](\mathscr{D}_1^*r+c_n)+\mathcal{M}_0\overline{\nu}_1^*+\mathcal{M}_0\overline{\nu}_2T\xi_1^*\\
&+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}(\overline{\nu}_2T\xi_1^*+\overline{\nu}_1^*)+\frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}(\nu_2T\xi_2^*+\nu_1^*)+\frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}\xi_3^*\\
&\quad+\frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}B_0\\
&\leq r.
\end{aligned}
$$

From this, we observe that $\overline{\Upsilon}_1$ maps the ball $B_r(0,\mathscr{B}_T^0)$ into itself. Finally, we demonstrate that $\overline{\Upsilon}_1$ is contraction on $B_r(0,\mathscr{B}_T^0)$. Due to this, let us take $x,\overline{x}\in B_r(0,\mathscr{B}_T^0)$, then from Remark 3.1, we receive

$$
\begin{aligned}
&\|(\overline{\Upsilon}_1x)(t)-(\overline{\Upsilon}_1\overline{x})(t)\| \leq \mathscr{D}_1^*\left(1+\frac{\mathcal{M}\widetilde{M}_1\widetilde{\mathcal{M}}_2T^\alpha}{\Gamma(\alpha+1)}\right)\cdot\\
&\cdot\Bigg[\Big(\overline{\nu}_1+\overline{\nu}_2T\xi_1+2L_{\mathscr{G}}(r)L_\zeta(1+TL_{e_1}(r))\Big)\left(\mathcal{M}_0+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)\\
&+\frac{\mathcal{M}T^\alpha}{\Gamma(\alpha+1)}\Big(\nu_1+\nu_2T\xi_2+2L_{\mathscr{F}}(r)L_\zeta(1+TL_{e_2}(r))\Big)\\
&+\frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}[\xi_3+2L_\zeta L_{e_3}(r)]\Bigg]\|x-\overline{x}\|_{\mathscr{B}_T^0}\\
&\leq \Lambda^{**}\|x-\overline{x}\|_{\mathscr{B}_T^0}.
\end{aligned}
$$

The verification is very much alike to the Theorem 3.1, so we bypass it. From the assumption (H6)* and in the perspective of the contraction mapping principle, we understand that $\overline{\Upsilon}_1$ includes a unique fixed point $x \in \mathscr{B}_T^0$ which is a mild solution of the structure (4.1)-(4.2) on $(-\infty, T]$. □

5 Application

Example 5.1:
To exemplify our theoretical results, we treat the FNIDS with SDD of the model

$$\begin{aligned}
&{}^C D_t^{\alpha}\Big[u(t,\xi) - g\Big(t, u(t-\sigma(u(t,0)),\xi), \int_0^t \eta_1(t,s,u(s-\sigma(u(s,0)),\xi))ds\Big)\Big] \\
&= \frac{\partial^2}{\partial \xi^2} u(t,\xi) + f\Big(t, u(t-\sigma(u(t,0)),\xi), \int_0^t \eta_2(t,s,u(s-\sigma(u(s,0)),\xi))ds\Big) \\
&\quad + \int_0^t \eta_3(t,s,u(s-\sigma(u(s,0)),\xi))ds,\ t \in \mathscr{I} = [0,T],\ \xi \in [0,\pi], \qquad (5.1)
\end{aligned}$$

$$u(t,0) = u(t,\pi) = 0, \quad t \in \mathscr{I}, \qquad (5.2)$$

$$u(\theta,\xi) = \varsigma(\theta,\xi), \quad \theta \in (-\infty,0], \quad \xi \in [0,\pi], \qquad (5.3)$$

where ${}^C D_t^{\alpha}$ is Caputo's fractional derivative of order $0 < \alpha < 1; \varsigma(\theta,\xi)$ is continuous and $\sigma \in C(\mathbb{R}, \mathscr{I}), \varsigma \in \mathscr{B}_h$. We consider $\mathbb{X} = L^2[0,\pi]$ with the norm $|\cdot|_{L^2}$ and determine the operator $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ by $\mathscr{A}w = w''$ with the domain

$$D(\mathscr{A}) = \{w \in \mathbb{X} : w, w' \text{ are absolutely continuous},\ w'' \in \mathbb{X},\ w(0) = w(\pi) = 0\}.$$

Then

$$\mathscr{A}w = \sum_{n=1}^{\infty} n^2 \langle w, w_n \rangle w_n, \quad w \in D(\mathscr{A}),$$

in which $w_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns)$, $n = 1, 2, \ldots,$. is that the orthogonal set of eigenvectors of $\mathscr{A}$. It is long acquainted that $\mathscr{A}$ c_0 semigroup $(\mathbb{T}(t))_{t\geq 0}$ in $\mathbb{X}$ and is come up with by

$$\mathbb{T}(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} \langle w, w_n \rangle w_n, \quad \text{for all} \quad w \in \mathbb{X}, \quad \text{and every} \quad t > 0.$$

