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AN ITERATIVE METHOD FOR AN EQUILIBRIUM POINT OF LINEAR QUADRATIC STOCHASTIC DIFFERENTIAL GAMES WITH STATE AND CONTROL-DEPENDENT NOISE*

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Abstract

We study a numerical algorithm for solving the coupled stochastic algebraic Riccati equations arising in the infinite time horizon nonzero-sum linear quadratic (LQ) differential games of stochastic systems. We construct a matrix sequence, which converges to the solution of the considered coupled stochastic algebraic Riccati equations and defines the Nash equilibrium point, which solves a stochastic control problem with state, control and external disturbance-dependent noise. Computer realizations of the introduced methods are numerically compared via Python.

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1 Introduction

Linear quadratic games based on the Riccati equations and their applications have been widely investigated in literatures [1, 2, 4, 3]. Some special kinds of stochastic differential games for *Itô* systems with state and control-dependent noise are investigated in [9, 10, 11]. The system with the state and control-dependent noise, where using the stochastic Nash game approach to solve stochastic H_2/H_∞ control with state, control and external disturbance-dependent noise is analysed in [10]. The existence of the Nash equilibrium for infinite time horizon nonzero-sum LQ stochastic differential games is equivalent to the solvability of four coupled stochastic Riccati algebraic equations [10].

The goal of the paper is to present a numerical algorithm for computing the Nash equilibrium point for a two-player game. We study numerical algorithms for solving the coupled stochastic algebraic Riccati equations arising in the infinite time horizon nonzero-sum linear quadratic (LQ) differential games of stochastic systems. We construct a matrix sequence, which converges to a solution of the considered coupled stochastic algebraic Riccati equations. This solution defines the Nash equilibrium point [10, Theorem 2]. Computer realizations of the introduced methods are numerically compared via Python. In our investigation we adapt ideas and algorithms derived by Ivanov in [6, 7].

A Nash equilibrium exists if and only if there exist real symmetric $n \times n$ solutions $(\tilde{X}_1, \tilde{X}_2, \tilde{F}_1, \tilde{F}_2)$ to the following four coupled stochastic algebraic Riccati equations:

$$
\begin{aligned}
\mathcal{R}_1(X_1, X_2) = {} & X_1 \bar{A}_0 + \bar{A}_0^T X_1 + \bar{A}_1^T X_1 \bar{A}_1 + \bar{Q}_1 \\
& -(X_1 B_1 + \bar{A}_1^T X_1 C_1)(R_{11} + C_1^T X_1 C_1)^{-1} \\
& \times (B_1^T X_1 + C_1^T X_1 P \bar{A}_1) \quad = \quad 0 \\
F_1 = {} & -(R_{11} + C_1^T X_1 C_1)^{-1} \, (B_1^T X_1 + C_1^T X_1 \bar{A}_1) \\
& R_{11} + C_1^T X_1 C_1 \ > \ 0 \, , \\
\mathcal{R}_2(X_1, X_2) = {} & X_2 \tilde{A}_0 + \tilde{A}_0^T X_2 + \tilde{A}_1^T X_2 \tilde{A}_1 + \bar{Q}_2 \\
& -(X_2 B_2 + \tilde{A}_1^T X_2 C_2)(R_{22} + C_2^T X_2 C_2)^{-1} \\
& \times (B_2^T X_2 + C_2^T X_2 \tilde{A}_1) \quad = \quad 0 \\
F_2 = {} & -(R_{22} + C_2^T X_2 C_2)^{-1} \, (B_2^T X_2 + C_2^T X_2 \tilde{A}_1) \\
& R_{22} + C_2^T X_2 C_2 \ > \ 0 \, ,
\end{aligned} \tag{1}
$$

where

$$\begin{cases} \bar{A}_0 = A_0 + B_2F_2\,, \quad \bar{A}_1 = A_1 + C_2F_2\,, \\ \tilde{A}_0 = A_0 + B_1F_1\,, \quad \tilde{A}_1 = A_1 + C_1F_1\,, \\ \bar{Q}_1 = Q_1 + F_2^TR_{12}F_2\,, \quad \bar{Q}_2 = Q_2 + F_1^TR_{21}F_1\,. \end{cases}$$

