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HIGHER-ORDER DIFFERENCES AND HIGHER-ORDER PARTIAL SUMS OF EULER'S PARTITION FUNCTION *

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Dedicated to Professor Mihail Megan
on the occasion of his 70th anniversary

Abstract

We provide generalizations for Euler's recurrence relation for the partition function $p(n)$ and the recurrence relation for the partial sums of the partition function $p(n)$. As a corollary, we derive an infinite family of inequalities for the partition function $p(n)$. We present few infinite families of determinant formulas for: the partition function $p(n)$, the finite differences of the partition function $p(n)$ and the higher-order partial sums of the partition function $p(n)$.

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1 Introduction

Let n be a positive integer. In order to indicate that $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ is a partition of n, i.e.,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k,$$

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we use the notation $\lambda \vdash n$. The number of all partitions of a positive integer n is denoted by $p(n)$. More details and proofs about partitions can be found in Andrews's book [1]. We denote by $S(n)$ the n-th partial sum of the partition function $p(n)$, i.e.,

$$S(n) = \sum_{k=0}^{n} p(k).$$

It is well-known that $S(n)$ counts the partitions of n into parts where the part 1 comes in two colours.

The following recurrence relation for the partial sums of the partition function $p(n)$,

$$\sum_{k=-\infty}^{\infty} (-1)^k S\left(n - k(3k-1)/2\right) = 1, \tag{1}$$

follows easily from Euler's recurrence relation for the partition function [1, Corollary 1.8, p. 12], namely

$$\sum_{k=-\infty}^{\infty} (-1)^k p\left(n - k(3k-1)/2\right) = 0. \tag{2}$$

In [3], the author presented the fastest known algorithm for the generation of the partitions of n. In the above mentioned work, the author produced this algorithm by introducing a special case of partitions with restrictions: the partition $\lambda \vdash n$ with the property

$$\lambda_1 \geq t \cdot \lambda_2 \qquad \text{and} \qquad \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_k,$$

where t is a positive integer such that $t \leq n$. We consider that the partition $[n]$ has this property and we denote the number of these partitions by $p^{(t)}(n)$. It is clear that

$$p^{(t)}(n) \geq 1 \qquad \text{and} \qquad p^{(1)}(n) = p(n).$$

Moreover, for $t \geq n$ we have $p^{(t)}(n) = 1$. By convention, we set

$$p^{(t)}(0) = 1, \qquad p^{(0)}(n) = p(n) \qquad \text{and} \qquad p^{(t)}(-n) = 0.$$

The formula

$$p^{(t)}(n) = p^{(t-1)}(n) - p^{(t-1)}(n-t) \tag{3}$$

has already been proved for $1 < t < n$ (see [3, Corollary 1]). It is clear that the relation (3) holds for any positive integer t and any positive integer n.

For all non-negative integers t and for all integers n, we define $a^{(t)}(n)$ by

$$a^{(t)}(n) = a^{(t-1)}(n) - a^{(t-1)}(n-t), \tag{4}$$

with

$$a^{(0)}(n) = \delta_{0,n},$$

where $\delta_{i,j}$ is Kronecker's delta. Note that the recurrence (3) for $p^{(t)}(n)$ is identical in form to the recurrence (4) for $a^{(t)}(n)$, while the initial conditions are different.

We shall use the integers $p^{(t)}(n)$ and $a^{(t)}(n)$ to prove:

Theorem 1. *Let n and t be two positive integers. The number of partitions of n into parts $> t$ is equal to $\nabla[p^{(t)}](n)$ and*

$$\sum_{k=0}^{n} a^{(t)}(k)p(n-k) = \nabla[p^{(t)}](n),$$

where $\nabla[f]$ denotes the first backward differences of the function f, i.e.,

$$\nabla[f](n) = f(n) - f(n-1).$$

Theorem 2. *Let n and t be two non-negative integers. Then*

$$\sum_{k=0}^{n} s_{t,k}p(n-k) = \binom{n+t}{t},$$

where

$$s_{0,n} = \sum_{k=0}^{n} a^{(k)}(k) \qquad \text{and} \qquad s_{t,n} = \sum_{k=0}^{n} s_{t-1,k}, \text{ for } t > 0.$$

Corollary 1. *Let n and t be two positive integers. Then*

$$\sum_{k=0}^{n} a^{(t)}(k)S(n-k) = p^{(t)}(n).$$

This result is immediate from Theorem 1 because

$$\begin{aligned} p^{(t)}(n) - p^{(t)}(0) &= \sum_{j=1}^{n} \nabla[p^{(t)}](j) \\ &= \sum_{j=1}^{n}\sum_{k=0}^{j} a^{(t)}(k)p(j-k) \\ &= \sum_{k=0}^{n} a^{(t)}(k)S(n-k) - a^{(t)}(0)p(0) \end{aligned}$$

and $p^{(t)}(0) = a^{(t)}(0)p(0) = 1$.

