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ON UNIFORM EXPONENTIAL SPLITTING FOR COCYCLES OF LINEAR OPERATORS IN BANACH SPACES *

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Dedicated to Professor Mihail Megan
on the occasion of his 70th anniversary

Abstract

The aim of this paper is to study a concept of uniform exponential splitting, as a generalization of uniform exponential dichotomy for a cocycle C over a semiflow S. We obtain some characterizations of this concept in terms of Datko's type, respectively Lyapunov functions.

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1 Introduction

The notion of cocycle over a semiflow comes naturally when one considers the linearization along an invariant manifold of a dynamical system generated by a nonlinear differential equation (see [5], Chapter 6).
Important results in the study of the asymptotical behaviors of the dynamical systems described by cocycles have been obtained by L. Barreira and C. Valls [3], S. N. Chow and H. Leiva [6], N. T. Huy [8], Y. Latushkin, S. Montgomery-Smith and T. Randolph [9], Y. Latushkin and R. Schnaubelt [10], M. Megan, A. L. Sasu and B. Sasu [11], M. Megan, C. Stoica and L. Buliga [12], M. Megan and C. Stoica [13], R. J. Sacker and G. R. Sell [15].
In this paper we study a general concept of uniform exponential splitting as a generalization of uniform exponential dichotomy property for the cocycles of linear operators. Characterizations of this concept are obtained from the point of view of the projectors families (invariant and strongly invariant). Some illustrative examples which motivate the use of this concept of exponential splitting are presented.

2 Cocycles over semiflows

Let us denote by X a metric space, by V a Banach space and by $\mathcal{B}(V)$ the Banach algebra of all bounded linear operators on V. The norm on V and on $\mathcal{B}(V)$ will be denoted by $\|\cdot\|$. Let I be the identity operator on V and we also shall denote by $\mathbb{R}_+$ the set of nonnegative real numbers and $Y = X \times V$.

Definition 1. A mapping $S : \mathbb{R}_+ \times X \to X$ is called a *semiflow* on X, if:

(s_1) $S(0, x) = x$, for every $x \in X$;

(s_2) $S(t_1, S(t_2, x)) = S(t_1 + t_2, x)$, for all $(t_1, t_2, x) \in \mathbb{R}^2_+ \times X$.

A trivial example of semiflow is given in

Example 1. If $X = \mathbb{R}_+$(with the euclidian metric) then

$$S : \mathbb{R}_+ \times X \to X, \quad S(t, x) = t + x$$

is a semiflow on $\mathbb{R}_+$.

Example 2. Let X be the metric space of all continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$ with the topology of uniform convergence on compact subset of $\mathbb{R}_+$. The mapping $S : \mathbb{R}_+ \times X \to X$ defined by $S(t, x) = x_t$, where

$$x_t(s) = x(t + s), \text{ for all } (t, s) \in \mathbb{R}^2_+$$

is a semiflow on X.

Definition 2. A mapping $C:\mathbb{R}_+\times X\to\mathcal{B}(V)$ is called *a cocycle over the semiflow* $S:\mathbb{R}_+\times X\to X$ on the space $Y=X\times V$ if

(c_1) $C(0,x)v=v$, for every $(x,v)\in Y$;

(c_2) $C(t_1,S(t_2,x))C(t_2,x)=C(t_1+t_2,x)$, for all $(t_1,t_2,x)\in\mathbb{R}_+^2\times X$.
Moreover, if

(c_3) there are $M\geq 1$ and $\omega>0$ such that

$$\|C(t,x)v\|\leq Me^{\omega t}\|v\|,\ \text{for all } (t,x,v)\in\mathbb{R}_+\times Y,$$

then we say that C has *exponential growth.*

The linear skew-product semiflow associated with the above cocycle is the dynamical system $\pi=(S,C)$ on $Y=X\times V$ defined by

$$\pi:\mathbb{R}_+\times Y\to Y,\ \pi(t,x,v)=(S(t,x),C(t,x)v)$$

Definition 3. A cocycle $C:\mathbb{R}_+\times X\to\mathcal{B}(V)$ is called *strongly measurable* if for all $x\in X$, the mapping $t\to C(t,x)$ is measurable.

Example 3. Let $E:\Delta=\{(t,s)\in\mathbb{R}_+^2:t\geq s\}\to\mathcal{B}(V)$ be an evolution operator on the space V (i.e. $E(t,t)=I$ and $E(t,s)E(s,t_0)=E(t,t_0)$, for all $t\geq s\geq t_0\geq 0$). If $X=\mathbb{R}_+$, then the mapping

$$C_E:\mathbb{R}_+\times X\to\mathcal{B}(V),\ C_E(t,x)=E(t+x,x)$$

is a cocycle over the semiflow S defined in Example 1.