The notations are : A_0, A_1 are real $n \times n$ matrices, Q_1, Q_2 are real symmetric $n \times n$ matrices, B_1, C_1 are real $n \times m_1$ matrices, B_2, C_2 are real $n \times m_2$ matrices, R_{11}, R_{21} are real $m_1 \times m_1$ matrices, and R_{12}, R_{22} are real $m_2 \times m_2$ matrices.

A matrix A is said to be stable if the all eigenvalues of A lie in the open left half plane. We write $X \geq Y$ or $X \geq Y$ if $X - Y$ is positive definite or $X - Y$ is positive semidefinite.

2 An algorithm

We rewrite the set of Riccati equations $\mathcal{R}_1(X_1, X_2) = 0$ and $\mathcal{R}_2(X_1, X_2) = 0$ as a common Riccati equation with block matrix coefficients:

$$\mathcal{A}_0^T\mathbf{X} + \mathbf{X}\mathcal{A}_0 + \Pi_1(\mathbf{X}) + Q - \mathcal{S}(\mathbf{X})\mathbb{R}(\mathbf{X})^{-1}\mathcal{S}(\mathbf{X})^T = 0 \tag{2}$$

where

$$\begin{aligned} \mathbb{R}(\mathbf{X}) &= \mathcal{R} + \mathcal{C}^T\mathbf{X}\mathcal{C}\,, \\ &= diag\,[R_{11} + C_1^TX_1C_1, R_{22} + C_2^TX_2C_2] \\ \mathcal{S}(\mathbf{X}) &= \mathbf{XB} + \mathcal{A}_1^T\mathbf{X}\mathcal{C} \\ &= diag\,[X_1B_1 + \bar{A}_1^TX_1C_1, X_2B_2 + \tilde{A}_1^TX_2C_2] \\ \Pi_1(\mathbf{X}) &= \mathcal{A}_1^T\mathbf{X}\mathcal{A}_1 = diag\,[\bar{A}_1^TX_1\bar{A}_1, \tilde{A}_1^TX_2\tilde{A}_1], \\ \mathcal{A}_0 &= diag\,[\bar{A}_0, \tilde{A}_0]\,, \quad \mathcal{A}_1 = diag\,[\bar{A}_1, \tilde{A}_1]\,, \\ \mathcal{B} &= diag\,[B_1, B_2]\,, \quad \mathcal{C} = diag\,[C_1, C_2]\,, \\ \mathcal{R} &= diag\,[R_{11}, R_{22}]\,, \quad \mathcal{Q} = diag\,[\bar{Q}_1, \bar{Q}_2] \quad . \end{aligned}$$

The introduced Riccati equation (1) is a Riccati type equation investigated in [7]. We can modify Lyapunov iteration (8) from [7]. We derive the following iteration suitable for the set of Riccati equations (1). We take

$\mathbf{X}^{(0)} = diag\,[X_1^{(0)}, X_2^{(0)}]$ and compute

$$\begin{aligned}
F_1^{(0)} &= -(R_{11} + C_1^T X_1^{(0)} C_1)^{-1} (B_1^T X_1^{(0)} + C_1^T X_1^{(0)} A_1)\,,\\
\tilde{A}_1 &= A_1 + C_1\, F_1^{(0)}\,,\\
F_2^{(0)} &= -(R_{22} + C_2^T X_2^{(0)} C_2)^{-1} (B_2^T X_2^{(0)} + C_2^T X_2^{(0)} \tilde{A}_1)\,,\\
\bar{A}_1 &= A_1 + C_2\, F_2^{(0)}\,,\\
F_1^{(0)} &= -(R_{11} + C_1^T X_1^{(0)} C_1)^{-1} (B_1^T X_1^{(0)} + C_1^T X_1^{(0)} \bar{A}_1)\,.
\end{aligned} \tag{3}$$