Taking into account (4), it is an easy exercise to show that the generating function for $a^{(t)}(n)$ is $(q;q)_t$, i.e.,

$$\sum_{n=0}^{\infty} a^{(t)}(n)q^n = (q;q)_t, \tag{5}$$

where $(A;q)_n$ is q-Pochhammer symbol, namely

$$(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}),$$

with $(A;q)_0 = 1$. Because $\nabla[p^{(n)}](n) = \delta_{0,n}$, the following result is a consequence of Theorem 1 and the pentagonal number theorem [1, Corollary 1.7, p. 11].

Corollary 2. *Let n and t be two nonnegative integers such that $n \le t$. Then*

$$a^{(t)}(n) = \begin{cases} (-1)^k, & \text{if } n = \frac{1}{2}(3k^2 \pm k),\ k \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}.$$

Now, we note that the recurrence (1) is the case $t \ge n$ in Corollary 1 and the recurrence (2) is the case $t \ge n$ in Theorem 1. We can see that for all non-negative integers t we have

$$a^{(t+n)}(n) = a^{(n)}(n)$$

and the integer $a^{(n)}(n)$ is the coefficient of q^n in the Euler function $(q;q)_\infty$. Moreover, $s_{0,n}$ is the n-th partial sum of the coefficients q^n from $(q;q)_\infty$, i.e.,

$$s_{0,n} = \begin{cases} (-1)^k, & \text{if } k + P_k \le n < P_{k+1},\ k \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

where P_k is the k-th pentagonal number, namely

$$P_k = \frac{1}{2}\left(3k^2 - k\right)$$

(see A078616 in [4]).

In this paper, using the integers $p^{(t)}(n)$ and $a^{(t)}(n)$, we give a new formulas for the partition function, the finite differences of the partition function and the partial sum of the partition function. As a corollary, we derive an infinite family of inequalities for the partition function. We consider this a good reason for someone to study the $p^{(t)}(n)$ and $a^{(t)}(n)$ numbers.

2 Proofs of theorems

The generating function of $p(n)$ is given by the reciprocal of Euler's function $(q;q)_\infty$, namely

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Using induction on t and the relation (3) it is an easy exercise to show that the generating function for $\nabla[p^{(t)}](n)$ is $(q;q)_t/(q;q)_\infty$, i.e.,

$$\sum_{n=0}^{\infty} \nabla[p^{(t)}](n)q^n = \frac{(q;q)_t}{(q;q)_\infty}.$$

Therefore, taking into account (5), we obtain

$$\sum_{n=0}^{\infty} \nabla[p^{(t)}](n)q^n = \left(\sum_{n=0}^{\infty} p(n)q^n\right)\left(\sum_{n=0}^{\infty} a^{(t)}(n)q^n\right).$$

Extracting coefficients of q^n we get

$$\nabla[p^{(t)}](n) = \sum_{k=0}^{n} a^{(t)}(k)p(n-k)$$

and Theorem 1 is proved.

Theorem 2 follows directly from

Lemma 1. *Let n be a non-negative integers. Then*

$$\sum_{k=0}^{n} s_{0,k}p(n-k) = 1.$$

Proof. Expanding the term $p^{(t-1)}(n)$ from the relation (3) and taking into account that $p^{(n)}(n) = 1$, we obtain the identity

$$p^{(t)}(n) = 1 + \sum_{k=t}^{n-1} p^{(k)}(n-1-k).$$

When $k \geq n$, we have $p^{(k)}(n) = 1$. For $\lfloor \frac{n}{2} \rfloor \leq t \leq n$, we get

$$p^{(t)}(n) = n - t + 1$$

and then

$$\nabla[p^{(t)}](n) = 1.$$

By Theorem 1, we get the relations

$$\sum_{k=n+1}^{2n} \left(a^{(n)}(k) - a^{(k)}(k) \right) p(2n-k) = 1, \;\; n > 0$$

that can be rewritten in the following way

$$L_n \cdot \begin{bmatrix} p(0) \\ p(1) \\ \vdots \\ p(n) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

where $L_n = [l_{i,j}]_{1 \leq i,j \leq n+1}$ is a square matrix with entries

$$l_{i,j} = a^{(i)}(2i+1-j) - a^{(2i+1-j)}(2i+1-j).$$

We have

$$\begin{aligned} a^{(t)}(t+n) - a^{(t+n)}(t+n) &= \sum_{k=0}^{n-1} \left(a^{(t+k)}(t+n) - a^{(t+k+1)}(t+n) \right) \\ &= \sum_{k=0}^{n-1} a^{(t+k)}(n-1-k) \qquad \text{(by relation (4))} \\ &= \sum_{k=0}^{n-1} a^{(t+n-1-k)}(k). \end{aligned}$$