Example 4. Let X be the metric space and let S be the semiflow defined in Example 2. If $a,b,c,d\in\mathbb{R}$ and $V=\mathbb{R}^3$ with the norm

$$\|(v_1,v_2,v_3)\|=|v_1|+|v_2|+|v_3|,$$

then the mapping $C:\mathbb{R}_+\times X\to\mathcal{B}(V)$ defined by

$$C(t,x)v=\begin{cases}(v_1e^{at+b\int_0^t x(s)ds},0,v_3e^{ct+d\int_0^t x(s)ds}), & t>0\\ (v_1,v_2,v_3), & t=0\end{cases}$$

is a cocycle over S on $Y=X\times V$.

Example 5. Let X be a locally compact metric space, $S : \mathbb{R}_+ \times X \to X$ a semiflow on X and V a Banach space. If $A : X \to \mathcal{B}(V)$ is a continuous mapping and $y(t, x, v)$ is the solution of the Cauchy problem

$$\begin{cases} y'(t) = A(S(t,x))y(t), & t \geq 0 \\ y(0) = v \end{cases}$$

then the mapping

$$C : \mathbb{R}_+ \times X \to \mathcal{B}(V),\ C(t,x)v = y(t,x,v)$$

is a cocycle over the semiflow S on $Y = X \times V$.

3 Uniform exponential splitting with invariant projectors

Let $C : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ be a cocycle over the semiflow $S : \mathbb{R}_+ \times X \to X$ on $Y = X \times V$.

Definition 4. A mapping $P : X \to \mathcal{B}(V)$ is called a *family of projectors* on the Banach space V if

(p_1) $P^2(x) = P(x)$, for every $x \in X$.
Moreover,

(p_2) if there is $M \geq 1$ such that

$$\|P(x)\| \leq M, \quad \text{for all } x \in X,$$

then we say that P *is bounded*;

(p_3) if $C(t,x)P(x) = P(S(t,x))C(t,x)$, for all $(t,x) \in \mathbb{R}_+ \times X$, then we say that P is *invariant for the cocycle* C.

Remark 1. If $P : X \to \mathcal{B}(V)$ is invariant for the cocycle $C : \mathbb{R}_+ \times X \to \mathcal{B}(V)$, then

$$Q : X \to \mathcal{B}(V) \quad \text{defined by} \quad Q(x) = I - P(x)$$

is a family of projectors (called the *complementary family of* P) which is also invariant for C.

Let $C : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ be a cocycle over the semiflow $S : \mathbb{R}_+ \times X \to X$ on $Y = X \times V$. We shall denote by

$$C_P(t,x) = C(t,x)P(x) \quad \text{and} \quad C_Q(t,x) = C(t,x)Q(x).$$

Definition 5. We say that the pair (C,P) has *uniform exponential splitting* if there exist three constants $N \geq 1$ and $\alpha, \beta \in \mathbb{R}, \alpha < \beta$ such that

(ues_1) $\|C_P(t,x)v\| \leq Ne^{\alpha t}\|P(x)v\|$

(ues_2) $e^{\beta t}\|Q(x)v\| \leq N\|C_Q(t,x)v\|$,

for all $(t,x,v) \in \mathbb{R}_+ \times Y$.

Remark 2. The constants α and β are called *splitting rates.*

Remark 3. As a particular case, for $\alpha < 0 < \beta$ we obtain the *uniform exponential dichotomy* property.

Remark 4. If the pair (C,P) has uniform exponential dichotomy, then it has uniform exponential splitting. The converse implication is not true, as the following example shows.

Example 6. Let C be the cocycle defined in Example 4. We consider the family of projectors $P : X \to \mathcal{B}(V)$ defined by $P(x)v = (v_1, v_2, 0)$. We observe that P is invariant for C and

$$C(t,x)P(x)v = \begin{cases} (v_1 e^{at+b\int_0^t x(s)ds}, 0, 0), & t > 0 \\ (v_1, v_2, 0), & t = 0 \end{cases}$$

$$C(t,x)Q(x)v = \begin{cases} (0, 0, v_3 e^{ct+d\int_0^t x(s)ds}), & t > 0 \\ (0, 0, v_3), & t = 0 \end{cases}$$