We construct the matrix sequence $\{\mathbf{X}^{(k)}\}_{k=0}^{\infty}$ as follow. Assume we know $\mathbf{X}^{(k)}$. We compute :

$$\begin{aligned}
&\tilde{A}_1 = A_1 + C_1 F_1^{(k-1)}\,, \qquad \bar{A}_1 = A_1 + C_2\, F_2^{(k-1)}\,,\\
&\mathcal{A}_1 = diag\,[\bar{A}_1, \tilde{A}_1]\,,\\
&\mathcal{S}(\mathbf{X}^{(k)}) = \mathbf{X}^{(k)}\mathbf{B} + \mathcal{A}_1^T \mathbf{X}^{(k)} \mathcal{C}\,,\\
&\mathcal{F}_{\mathbf{X}^{(k)}} = -(\mathbb{R}(\mathbf{X}^{(k)}))^{-1}\, \mathcal{S}(\mathbf{X}^{(k)})^T\\
&\qquad = diag[F_1(\mathbf{X}^{(k)}), F_2(\mathbf{X}^{(k)})] = diag[F_1^{(k)}, F_2^{(k)}]\,,\\
&\tilde{A}_0 = A_0 + B_1 F_1^{(k)}\,, \qquad \bar{A}_0 = A_0 + C_2\, F_2^{(k)}\,,\\
&\mathcal{A}_0 = diag\,[\bar{A}_0, \tilde{A}_0]\,,\\
&\bar{Q}_1 = Q_1 + (F_2^{(k)})^T R_{12} F_2^{(k)}\,, \qquad \bar{Q}_2 = Q_2 + (F_1^{(k)})^T R_{21} F_1^{(k)}\,,\\
&\mathcal{Q} = diag\,[\bar{Q}_1, \bar{Q}_2]\,.
\end{aligned} \tag{4}$$

We ready to apply the iteration

$$\begin{aligned}
&(\mathcal{A}_0 + \mathbf{B}\mathcal{F}_{\mathbf{X}^{(k)}})^T \mathbf{X}^{(k+1)} + \mathbf{X}^{(k+1)} (\mathcal{A}_0 + \mathbf{B}\mathcal{F}_{\mathbf{X}^{(k)}})\\
&\quad + \mathcal{T}_{\mathbf{X}^{(k)}} + \Pi_{\mathbf{X}^{(k)}}(\mathbf{X}^{(k)}) = \ 0\,,
\end{aligned} \tag{5}$$

where

$$\mathcal{T}_{\mathbf{Z}} = \begin{pmatrix} \mathcal{I} \\ \mathcal{F}_{\mathbf{X}^{(k)}} \end{pmatrix}^T \begin{pmatrix} \mathcal{Q} & 0 \\ 0 & \mathcal{R} \end{pmatrix} \begin{pmatrix} \mathcal{I} \\ \mathcal{F}_{\mathbf{X}^{(k)}} \end{pmatrix},$$

and

$$\Pi_{\mathbf{X}^{(k)}}(\mathbf{X}^{(k)}) = \begin{pmatrix} \mathcal{I} \\ \mathcal{F}_{\mathbf{X}^{(k)}} \end{pmatrix}^T \begin{pmatrix} \mathcal{A}_1^T \mathbf{X}^{(k)} \mathcal{A}_1 & \mathcal{A}_1^T \mathbf{X}^{(k)} \mathcal{C} \\ \mathcal{C}^T \mathbf{X}^{(k)} \mathcal{A}_1 & \mathcal{C}^T \mathbf{X}^{(k)} \mathcal{C} \end{pmatrix} \begin{pmatrix} \mathcal{I} \\ \mathcal{F}_{\mathbf{X}^{(k)}} \end{pmatrix}.$$