For each $w \in \mathbb{X}$,

$$\mathscr{A}^{\frac{-1}{2}} w = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) (w, w_n) w_n$$

and $\|\mathscr{A}^{\frac{-1}{2}}\| = 1$. The operator $\mathscr{A}^{\frac{1}{2}}$ is given by $\mathscr{A}^{\frac{1}{2}}w = \sum_{n=1}^{\infty} n(w, w_n)w_n$ on the space $D(\mathscr{A}^{\frac{1}{2}}) = \{w(\cdot) \in \mathbb{X} : \sum_{n=1}^{\infty} n(w, w_n)w_n \in \mathbb{X}\}$.

For phase space, we select $h = e^{2s}$, $s < 0$, then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2} < \infty$, for $t \leq 0$ and determine

$$\|\varsigma\|_{\mathscr{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta\in[s,0]} \|\varsigma(\theta)\|_{L^2} ds.$$

Hence, for $(t, \varsigma) \in [0, b] \times \mathscr{B}_h$, where $\varsigma(\theta)(x) = \varsigma(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times [0, \pi]$. Set

$$\begin{aligned} x(t)(\xi) &= u(t, \xi), \quad t \in \mathscr{I}, \quad \xi \in [0, \pi], \\ \mathscr{G}(t, \varsigma, x)(\xi) &= g\left(t, \varsigma(0, \xi), \int_0^t \eta(t, s, \varsigma(0, \xi))ds\right), \quad t \in \mathscr{I}, \quad \xi \in [0, \pi], \end{aligned}$$

$$\begin{aligned} \mathscr{F}(t, \varsigma, x)(\xi) &= f\left(t, \varsigma(0, \xi), \int_0^t \eta(t, s, \varsigma(0, \xi))ds\right),\ t \in \mathscr{I},\ \xi \in [0, \pi], \\ e_i(t, s, \varsigma)(\xi) &= \eta_i(t, s, \varsigma(0, \xi)), i = 1, 2, 3; \\ \varrho(t, \varsigma) &= t - \sigma(\varsigma(0, 0)). \end{aligned}$$

Consequently, with the above decisions, the framework (5.1)-(5.3) can be composed to the theoretical model (1.1)-(1.2). If suppose that (H1)-(H6) are fulfilled, then from Banach contraction principle, the system (5.1)-(5.3) has a unique mild solution on $(-\infty, T]$.

Example 5.2: Now, let as consider the FNIDS with SDD of the problem

$$
\begin{aligned}
{}^{C}D_t^{\alpha}\Bigg[v(t,x) - \bigg\{ \int_{-\infty}^{t} & e^{2(s-t)} \frac{v(s-\varrho_1(s)\varrho_2(\|v(s)\|),x)}{49} ds \\
&+ \int_0^t \sin(t-s) \int_{-\infty}^{s} e^{2(\tau-s)} \frac{v(\tau-\varrho_1(\tau)\varrho_2(\|v(\tau)\|),x)}{36} d\tau ds \bigg\}\Bigg] \\
&= \frac{\partial^2}{\partial x^2} v(t,x) + \int_{-\infty}^{t} e^{2(s-t)} \frac{v(s-\varrho_1(s)\varrho_2(\|v(s)\|),x)}{9} ds \\
&+ \int_0^t \sin(t-s) \int_{-\infty}^{s} e^{2(\tau-s)} \frac{v(\tau-\varrho_1(\tau)\varrho_2(\|v(\tau)\|),x)}{25} d\tau ds \\
&+ \int_0^t \sin(t-s) \int_{-\infty}^{s} e^{2(\tau-s)} \frac{v(\tau-\varrho_1(\tau)\varrho_2(\|v(\tau)\|),x)}{16} d\tau ds,
\end{aligned} \tag{5.4}
$$

$$v(t,0) = 0 = v(t,\pi), \quad t \in [0,T], \tag{5.5}$$

$$v(t,x) = \varsigma(t,x), \quad t \le 0, \quad x \in [0,\pi], \tag{5.6}$$

In perspective of Example 5.1, we set

$$v(t)(x) = v(t,x), \quad \varrho(t,\varsigma) = \varrho_1(t)\varrho_2(\|\varsigma(0)\|),$$