It follows that

$$\|C_P(t,x)v\| = \begin{cases} |v_1| e^{at+b\int_0^t x(s)ds}, & t > 0 \\ |v_1| + |v_2|, & t = 0 \end{cases}$$

and

$$\|C_Q(t,x)v\| = \begin{cases} |v_3| e^{ct+d\int_0^t x(s)ds}, & t > 0 \\ |v_3|, & t = 0 \end{cases}$$

If we suppose that $b \leq 0 \leq d$ and $a < c$ then for $N = 1$ and for all $\alpha, \beta \in \mathbb{R}$ with $a \leq \alpha < \beta \leq c$ the inequalities (ues_1) and (ues_2) are equivalent with

$$|v_1| e^{at + b\int_0^t x(s)ds} \leq N e^{\alpha t}(|v_1| + |v_2|)$$

and

$$e^{\beta t}|v_3| \leq N|v_3| e^{ct + d\int_0^t x(s)ds}.$$

So the pair (C, P) has uniform exponential splitting.
If we suppose that $b \leq 0 \leq d$ and $a < 0 < c$ then for $N = 1$ and for all $\alpha, \beta \in \mathbb{R}$ with $a \leq \alpha < 0 < \beta \leq c$ the inequalities (ues_1) and (ues_2) hold. In this case we obtain that (C, P) is uniformly exponentially dichotomic.
If the inequalities $b \leq 0 \leq d$ and $a < c$ don't hold then there is not $N \geq 1$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ such that the relations (ues_1) and (ues_2) hold. In this case (C, P) doesn't have uniform exponential splitting.

Proposition 1. *The pair (C, P) has uniform exponential splitting if and only if there exist $N \geq 1$ and $\alpha, \beta \in \mathbb{R}, \alpha < \beta$ such that*

(ues_1') $\|C_P(t + s, x)v\| \leq N e^{\alpha t}\|C_P(s, x)v\|$

(ues_2') $e^{\beta t}\|C_Q(s, x)v\| \leq N\|C_Q(t + s, x)v\|,$

for all $(t, s, x, v) \in \mathbb{R}_+^2 \times Y$.

Proof. Necessity. It results for $x \to S(s, x)$ and $v \to C(s, x)v$.
Sufficiency. It is immediate for $s = 0$. □

For the particular case of uniform exponential dichotomy, we obtain

Proposition 2. *The pair (C, P) has uniform exponential dichotomy if and only if there are $N \geq 1$ and $\nu > 0$ such that*

(ued_1) $\|C_P(t, x)v\| \leq N e^{-\nu t}\|P(x)v\|$

(ued_2) $e^{\nu t}\|Q(x)v\| \leq N\|C_Q(t, x)v\|,$

for all $(t, x, v) \in \mathbb{R}_+ \times Y$.

Proof. Necessity. It results for $\nu = \min\{-\alpha, \beta\}$.
Sufficiency. It is immediate for $\beta = \nu = -\alpha$.

□

We consider $\gamma = \dfrac{\alpha+\beta}{2}$ and $\delta = \dfrac{\beta-\alpha}{2}$, where $\alpha < \beta$ and we define the cocycle $C^\gamma : \mathbb{R}_+ \times X \to \mathcal{B}(V)$, by $C^\gamma(t,x) = e^{-\gamma t}C(t,x)v$.

Proposition 3. *The pair (C,P) has uniform exponential splitting with the splitting rates α, β if and only if (C^γ, P) has an uniform exponential dichotomy with the dichotomy rate γ.*

Proof. Necessity. To prove (ued_1), we observe that

$$\|C_P^\gamma(t,x)v\| = e^{-\gamma t}\|C_P(t,x)v\| \leq Ne^{-\gamma t}e^{\alpha t}\|P(x)v\| =$$
$$= Ne^{(\alpha-\gamma)t}\|P(x)v\| = Ne^{-\delta t}\|P(x)v\|, \quad \text{for all} \quad (t,x,v) \in \mathbb{R}_+ \times Y.$$

Similarly,

$$e^{\delta t}\|Q(x)v\| \leq Ne^{\delta t}e^{-\beta t}\|C_Q(t,x)v\| = Ne^{(\delta-\beta)t}\|C_Q(t,x)v\| =$$
$$= Ne^{-\gamma t}\|C_Q(t,x)v\| = N\|C_Q^\gamma(t,x)v\|,$$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$ and hence we obtain (ued_2).
Sufficiency. We have that

$$\|C_P(t,x)v\| = e^{\gamma t}\|C_P^\gamma(t,x)v\| \leq Ne^{(\gamma-\delta)t}\|P(x)v\| = Ne^{\alpha t}\|P(x)v\|,$$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$, so we proved (ues_1). Similarly,

$$e^{\beta t}\|Q(x)v\| = e^{(\gamma+\delta)t} = \|Q(x)v\| = e^{\delta t}e^{\gamma t}\|Q(x)v\| \leq$$
$$\leq Ne^{\gamma t}\|C_Q^\gamma(t,x)v\| = N\|C_Q(t,x)v\|.$$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$ and hence we obtain (ues_2). □