Under the assumptions that the $\mathcal{A}_0, \mathcal{A}_1, \mathcal{Q}$ are given constant matrices the convergence properties of iteration (5) are derived in the following theorem:

Theorem 1 *[7, Theorem 2.10] Assume there exist Hermitian matrices* $\hat{\mathbf{X}}$ *and* $\mathbf{X}_0$ *such that* $\mathcal{R}(\hat{\mathbf{X}}) \geq 0$ *and* $\mathbf{X}_0 > \hat{\mathbf{X}}\,, \mathcal{R}(\mathbf{X}_0) < 0$ *and* $\mathcal{A}_0 + \mathbf{B}\mathcal{F}_{\mathbf{X}^{(0)}}$ *is stable, where* $\mathcal{F}_{\mathbf{X}^{(0)}} = -(\mathbb{R}(\mathbf{X}^{(0)}))^{-1}\,\mathcal{S}(\mathbf{X}^{(0)})^T$*. Then for the matrix sequence* $\{\mathbf{X}^{(s)}\}$ *defined by (5) are satisfied*

(i) $\mathbf{X}^{(s)} > \mathbf{X}^{(s+1)}\,, \mathbf{X}^{(s)} > \hat{\mathbf{X}}\,, \mathcal{R}(\mathbf{X}^{(s)}) < 0\,, \quad s = 0, 1, 2, \ldots$*;*

(ii) $\mathcal{A}_0 + \mathbf{B}\mathcal{F}_{\mathbf{X}^{(s)}}$ *is stable for* $s = 0, 1, 2, \ldots$*;*

(iii) $\lim_{s\to\infty}\ \mathbf{X}^{(s)} = \tilde{\mathbf{X}}$ *is a solution of* $\mathcal{R}(\mathbf{X}) = 0$ *with* $\tilde{\mathbf{X}} > \hat{\mathbf{X}}$*. Moreover, if* $\mathbf{X}^{(0)} > \mathbf{X}$ *for all solutions* $\mathbf{X}$ *of* $\mathcal{R}(\mathbf{X}) = 0$*, then* $\tilde{\mathbf{X}}$ *is the maximal solution;*

(iv) *the eigenvalues of* $\mathcal{A}_0 + \mathbf{B}\mathcal{F}_{\tilde{\mathbf{X}}}$ *lie in the closed left half plane. In addition, if* $\mathcal{R}(\hat{X}) > 0$*, then all eigenvalues of* $\mathcal{A}_0 + \mathbf{B}\mathcal{F}_{\tilde{\mathbf{X}}}$ *lie in the open left half plane.*

Remark 1 *The introduced approach can be applied for the infinite time horizon stochastic* H_2/H_∞ *control problem to find the Nash equilibrium point [8, 10]. Theorem 3 [10] confirms that the existence the Nash equilibrium point is obtained via the solution of the special four coupled stochastic algebraic Riccati equations derived in [10]. The solution can be found applying through formulas (3)-(5).*

3 Numerical examples

We carry out some numerical experiments for computing the stabilizing solution to block Riccati equation (1). We apply the algorithm described by (3)-(5). We use Python in an easy-to-use Anaconda environment where problems and solutions are expressed in most effective way. Python is a programming language that lets you work more quickly and integrate your systems more effectively. The Python programming language is freely available and makes solving a computer problem almost as easy as writing out the problems. Python can be used for processing text, numbers, and scientific data and applications.