we utilize

$$
\begin{aligned}
\mathscr{G}(t,\varsigma,\mathscr{H}\varsigma)(x) &= \int_{-\infty}^{0} e^{2(s)} \frac{\varsigma}{49} ds + (\mathscr{H}\varsigma)(x), \\
\mathscr{F}(t,\varsigma,\overline{\mathscr{H}}\varsigma)(x) &= \int_{-\infty}^{0} e^{2(s)} \frac{\varsigma}{9} ds + (\overline{\mathscr{H}}\varsigma)(x), \\
\int_0^t e_3(t,s,\varsigma)(x) ds &= \int_0^t \sin(t-s) \int_{-\infty}^{0} e^{2(\tau)} \frac{\varsigma}{16} d\tau ds,
\end{aligned}
$$

where

$$
\begin{aligned}
(\mathscr{H}\varsigma)(x) &= \int_0^t \sin(t-s) \int_{-\infty}^{0} e^{2(\tau)} \frac{\varsigma}{36} d\tau ds, \\
(\overline{\mathscr{H}}\varsigma)(x) &= \int_0^t \sin(t-s) \int_{-\infty}^{0} e^{2(\tau)} \frac{\varsigma}{25} d\tau ds,
\end{aligned}
$$

then using these configurations, the system (5.4)-(5.6) is generally composed in the abstract type of issue (1.1)- (1.2).

To treat this system we have a tendency to assume that $\varrho_i : [0,\infty) \to [0,\infty)$, $i = 1,2$ are continuous. Now, we will see that for $t \in [0,1], \varsigma, \overline{\varsigma} \in \mathscr{B}_h$, we get

$$\|(\mathscr{A})^{\frac{1}{2}}\mathscr{G}(t,\varsigma,\mathscr{H}\varsigma)\|_{\mathbb{X}}$$
$$\leq \left(\int_0^\pi \left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varsigma}{49}\right\| ds + \int_0^t \|\sin(t-s)\| \int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varsigma}{36}\right\| d\tau ds\right)^2 dx\right)^{\frac{1}{2}}$$
$$\leq \left(\int_0^\pi \left(\frac{1}{49}\int_{-\infty}^0 e^{2(s)} \sup\|\varsigma\| ds + \frac{1}{36}\int_{-\infty}^0 e^{2(s)} \sup\|\varsigma\| ds\right)^2 dx\right)^{\frac{1}{2}}$$
$$\leq \frac{\sqrt{\pi}}{49}\|\varsigma\|_{\mathscr{B}_h} + \frac{\sqrt{\pi}}{36}\|\varsigma\|_{\mathscr{B}_h}$$
$$\leq \overline{\nu}_1\|\varsigma\|_{\mathscr{B}_h} + \overline{\nu}_2\|\varsigma\|_{\mathscr{B}_h},$$

where $\overline{\nu}_1 + \overline{\nu}_2 = \frac{85\sqrt{\pi}}{1764}$, and

$$\|(\mathscr{A})^{\frac{1}{2}}\mathscr{G}(t,\varsigma,\mathscr{H}\varsigma) - (\mathscr{A})^{\frac{1}{2}}\mathscr{G}(t,\overline{\varsigma},\mathscr{H}\overline{\varsigma})\|_{\mathbb{X}}$$
$$\leq \Bigg(\int_0^\pi \Bigg(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varsigma}{49} - \frac{\overline{\varsigma}}{49}\right\| ds$$
$$+ \int_0^t \|\sin(t-s)\| \int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varsigma}{36} - \frac{\overline{\varsigma}}{36}\right\| d\tau ds\Bigg)^2 dx\Bigg)^{\frac{1}{2}}$$
$$\leq \left(\int_0^\pi \left(\frac{1}{49}\int_{-\infty}^0 e^{2(s)} \sup\|\varsigma - \overline{\varsigma}\| ds + \frac{1}{36}\int_{-\infty}^0 e^{2(s)} \sup\|\varsigma - \overline{\varsigma}\| ds\right)^2 dx\right)^{\frac{1}{2}}$$
$$\leq \frac{\sqrt{\pi}}{49}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} + \frac{\sqrt{\pi}}{36}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h}$$
$$\leq \overline{\nu}_1\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} + \overline{\nu}_2\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h}.$$