We consider a cocycle $C : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ which is strongly measurable and a family of projectors $P : X \to \mathcal{B}(V)$ invariant for the cocycle C.
The next result is a theorem of Datko's type for uniform exponential splitting.

Theorem 1. *The pair (C,P) has uniform exponential splitting if and only if there exist $D \geq 1$, $\omega > 0$ and μ, ν with $\mu < \nu$ such that the following conditions hold*

$$(ues_1'') \quad \int_t^\infty e^{\mu(t-\tau)}\|C_P(\tau,x)v\|d\tau \leq D\|C_P(t,x)v\|$$

(ues_2'') $\displaystyle\int_0^t e^{\nu(t-\tau)}\|C_Q(\tau,x)v\|d\tau \le D\|C_Q(t,x)v\|$

(ues_3'') $\|C_P(t,x)v\| \le De^{\omega t}\|P(x)v\|$

(ues_4'') $e^{-\omega t}\|Q(x)v\| \le D\|C_Q(t,x)v\|,$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$.

Proof. Necessity. It follows immediately for

$$\omega = \begin{cases} \beta, & \beta > 0 \\ 1, & \beta = 0 \\ -\beta, & \beta < 0 \end{cases}$$

and $\alpha < \mu < \nu < \beta$.
Sufficiency. First, we shall prove that the condition (ues_1) from Definition 5 holds. If $t \ge 1$ then

$$\begin{aligned} e^{-\mu t}\|C_P(t,x)v\| &= \int_{t-1}^{t} e^{-\mu t}\|C_P(t,x)v\|d\tau = \\ &= \int_{t-1}^{t} e^{-\mu t}De^{\omega(t-\tau)}\|C_P(\tau,x)v\|d\tau = \\ &= D\int_{t-1}^{t} e^{-\mu(t-\tau)}e^{\omega(t-\tau)}e^{-\mu\tau}\|C_P(\tau,x)v\|d\tau \le \\ &\le De^{|\omega-\mu|}\int_{t-1}^{t} e^{-\mu\tau}\|C_P(\tau,x)v\|d\tau \le \\ &\le De^{|\omega-\mu|}\int_{0}^{+\infty} \|C_P(\tau,x)v\|d\tau \le D^2e^{|\omega-\mu|}\|P(x)v\|, \end{aligned}$$

for all $(x,v) \in Y$. For $t \in [0,1)$ we have that

$$\begin{aligned} \|C_P(t,x)v\| &\le De^{\omega t}\|P(x)v\| = De^{(\omega-\mu)t}e^{\mu t}\|P(x)v\| \le \\ &\le De^{|\omega-\mu|}e^{\mu t}\|P(x)v\|, \end{aligned}$$

for all $(x, v) \in Y$. Next, we shall prove the condition (ues_2) from Definition 5. If $t \geq 1$ then

$$e^{\nu t}\|Q(x)v\| = \int_0^1 e^{\nu t}\|Q(x)v\|d\tau \leq \int_0^1 e^{\nu t} D e^{\omega\tau}\|C_Q(\tau, x)v\|d\tau =$$

$$= D\int_0^1 e^{(\omega+\nu)\tau} e^{\nu(t-\tau)}\|C_Q(\tau, x)v\|d\tau \leq$$

$$\leq De^{\omega+\nu}\int_0^1 e^{\nu(t-\tau)}\|C_Q(\tau, x)v\|d\tau \leq$$

$$\leq De^{\omega+\nu}\int_0^t e^{\nu(t-\tau)}\|C_Q(\tau, x)v\|d\tau \leq D^2 e^{\omega+\nu}\|C_Q(t, x)v\|,$$

for all $(x, v) \in Y$. For $t \in [0, 1)$ we obtain that

$$e^{\nu t}\|Q(x)v\| \leq e^{\nu t} e^{\omega t} D\|C_Q(t, x)v\| = e^{(\nu+\omega)t} D\|C_Q(t, x)v\| \leq$$
$$\leq De^{|\nu+\omega|}\|C_Q(t, x)v\|,$$

for all $(x, v) \in Y$.