Then we get

$$l_{i,j} = \sum_{k=0}^{i-j} a^{(2i-j-k)}(k)$$

and

$$l_{i+1,j+1} = \sum_{k=0}^{i-j} a^{(2i+1-j-k)}(k).$$

For $k \leq i-j$, we have $2i+1-j-k < k$. By (4), we get

$$a^{(2i+1-j-k)}(k) = a^{(2i-j-k)}(k).$$

Thus, we deduce that $l_{i,j} = l_{i+1,j+1}$, i.e., L_n is a Toeplitz matrix. For $i < j$, we have $2i + 1 - j < i$. So, we get $l_{i,j} = 0$. On the other hand, for $k < i$, we have $k < 2i - 1 - k$. Thus, we obtain

$$l_{i,1} = \sum_{k=0}^{i-1} a^{(k)}(k)$$

or

$$L_n = \begin{bmatrix} s_{0,0} & & & \\ s_{0,1} & s_{0,0} & & \\ \vdots & \ddots & \ddots & \\ s_{0,n} & \dots & s_{0,1} & s_{0,0} \end{bmatrix}.$$

The lemma is proved. $\square$

We are to prove the Theorem 2 by induction on t. For $t = 0$ we obtain Lemma 1. The base case of induction is finished. We suppose that the relation

$$\sum_{k=0}^{n} s_{t',k} p(n-k) = \binom{n+t'}{t'}$$

is true for any non-negative integers t', $t' < t$. We can write

$$\begin{aligned} \sum_{k=0}^{n} s_{t,k} p(n-k) &= \sum_{k=0}^{n} \sum_{i=0}^{k} s_{t-1,i} p(n-k) \\ &= \sum_{k=0}^{n} \sum_{i=0}^{n-k} s_{t-1,i} p(n-i) \\ &= \sum_{k=0}^{n} \binom{n-k+t-1}{t-1}. \end{aligned}$$

Taking into account the relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

Theorem 2 is proved.

3 Formulas involving Euler's partition function

The relation proved in Theorem 1 can be rewritten in the following way

$$\begin{bmatrix} a^{(t)}(0) & & & \\ a^{(t)}(1) & a^{(t)}(0) & & \\ \vdots & \ddots & \ddots & \\ a^{(t)}(n) & \dots & a^{(t)}(1) & a^{(t)}(0) \end{bmatrix} \cdot \begin{bmatrix} p(0) \\ p(1) \\ \vdots \\ p(n) \end{bmatrix} = \begin{bmatrix} \nabla[p^{(t)}](0) \\ \nabla[p^{(t)}](1) \\ \vdots \\ \nabla[p^{(t)}](n) \end{bmatrix}.$$

We then immediately have

Corollary 3. *Let n and t be two positive integers. Then*

$$p(n) = \begin{vmatrix} 1 & & & & \nabla[p^{(t)}](0) \\ a^{(t)}(1) & 1 & & & \nabla[p^{(t)}](1) \\ a^{(t)}(2) & a^{(t)}(1) & 1 & & \nabla[p^{(t)}](2) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a^{(t)}(n) & \dots & a^{(t)}(2) & a^{(t)}(1) & \nabla[p^{(t)}](n) \end{vmatrix}.$$

For $0 \le k \le n \le t$, we have $\nabla[p^{(t)}](k) = \delta_{0,k}$. Taking into account Corollaries 2 and 3, we obtain that

$$\begin{array}{l} 1 \\ 2 \\ \\ \\ 5 \\ \\ 7 \\ \\ \vdots \end{array} \qquad p(n) = (-1)^n \begin{vmatrix} -1 & 1 & & & & & & & \\ -1 & -1 & 1 & & & & & & \\ 0 & -1 & -1 & 1 & & & & & \\ 0 & 0 & -1 & -1 & 1 & & & & \\ 1 & 0 & 0 & -1 & -1 & 1 & & & \\ 0 & 1 & 0 & 0 & -1 & -1 & 1 & & \\ 1 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & \\ \vdots & & & & & & & & \ddots \end{vmatrix}_{(n\times n)}.$$

This formula can be easily derived by (2). We can see that $p(n)$ is the determinant of the $n \times n$ truncation of the infinite-dimensional Toeplitz matrix. The only non-zero diagonals of this matrix are those which start on a row labeled by a generalized pentagonal number. The superdiagonal is taken to start on row 0. On these diagonals, the matrix element is $(-1)^k$.