We rewrite (1) in the form

$$\begin{aligned} &\mathcal{A}_0^T\mathbf{X} + \mathbf{X}\mathcal{A}_0 + \mathcal{Q} + \mathcal{A}_1^T\mathbf{X}\mathcal{A}_1 - (\mathbf{X}\mathbf{B} + \mathcal{A}_1^T\mathbf{X}\mathcal{C}) \\ &\times (\mathcal{R} + \mathcal{C}^T\mathbf{X}\mathcal{C})^{-1}(\mathbf{X}\mathbf{B} + \mathcal{A}_1^T\mathbf{X}\mathcal{C})^T = 0\,. \end{aligned} \tag{6}$$

We represent iteration (5) in the form suitable for the computations:

$$\begin{aligned}&(\mathcal{A}_0+\mathbf{B}\mathcal{F}_{\mathbf{X}^{(k)}})^T\mathbf{X}^{(k+1)}+\mathbf{X}^{(k+1)}(\mathcal{A}_0+\mathbf{B}\mathcal{F}_{\mathbf{X}^{(k)}})+\mathcal{Q}\\&+\mathcal{F}^T_{\mathbf{X}^{(k)}}\,\mathcal{R}\,\mathcal{F}_{\mathbf{X}^{(k)}}+(\mathcal{A}_1+\mathcal{C}\mathcal{F}_{\mathbf{X}^{(k)}})^T\mathbf{X}^{(k)}(\mathcal{A}_1+\mathcal{C}\mathcal{F}_{\mathbf{X}^{(k)}})=0\,,\end{aligned} \tag{7}$$

i.e. we call it the block Lyapunov iteration.

The numerical experiments are constructed following the approach derived in [5] and the block Lyapunov iteration (7) is applied instead of (5).

We consider a two-player game and two numerical examples. The matrix coefficients A, B_i, Q_i and R_{ij} for $i, j = 1, 2$ are defined using the Python description.

Example 1 *The matrix coefficients are:*

n=3; m1=2; m2=3;

A_0 *= np.matrix([[-1.5, 0.17,-0.049],[0.07, -1.42, -0.027],[0.04, -0.11,-1.47]])*

A_1 *= np.matrix([[0.7, 0.19,-0.04],[0.24, 0.9,0.9],[0.3, 0.1,0.15]])*

Q_1*=0.3*np.matlib.identity(n)*

Q_2*=0.025*np.matlib.identity(n)*

B_1 *= np.matrix([[0.0, 0.],[0.05, 0.1],[0.04, 0.15]]);*

C_1 *= np.matrix([[0., 0.1],[1.1, 0],[0., 0.02]]);*

B_2 *= np.matrix([[0.1, 0.5 , 0.4],[0., 0, 0.08],[0., 0., 2.2]])*

C_2 *= np.matrix([[0.1, 0. , 0.],[0., 1.5, 0.0],[0.1, 0.05, 0.0]])*

R_{11} *= np.matlib.identity(m1);* R_{11}*[0,0]=4.0;* R_{11}*[m1-1,m1-1]=5.0;*

R_{21} *= np.matlib.identity(m1)/2.;* R_{21}*[1,1]=10.;*

R_{22} *= np.matlib.identity(m2);* R_{22}*[0,0]=2.;* R_{22}*[m2-1,m2-1]=8.;*

R_{12} *= np.matlib.identity(m2)/2.;* R_{12}*[1,1]=2.;* R_{12}*[m2-1,m2-1]=3.;*

We execute Example 1 for $n = 3$ and $tol = 1.0e-8$. We take $X_1^{(0)} = diag[6,6,6]$, and $X_2^{(0)} = diag[9,9,9]$. Thus, we obtain $\mathcal{R}_1(X_1^{(0)}, X_2^{(0)}) < 0$, and $\mathcal{R}_2(X_1^{(0)}, X_2^{(0)}) < 0$. We take $\hat{X}_1^{(0)} = \hat{X}_2^{(0)} = diag[0.0002, 0.0002, 0.0002]$, and $\mathcal{R}_1(\hat{X}_1, \hat{X}_2) > 0$, and $\mathcal{R}_2(\hat{X}_1, \hat{X}_2) > 0$. In addition, the matrix $\mathcal{A}_0 + \mathbf{B}\mathcal{F}_{\mathbf{X}^{(0)}}$ is stable. Thus the conditions of Theorem 1 are satisfied. We execute iteration (7) with the initial matrices $X_1^{(0)}$ and $X_2^{(0)}$. We obtain the solution $\tilde{X}_1$, $\tilde{X}_2$ after 25 iteration steps. The matrix $\mathcal{A}_0 + \mathbf{B}\mathcal{F}_{\tilde{X}}$ is stable. We have