□

As a particular case, we obtain a characterization of uniform exponential dichotomy for cocycles of linear operators.

Corollary 1. *The pair (C, P) has uniform exponential dichotomy if and only if there exist $D \geq 1$, $\omega, \nu > 0$ such that the following conditions hold*

(i) $\displaystyle\int_t^\infty e^{\nu(\tau-t)}\|C_P(\tau, x)v\|d\tau \leq D\|C_P(t, x)v\|$

(ii) $\displaystyle\int_0^t e^{\nu(t-\tau)}\|C_Q(\tau, x)v\|d\tau \leq D\|C_Q(t, x)v\|$

(iii) $\|C_P(t, x)v\| \leq De^{\omega t}\|P(x)v\|$

(iv) $e^{-\omega t}\|Q(x)v\| \leq D\|C_Q(t, x)v\|$,

for all $(t, x, v) \in \mathbb{R}_+ \times Y$.

Definition 6. A mapping $L : \mathbb{R}_+ \times Y \to \mathbb{R}_+$ is called a *Lyapunov function* for the pair (C, P) if there exist $c < d$ such that the following inequalities hold

$$(l_1)\;\; L(t, x, P(x)v) + \int_0^t e^{-c\tau} \|C_P(\tau, x)v\| d\tau \leq L(0, x, P(x)v)$$

$$(l_2)\;\; L(0, x, Q(x)v) + \int_0^t e^{d(t-\tau)} \|C_Q(\tau, x)v\| d\tau \leq L(t, x, Q(x)v),$$

for all $(t, x, v) \in \mathbb{R}_+ \times Y$.

The next result is a theorem of Lyapunov type for the property of uniform exponential splitting of cocycles of linear operators.

Theorem 2. *The pair (C, P) has uniform exponential splitting if and only if there exist a Lyapunov function $L : \mathbb{R}_+ \times Y \to \mathbb{R}_+$ for (C, P) and the constants $M \geq 1$, $\omega > 0$ such that the following conditions hold*

(i) $L(0, x, P(x)v) \leq M\|P(x)v\|$

(ii) $L(t, x, Q(x)v) \leq M\|C_Q(t, x)v\|$

(ues_3'') $\|C_P(t, x)v\| \leq De^{\omega t}\|P(x)v\|$

(ues_4'') $e^{-\omega t}\|Q(x)v\| \leq D\|C_Q(t, x)v\|,$

for all $(t, x, v) \in \mathbb{R}_+ \times Y$.

Proof. Necessity. Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ be as in Definition 5 and we consider ω, μ, ν as in Theorem 1 and $L : \mathbb{R}_+ \times Y \to \mathbb{R}_+$,

$$L(t, x, v) = \int_t^\infty e^{-\mu\tau} \|C_P(\tau, x)v\| d\tau + \int_0^t e^{\nu(t-\tau)} \|C_Q(\tau, x)v\| d\tau.$$

We shall prove (l_1) and (l_2). We observe that

$$L(t,x,P(x)v)+\int\limits_0^t e^{-\mu\tau}\|C_P(\tau,x)v\|d\tau=$$

$$=\int\limits_t^\infty e^{-\mu\tau}\|C_P(\tau,x)v\|d\tau+\int\limits_0^t e^{-\mu\tau}\|C_P(\tau,x)v\|d\tau=$$

$$=\int\limits_0^\infty e^{-\mu\tau}\|C_P(\tau,x)v\|d\tau=L(0,x,P(x)v),$$

for all $(t,x,v)\in\mathbb{R}_+\times Y$. For (l_2), we have that

$$L(0,x,Q(x)v)+\int\limits_0^t e^{\nu(t-\tau)}\|C_Q(\tau,x)v\|d\tau=\int\limits_0^t e^{\nu(t-\tau)}\|C_Q(\tau,x)v\|d\tau=$$

$$=L(t,x,Q(x)v),$$

for all $(t,x,v)\in\mathbb{R}_+\times Y$. Using Datko's theorem, we have that

$$L(0,x,P(x)v)=\int\limits_0^\infty e^{-\mu\tau}\|C_P(\tau,x)v\|d\tau\le D\|P(x)v\|,$$

and

$$L(t,x,Q(x)v)=\int\limits_0^t e^{\nu(t-\tau)}\|C_Q(\tau,x)v\|d\tau\le D\|C_Q(t,x)v\|,$$

for all $(t,x,v)\in\mathbb{R}_+\times Y$.