The relation proved in Theorem 2 can be rewritten in the following way

$$L_n^{(t)} \cdot \begin{bmatrix} p(0) \\ p(1) \\ \vdots \\ p(n) \end{bmatrix} = \begin{bmatrix} 1 \\ \binom{1+t}{t} \\ \vdots \\ \binom{n+t}{t} \end{bmatrix},$$

where

$$L_n^{(t)} = [s_{t,i-j}]_{1\le i,j\le n+1}$$

is a triangular Toeplitz matrix with

$$\det L^{(t)}(n) = 1.$$

We then immediately have

Corollary 4. *Let n and t be two non-negative integers. Then*

$$p(n) = \begin{vmatrix} 1 & & & & 1 \\ s_{t,1} & 1 & & & \binom{1+t}{t} \\ s_{t,2} & s_{t,1} & 1 & & \binom{2+t}{t} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{t,n} & \dots & s_{t,2} & s_{t,1} & \binom{n+t}{t} \end{vmatrix}.$$

For the higher-order differences of the partition function, we have the following result.

Theorem 3. *Let n, t and u be three non-negative integers such that $t \ge u$. Then*

$$\sum_{k=0}^{n} s_{t,k} \nabla^u[p](n-k) = \binom{n+t-u}{t-u},$$

where $\nabla^u[f]$ is u-th order backward differences of the function f.

Proof. To prove the theorem we use induction on u and the relation

$$\nabla^u[p](n-k) = \nabla^{u-1}[p](n-k) - \nabla^{u-1}[p](n-1-k).$$

For the case $u = 0$ we consider Theorem 2. □

The next corollary follows easily by this theorem.

Corollary 5. *Let n, t and u be three non-negative integers such that $t \ge u$. Then*

$$\nabla^u[p](n) = \begin{vmatrix} 1 & & & & 1 \\ s_{t,1} & 1 & & & \binom{1+t-u}{t-u} \\ s_{t,2} & s_{t,1} & 1 & & \binom{2+t-u}{t-u} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{t,n} & \dots & s_{t,2} & s_{t,1} & \binom{n+t-u}{t-u} \end{vmatrix}.$$

The case $t = u$ of this corollary can be written as follows.

Corollary 6. *Let n and t be two non-negative integers, $n > 0$. Then*

$$\nabla^{t+1}[p](n) = (-1)^n \begin{vmatrix} s_{t,1} & 1 & & \\ s_{t,2} & s_{t,1} & 1 & \\ \vdots & \ddots & \ddots & 1 \\ s_{t,n} & \dots & s_{t,2} & s_{t,1} \end{vmatrix}.$$

We define the higher-order partial sums of the partition function by

$$S^{(u)}(n) = \sum_{k=0}^{n} S^{(u-1)}(k),$$

with $S^{(0)}(n) = p(n)$. It is clear that $S^{(1)}(n) = S(n)$. We remark that $S^{(u)}(n)$ counts the partitions of n into parts where the part 1 comes in $u+1$ colours. We have the following result.

Theorem 4. *Let n, t and u be three non-negative integers. Then*

$$\sum_{k=0}^{n} s_{t,k} S^{(u)}(n-k) = \binom{n+t+u}{t+u}.$$

Proof. To prove the theorem we use induction on u. For the case $u = 0$ we consider Theorem 2. □

Corollary 7. *Let n, t and u be three non-negative integers. Then*

$$S^{(u)}(n) = \begin{vmatrix} 1 & & & & 1 \\ s_{t,1} & 1 & & & \binom{1+t+u}{t+u} \\ s_{t,2} & s_{t,1} & 1 & & \binom{2+t+u}{t+u} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{t,n} & \dots & s_{t,2} & s_{t,1} & \binom{n+t+u}{t+u} \end{vmatrix}.$$