$$\tilde{X}_1 = \begin{pmatrix} 0.13952043 & 0.04144027 & 0.02188102 \\ 0.04144027 & 0.15624824 & 0.03732627 \\ 0.02188102 & 0.03732627 & 0.14154421 \end{pmatrix}$$

and

$$\tilde{X}_2 = \begin{pmatrix} 0.0120035 & 0.003909 & 0.00222309 \\ 0.003909 & 0.01359531 & 0.0036183 \\ 0.00222309 & 0.0036183 & 0.01226303 \end{pmatrix}.$$

The pair $(F_1(\tilde{\mathbf{X}}), F_2(\tilde{\mathbf{X}}))$ defines the Nash equilibrium point with

$$F_1(\tilde{\mathbf{X}}) = \begin{pmatrix} -0.02010912 & -0.04090623 & -0.0385815 \\ -0.00398803 & -0.00569488 & -0.00583653 \end{pmatrix}$$

and

$$F_1(\tilde{\mathbf{X}}) = \begin{pmatrix} -0.00138726 & -0.00071184 & -0.00050222 \\ -0.01597642 & -0.02063644 & -0.01883039 \\ -0.00125061 & -0.00132644 & -0.00351967 \end{pmatrix}.$$

We execute additional example for different values of n.

Example 2 *The matrix coefficients are:*
m1=2; m2=3;
A_0 *= np.random.randn(n,n)/100 -1.5*np.matlib.identity(n);*
A_1 *= abs(np.random.randn(n,n))/10*
B_1 *= np.matrix([[0.0, 0.],[0.05, 0.1],[0.04, 0.15]]);*
C_1 *= np.matrix([[0., 0.1],[1.1, 0],[0., 0.02]]);*
for i in range (0,n-3):
h=np.matrix([uniform(-0.5, 0.5),uniform(-0.5, 0.5)])
B_1*=np.concatenate((*B_1*,h/10))*
h=np.matrix([uniform(-0.5, 0.5),uniform(-0.5, 0.5)])
C_1*=np.concatenate((*C_1*,h/10))*
B_2 *= np.matrix([[0.1, 0.5 , 0.4],[0., 0, 0.08],[0., 0., 2.2]])*
C_2 *= np.matrix([[0.1, 0. , 0.],[0., 1.5, 0.0],[0.1, 0.05, 0.0]])*
for i in range (0,n-3):
h=np.matrix([uniform(-0.5, 0.5),uniform(-0.5, 0.5),uniform(-0.5, 0.5)])
B_2*=np.concatenate((*B_2*,h/10))*
h=np.matrix([uniform(-0.5, 0.5),uniform(-0.5, 0.5),uniform(-1.5, 0.5)])
C_2*=np.concatenate((*C_2*,h/10))*
Matrices $Q_1, Q_2, R_{11}, R_{21}, R_{22}, R_{12}$ *are the same as in Example 1. We execute 100 runs for each value of* n*.*

Table 1 presents the computational results for different values of n.

Table 1. Example 2. Numerical results for different values of n.

	Iteration (7)	
n	*maxIt*	*avIt*
5	12	10.15
10	20	14.6
15	35	26.9
25	287	122.6

4 Conclusion

We have made numerical experiments for computing the stabilizing solution to to block Riccati equation (1). The numerical experiments confirm the effectiveness of the block Lyapunov iteration.

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