Sufficiency. We suppose that there exists a Lyapunov function $L:\mathbb{R}_+\times Y\to\mathbb{R}_+$ for (C,P). Then from the condition (l_1) we obtain that

$$\int_0^t e^{-c\tau}\|C_P(\tau,x)v\|d\tau\le L(0,x,P(x)v)\le M\|P(x)v\|,$$

for all $(t,x,v)\in\mathbb{R}_+\times Y$. If $t\ge 1$ then

$$e^{-ct}||C_P(t,x)v||=\int\limits_{t-1}^t e^{-ct}||C_P(t,x)v||d\tau\le$$

$$\leq D\int_{t-1}^{t} e^{-ct}e^{\omega(t-\tau)}||C_P(\tau,x)v||d\tau \leq D\int_{t-1}^{t} e^{(\omega-c)(t-\tau)}e^{-c\tau}||C_P(\tau,x)v||d\tau \leq$$

$$\leq De^{|\omega-c|}\int_{0}^{t} e^{-c\tau}||C_P(\tau,x)v||d\tau \leq DMe^{|\omega-c|}||P(x)v||.$$

If $t \in [0,1)$ then

$$||C_P(t,x)v|| \leq De^{\omega t}||P(x)v|| = De^{(\omega-c)t}e^{ct}||P(x)v|| \leq DMe^{|\omega-c|}e^{ct}||P(x)v||.$$

Now, using (l_2) we have that

$$\int_{0}^{t} e^{d(t-\tau)}\|C_Q(\tau,x)v\|d\tau \leq L(t,x,Q(x)v) \leq M\|C_Q(t,x)v\|,$$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$. In what follows, from

$$\int_{0}^{t} e^{d(t-\tau)}\|C_Q(\tau,x)v\|d\tau \leq M\|C_Q(t,x)v\|,$$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$ and from (ues_4''), as in sufficiency of Theorem 1, we obtain that

$$e^{dt}\|Q(x)v\| \leq DMe^{|d+\omega|}\|C_Q(t,x)v\|,$$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$. □

4 Uniform exponential splitting with strongly invariant projectors

Let $P : X \to \mathcal{B}(V)$ a family of projectors on the Banach space V which is invariant for the cocycle $C : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ over the semiflow $S : \mathbb{R}_+ \times X \to X$ on $Y = X \times V$, where X is a metric space.

Definition 7. We say that the family of projectors $P : X \to \mathcal{B}(V)$ is *strongly invariant* for the cocycle C if for all $(t,x) \in \mathbb{R}_+ \times X$ the bounded linear operator $C(t,x)$ is an isomorfism from $KerP(x)$ to $KerP(S(t,x))$.

Remark 5. If the family of projectors $P : X \to \mathcal{B}(V)$ is strongly invariant for the cocycle $C : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ over the semiflow $S : \mathbb{R}_+ \times X \to X$, then there exists $D : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ such that for all $(t, x) \in \mathbb{R}_+ \times X$ the bounded linear operator $D(t, x)$ is an isomorphism from $KerP(S(t, x))$ to $KerP(x)$ and

(i_1) $C(t, x)D(t, x)Q(S(t, x)) = Q(S(t, x))$

and

(i_2) $D(t, x)C(t, x)Q(x) = Q(x)$,

for all $(t, x) \in \mathbb{R}_+ \times X$, where $Q(x) = I - P(x)$.

Remark 6. If $P : X \to \mathcal{B}(V)$ is strongly invariant for the cocycle $C : \mathbb{R}_+ \times X \to \mathcal{B}(V)$, then the isomorphism D has the property

(i_3) $Q(x)D(t, x)Q(S(t, x)) = D(t, x)Q(S(t, x))$,

for every $(t, x) \in \mathbb{R}_+ \times X$.

Theorem 3. *If $P : X \to \mathcal{B}(V)$ is strongly invariant for the cocycle $C : \mathbb{R}_+ \times X \to \mathcal{B}(V)$, then the isomorphism D satisfies the following evolution property:*

(i_4) $D(t + s, x)Q(S(t + s, x)) = D(s, x)D(t, S(s, x))Q(S(t + s, x))$,

for all $(t, s, x) \in \mathbb{R}_+^2 \times X$.