Corollary 8. *Let n and u be two non-negative integers. Then*

$$S^{(u)}(n) = \begin{vmatrix} \binom{n+0+u}{0+u} & s_{0,1} & \dots & s_{0,n} \\ \binom{n+1+u}{1+u} & s_{1,1} & \dots & s_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+n+u}{n+u} & s_{n,1} & \dots & s_{n,n} \end{vmatrix}.$$

Proof. By Theorem 4 we get

$$A \cdot \begin{bmatrix} S^{(u)}(n) \\ S^{(u)}(n-1) \\ \vdots \\ S^{(u)}(0) \end{bmatrix} = \begin{bmatrix} \binom{n+0+u}{0+u} \\ \binom{n+1+u}{1+u} \\ \vdots \\ \binom{n+n+u}{n+u} \end{bmatrix},$$

where

$$A = \begin{bmatrix} s_{0,0} & s_{0,1} & \cdots & s_{0,n} \\ s_{1,0} & s_{1,1} & \cdots & s_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n,0} & s_{n,1} & \cdots & s_{n,n} \end{bmatrix}.$$

Taking into account Theorem 2 we perform the following transformations on the matrix A:

$$\text{Step 1.} \quad s^{(1)}_{i,j} = \begin{cases} s_{i,j}, & \text{if } i = 0, \\ s_{i,j} - s_{0,j}, & \text{otherwise} \end{cases}$$

$$\text{Step 2.} \quad s^{(2)}_{i,j} = \begin{cases} s^{(1)}_{i,j}, & \text{if } i = 1, \\ s^{(1)}_{i,j} - 2s^{(1)}_{1,j}, & \text{otherwise} \end{cases}$$

$$\vdots$$

$$\text{Step } n. \quad s^{(n)}_{i,j} = \begin{cases} s^{(n-1)}_{i,j}, & \text{if } i = n-1, \\ s^{(n-1)}_{i,j} - ns^{(n-1)}_{n-1,j}, & \text{otherwise.} \end{cases}$$

Thus, we obtain an upper triangular matrix with $s_{0,0}$ entries on the main diagonal. We deduce that $\det A = 1$. The proof is finished. □

For instance,

$$p(4) = S^{(0)}(4) = \begin{vmatrix} 1 & 0 & -1 & -1 & -1 \\ 5 & 1 & 0 & -1 & -2 \\ 15 & 2 & 2 & 1 & -1 \\ 35 & 3 & 5 & 6 & 5 \\ 70 & 4 & 9 & 15 & 20 \end{vmatrix}.$$

4 An infinite family of inequalities

To show the efficiency of the algorithm presented in [3] we had to prove the following inequality: for $n > 0$

$$p(n) - p(n-1) - p(n-2) + p(n-5) \leq 0.$$

In [2], this inequality is the second entry of an infinite family of inequalities for the partition function $p(n)$. The following inequality

$$p(n) - p(n-1) - p(n-2) + p(n-3) \geq 0$$

is also the second entry of the infinite family of inequalities given by the following corollary.

Corollary 9. *Let n and t be two positive integers. Then*

$$\sum_{k=0}^{n} a^{(t)}(k) p(n-k) \geq 0,$$

with strict inequality if and only if $t < n$.

Proof. The inequality

$$\nabla[p^{(t)}](n) \geq 0$$

is trivial. For $t \geq n$, we have $p^{(t)}(n) = 1$ and then we obtain

$$\nabla[p^{(t)}](n) = 0.$$

According to Theorem 1, it is sufficient to prove the strict inequality by induction on t. For $t = n-1$, we obtain

$$\nabla[p^{(n-1)}](n) = 2 - 1 > 0.$$

The base case of induction is finished. We suppose that the relation

$$\nabla[p^{(t')}](n)) > 0$$

is true for any positive integer t', $t < t'$. By relation (3), we can write

$$\nabla[p^{(t)}](n) = \nabla[p^{(t+1)}](n) + \nabla[p^{(t)}](n-t-1).$$

Taking into account that

$$\nabla[p^{(t+1)}](n) > 0,$$

we obtain

$$\nabla[p^{(t)}](n) > 0$$

and the corollary is proved. □

Finally, we remark few specializations of Corollary 9:

a) $p(n) - p(n-1) - p(n-2) + p(n-4) + p(n-5) - p(n-6) \geq 0$;

b) $p(n) - p(n-1) - p(n-2) + 2p(n-5)$

$$-p(n-8) - p(n-9) + p(n-10) \geq 0;$$

c) $p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-6) + p(n-7) - p(n-8)$

$$-p(n-9) - p(n-10) + p(n-13) + p(n-14) - p(n-15) \geq 0.$$

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