Similarly, we conclude

$$\|\mathscr{F}(t,\varsigma,\overline{\mathscr{H}}\varsigma)\|_{\mathbb{X}}$$
$$\leq \left(\int_0^\pi \left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varsigma}{9}\right\| ds + \int_0^t \|\sin(t-s)\| \int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varsigma}{25}\right\| d\tau ds\right)^2 dx\right)^{\frac{1}{2}}$$
$$\leq \left(\int_0^\pi \left(\frac{1}{9}\int_{-\infty}^0 e^{2(s)} \sup\|\varsigma\| ds + \frac{1}{25}\int_{-\infty}^0 e^{2(s)} \sup\|\varsigma\| ds\right)^2 dx\right)^{\frac{1}{2}}$$
$$\leq \frac{\sqrt{\pi}}{9}\|\varsigma\|_{\mathscr{B}_h} + \frac{\sqrt{\pi}}{25}\|\varsigma\|_{\mathscr{B}_h}$$
$$\leq \nu_1\|\varsigma\|_{\mathscr{B}_h} + \nu_2\|\varsigma\|_{\mathscr{B}_h},$$

where $\nu_1 + \nu_2 = \frac{34\sqrt{\pi}}{225}$, and

$$
\begin{aligned}
&\|\mathscr{F}(t,\varsigma,\overline{\mathscr{H}}\varsigma) - \mathscr{F}(t,\overline{\varsigma},\overline{\mathscr{H}}\overline{\varsigma})\|_{\mathbb{X}} \\
&\leq \left(\int_0^{\pi}\left(\int_{-\infty}^{0} e^{2(s)}\left\|\frac{\varsigma}{9} - \frac{\overline{\varsigma}}{9}\right\| ds + \int_0^t \|\sin(t-s)\| \int_{-\infty}^{0} e^{2(\tau)}\left\|\frac{\varsigma}{25} - \frac{\overline{\varsigma}}{25}\right\| d\tau ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^{\pi}\left(\frac{1}{9}\int_{-\infty}^{0} e^{2(s)} \sup\|\varsigma - \overline{\varsigma}\| ds + \frac{1}{25}\int_{-\infty}^{0} e^{2(s)} \sup\|\varsigma - \overline{\varsigma}\| ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{9}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} + \frac{\sqrt{\pi}}{25}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} \\
&\leq \nu_1\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} + \nu_2\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h}.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
&\left\|\int_0^t \left[e_3(t,s,\varsigma) - e_3(t,s,\overline{\varsigma})\right] ds\right\| \\
&\leq \left(\int_0^{\pi}\left(\int_0^t \|\sin(t-s)\| \int_{-\infty}^{0} e^{2(\tau)}\left\|\frac{\varsigma}{16} - \frac{\overline{\varsigma}}{16}\right\| d\tau ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{16}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} \\
&\leq \xi_3\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h},
\end{aligned}
$$

where $\xi_3 = \frac{\sqrt{\pi}}{16}$.

Thus the conditions (H1)-(H5) are satisfied. Moreover, we assume that $\mathscr{D}_1^* = 1, \mathcal{M}_0 = 1, \mathcal{M} = 1, \mathcal{M}_{\frac{1}{2}} = 1, T = 1, \xi_1 = 1, \xi_2 = 1$ and $L_\zeta = 1$. Moreover, the proper choice of constants $L_{\mathscr{G}}(r)$, $\mathcal{L}_{\mathscr{F}}(r)$ and $L_{e_i}(r)$, for $i = 1, 2, 3$; we get

$$
\begin{aligned}
\Lambda^* = \mathscr{D}_1^*\Bigg[&(\overline{\nu}_1 + \overline{\nu}_2 T\xi_1 + 2L_{\mathscr{G}}(r)L_\zeta(1 + TL_{e_1}(r)))\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right) \\
&+ \frac{\mathcal{M}T^{\alpha}}{\Gamma(\alpha+1)}(\nu_1 + \nu_2 T\xi_2 + 2L_{\mathscr{F}}(r)L_\zeta(1 + TL_{e_2}(r))) + \frac{\mathcal{M}T^{\alpha+1}}{\Gamma(\alpha+1)}[\xi_3 + 2L_\zeta L_{e_3}(r)]\Bigg] \\
&< 1.
\end{aligned}
$$

In this manner the condition (H6) holds. Subsequently by Theorem 3.1, we have a tendency to understand that the model (5.4)-(5.6) has a unique mild solution in $[0, 1]$.

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7 Conclusion

In this manuscript, we have studied the existence and controllability results for FNIDS with SDD in Banach space. More precisely, by utilizing the semigroup theory, fractional powers of operators and Banach contraction fixed point theorem, we investigate the FNIDS with SDD in Banach space. To validate the obtained theoretical results, two examples are analyzed. The FDEs are very efficient to describe the real-life phenomena; thus, it is essential to extend the present study to establish the other qualitative and quantitative properties such as stability and approximate controllability.

There are two direct issues which require further study. First, we will investigate the approximate controllability of fractional neutral stochastic integro-differential systems with state-dependent delay both in the case of a non-compact operator and a normal topological space. Secondly, we will be devoted to studying the approximate controllability of a new class of impulsive fractional stochastic differential equations with state-dependent delay in Hilbert space.

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