Proof. Indeed, we have that

$$\begin{aligned}
&D(t+s,x)Q(S(t+s,x)) = Q(x)D(t+s,x)Q(S(t+s,x)) = \\
&= D(s,x)C(s,x)Q(x)D(t+s,x)Q(S(t+s,x)) = \\
&= D(s,x)Q(S(s,x))C(s,x)D(t+s,x)Q(S(t+s,x)) = \\
&= D(s,x)D(t,S(s,x))C(t,S(s,x))Q(S(s,x))C(s,x)D(t+s,x)Q(S(t+s,x)) = \\
&= D(s,x)D(t,S(s,x))C(t+s,x)Q(x)D(t+s,x)Q(S(t+s,x)) = \\
&= D(s,x)D(t,S(s,x))C(t+s,x)D(t+s,x)Q(S(t+s,x)) = \\
&= D(s,x)D(t,S(s,x))Q(S(t+s,x)),
\end{aligned}$$

for all $(t, s, x) \in \mathbb{R}_+^2 \times X$. □

Proposition 4. *Let $P : X \to \mathcal{B}(V)$ be a family of projectors which is strongly invariant for the cocycle $C : \mathbb{R}_+ \times X \to \mathcal{B}(V)$. Then the pair (C, P) has uniform exponential splitting if and only if there exist $N \geq 1$, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ such that*

$(ues_1^{'''})$ $\|C_P(t,x)v\| \leq Ne^{\alpha t}\|P(x)v\|$

$(ues_2^{'''})$ $e^{\beta t}\|D(t,x)Q(S(t,x))v\| \leq N\|Q(S(t,x))v\|,$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$.

Proof. It is sufficient to prove $(ues_2) \Leftrightarrow (ues_2^{'''})$. We shall prove the implication $(ues_2^{'''}) \Rightarrow (ues_2)$.

$$e^{\beta t}\|Q(x)v\| = e^{\beta t}\|D(t,x)Q(S(t,x))C(t,x)Q(x)v\| \leq N\|C(t,x)Q(x)v\|,$$

$(t,x,v) \in \mathbb{R}_+ \times Y$.
For the converse implication $(ues_2) \Rightarrow (ues_2^{'''})$, we observe that

$$\begin{aligned} e^{\beta t}\|D(t,x)Q(S(t,x))v\| = e^{\beta t}\|Q(x)D(t,x)Q(S(t,x))v\| \leq \\ \leq N\|C(t,x)Q(x)D(t,x)Q(S(t,x))v\| = N\|Q(S(t,x))v\|, \end{aligned}$$

$(t,x,v) \in \mathbb{R}_+ \times Y$. □

Theorem 4. *Let* $P : X \to \mathcal{B}(V)$ *be a strongly invariant family of projectors for the cocycle* $C : \mathbb{R}_+ \times X \to \mathcal{B}(V)$. *Then the pair* (C,P) *has uniform exponential splitting if and only if there exist* $D \geq 1$, $\omega > 0$ *and* μ, ν *with* $\mu < \nu$ *such that the following conditions hold*

$$(ues_1^{''}) \quad \int_t^\infty e^{\mu(t-\tau)}\|C(\tau,x)P(x)v\|d\tau \leq D\|C(t,x)P(x)v\|$$

$$(ues_2^{''''}) \quad \int_0^t e^{\nu(t-\tau)}\|D(t-\tau,S(\tau,x))Q(S(t,x))v\|d\tau \leq D\|Q(S(t,x))v\|$$

$(ues_3^{''})$ $\|C_P(t,x)v\| \leq De^{\omega t}\|P(x)v\|$

$(ues_4^{'''})$ $e^{-\omega t}\|D(t,x)Q(S(t,x))v\| \leq D\|Q(S(t,x))v\|,$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$.

Proof. Necessity. We consider $\alpha < \mu < \nu < \beta$ and $D \geq N\left(\dfrac{1}{\mu-\alpha}+\dfrac{1}{\beta-\nu}\right)$. We have that

$$\int_t^\infty e^{\mu(t-\tau)}\|C(\tau,x)P(x)v\|d\tau \leq$$
$$\leq N\int_t^\infty e^{\mu(t-\tau)}e^{\alpha(\tau-t)}d\tau\|C(t,x)P(x)v\| =$$
$$= N\int_t^\infty e^{(\alpha-\mu)(\tau-t)}d\tau\|C(t,x)P(x)v\| \leq \frac{N}{\mu-\alpha}\|C(t,x)P(x)v\|,$$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$. If in (ues_2''') we consider $t \to t-\tau$ and $x \to S(\tau,x)$, then

$$\int_0^t e^{\nu(t-\tau)}\|D(t-\tau,S(\tau,x))Q(S(t,x))v\|d\tau \leq$$
$$\leq N\int_0^t e^{\nu(t-\tau)}e^{-\beta(t-\tau)}\|Q(S(t,x))v\|d\tau = N\int_0^t e^{(\nu-\beta)(t-\tau)}d\tau\|Q(S(t,x))v\| \leq$$
$$\leq \frac{N}{\beta-\nu}\|Q(S(t,x))v\|,$$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$. We shall prove $(ues_4^{''}) \Rightarrow (ues_4^{'''})$. We observe that

$$e^{-\omega t}\|D(t,x)Q(S(t,x))v\| = e^{-\omega t}\|Q(x)D(t,x)Q(S(t,x))v\| \leq$$
$$\leq D\|C(t,x)Q(x)D(t,x)Q(S(t,x))v\| =$$
$$= D\|Q(S(t,x))C(t,x)D(t,x)Q(S(t,x))v\| =$$
$$= D\|Q(S(t,x))Q(S(t,x))v\| = D\|Q(S(t,x))v\|,$$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$.

Sufficiency. From $(ues_1^{''})$ and $(ues_3^{''})$ as in sufficiency of Theorem 1, it results $(ues_1^{'''})$. We shall prove the condition $(ues_2^{'''})$ from Proposition 4. We consider $t \geq 1$. We observe that if in (i_4) and (i_3) $t \to t-\tau$ and $x \to S(\tau,x)$, then

$$\begin{aligned}
e^{\nu t}\|D(t,x)Q(S(t,x))v\| &= \int_0^1 e^{\nu t}\|D(t,x)Q(S(t,x))v\|d\tau = \\
&= \int_0^1 e^{\nu t}\|D(\tau,x)Q(S(\tau,x))D(t-\tau,S(\tau,x))Q(S(t,x))v\|d\tau \leq \\
&\leq D\int_0^1 e^{\nu t}e^{\omega\tau}\|Q(S(\tau,x))D(t-\tau,S(\tau,x))Q(S(t,x))v\|d\tau = \\
&= D\int_0^1 e^{(\omega+\nu)\tau}e^{\nu(t-\tau)}\|D(t-\tau,S(\tau,x))Q(S(t,x))v\|d\tau \leq \\
&\leq De^{|\omega+\nu|}\int_0^t e^{\nu(t-\tau)}\|D(t-\tau,S(\tau,x))Q(S(t,x))v\|d\tau \leq \\
&\leq D^2e^{|\omega+\nu|}\|Q(S(t,x))v\|,
\end{aligned}$$

for all $(x,v)\in Y$. If $t\in[0,1)$ then we have that

$$\begin{aligned}
e^{\nu t}\|D(t,x)Q(S(t,x))v\| &\leq De^{\nu t}e^{\omega t}\|Q(S(t,x))v\| \leq \\
&\leq De^{|\omega+\nu|}\|Q(S(t,x))v\| \leq N\|Q(S(t,x))v\|,
\end{aligned}$$

for all $(x,v)\in Y$. We shall prove $(ues_4^{'''})\Rightarrow(ues_4^{''})$. We observe that

$$\begin{aligned}
e^{-\omega t}\|Q(x)v\| &= e^{-\omega t}\|D(t,x)C(t,x)Q(x)v\| = \\
&= e^{-\omega t}\|D(t,x)Q(S(t,x))C(t,x)v\| \leq \\
&\leq D\|Q(S(t,x))C(t,x)v\| = D\|C(t,x)Q(x)v\|,
\end{aligned}$$

for all $(t,x,v)\in\mathbb{R}_+\times Y$. □

As a consequence, we obtain

Corollary 2. *Let $P: X\to\mathcal{B}(V)$ be a strongly invariant family of projectors for the cocycle $C:\mathbb{R}_+\times X\to\mathcal{B}(V)$. Then the pair (C,P) is uniformly exponentially dichotomic if and only if there exist $D\geq 1$, $\nu,\omega>0$ such that*

(i) $\displaystyle\int_t^\infty e^{\nu(\tau-t)}\|C(\tau,x)P(x)v\|d\tau \leq D\|C(t,x)P(x)v\|$

(ii) $\int_0^t e^{\nu(t-\tau)}\|D(t-\tau, S(\tau,x))Q(S(t,x))v\|d\tau \leq D\|Q(S(t,x))v\|$

(iii) $\|C_P(t,x)v\| \leq De^{\omega t}\|P(x)v\|$

(iv) $e^{-\omega t}\|D(t,x)Q(S(t,x))v\| \leq D\|Q(S(t,x))v\|,$

for all $(t,x,v) \in \mathbb{R}_+ \times Y$.

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