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FROM STABILITY TO CHAOS
TO BE OR NOT TO BE UNSTABLE

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Rezumat. *"Eşti inginer dacă nu ai deloc viaţă şi poţi să dovedeşti matematic acest fapt. Eşti instabil când cazi şi poţi dovedi matematic aceasta!" Articolul de faţă reprezintă o încercare originală de a demonstra ca un robot poate fi modificat astfel încât sa devină instabil. Acest experiment mi-a venit în minte când am aflat despre studiul stabilităţii în cursurile mele de Matematici avansate. Fiind un student curios, am încercat să modific un robot astfel încât să îi pot studia stabilitatea. Ca o remarcă: în acest articol termenul instabil trebuie tradus prin cade.*

Abstract. *"You are an engineer if you have no life and you can prove it mathematically! You are unstable if you fall and you can prove it mathematically!" The present paper is an original attempt of proving that a robot can be modified such that it can become unstable. This experiment purely came into my mind since I heard about the study of Stability in my Advanced Mathematics lectures. Being an inquisitive student I tried to modify a robot in order to study its Stability. As a remark, during this paper the term unstable should be interpreted as it falls.*

Keywords: stability, Lyapunov, directed graphs, nondirected graphs, eigenvalue

1. Introduction

According to different encyclopedias, stability represents the property of a body that causes it to return to its original position or motion as a result of the action of the so-called restoring forces, or torques, once the body has been disturbed from a condition of equilibrium or steady motion.

There are a lot of kinds of stability around us, either in nature, or in science. I want to write down few of them:

- Aircraft flight stability
- Atmospheric stability
- BIBO stability (Bounded Input, Bounded output) – in signal processing and control theory, part of electrical engineering
- Directional stability
- Numerical stability, a property of numerical algorithms which describes how errors in the input data propagate into the algorithm
- Stability theory, the study of the stability of solutions to differential equations and dynamical systems

… and many other kinds of stability.

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What does being unstable mean? Having a moving object following a precise trajectory, if we disturb it a little, then there are two kinds of “outputs”: either it follows the same path, or its path is totally different from the predicted one. For example, let us follow the next experiment: take a ball and place it on a couple roof just on the edge. When it starts rolling, with a very small perturbation (let us call it ε) its trajectory will not be the same (straight line on the edge), but it will fall (it will become unstable) either on the right side, or the left side of the roof.

Having now in mind what the stability is and what does mean to be unstable, I will state my problem: I have a KSR1 – robot car, designed by “Velleman Components”; it is a voice-controlled robot car that uses a microphone as a detector; the car changes directions when the sensor detects noise or when the car hits an object. Theoretically, the robot car is stable. It has three wheels: two of them are placed on the rear part of it and they are fixed on the same axle-tree. The third wheel, placed on the front of the car robot is fastened using a spring, a wheel bracket and a nylon nut, everything being fixed on a screw. The front wheel is such way mobile. The car robot starts and when it “hears” a noise, the direction of the motor DC3V from clockwise becomes counterclockwise and the robot moves backward. Because of the spring, there appears a friction force acting on the front wheel, so it will develop an angular movement. After a defined period of time of two seconds, the car robot’s motor is changing again its direction of rotation and it goes forward. During this period of time the car robot is “depth”.

Initially, the board of the robot had a screw placed on the left side of the front wheel and the angle made by the bracket was small enough, such that when it goes forward, it goes back on the straight line trajectory. Placing the machine in these conditions it is stable, nothing unpredictable happened.

What did I do in order to try its instability was to remove that screw and to add a cut washer on the screw sustaining the front wheel.

I started the engine, and I made a noise. The microphone heard the noise, the car robot reversed its direction, the front wheeled rotated with an angle which initially was not enough in order to produce the fall. But I insisted and finally the KSR1 became unstable.

Let us see now how can we prove it mathematically.

2. Experimental details – methods

2.1. THE PLANES' METHOD

My research had few steps, each step totally different from the previous one. The first one was to write a system of three equations, which represent the planes' equations through which the car robot pass when it is moving and falling.

The system is as follows:

$$\begin{cases} 0 = y - 1 \\ 0 = y + z - 1 \\ 0 = x + y + z - 1 \end{cases}$$

Unfortunately it has no big meaning, because the system needed to have three differential equations as components. But for the moment let us think there are the three desired differential equations of x, y and z with respect to the time. From this point I tried two different methods in order to study the stability of the system: the Lyapunov functions method, and the eigenvalues method.

2.2. LYAPUNOV FUNCTIONS' METHOD

Lyapunov says that: take a function v, positive definite, as you feel, being the energy of the system. Then differentiate it with respect to the time. If what did you obtain is positive, then the system is unstable.

I tried the following Lyapunov functions (as I felt):

$$v = x^2 + y^2 + z^2$$

I differentiate the function with respect to time and I obtained:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} + \frac{dv}{dy}\frac{dy}{dt} + \frac{dv}{dz}\frac{dz}{dt} = (x + y + z)(y + z - 1)$$

But for the above equation we cannot establish if it is positive or not, because of the lack of conditions. I have used Maple 10, Mathematics Specialized Software and it throws this kind of error.

I tried a second Lyapunov function (also as I felt):

$$v = x^3 + y^3 + z^3$$

The result was:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} + \frac{dv}{dy}\frac{dy}{dt} + \frac{dv}{dz}\frac{dz}{dt} = (y + z)^3 - 3\left(x^2 + y^2 + z^2\right) + 3x\left(xy + z^2\right)$$

As you can see, we have the same problem in defining if the equations is or it is not positive.

2.3. THE EIGENVALUES' METHOD

The second method I tried is the eigenvalues[1] one. The prerequisite of this method is to have a matrix (to find its eigenvalues).

[1] Eigenvalues are obtained by solving the equation P(λ)=0.

If at least one eigenvalue of your matrix is positive, then the system can be said to be unstable.

I took our system of equations and I built a matrix using it.

$$A := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

having its characteristic polynomial:

$$P(\lambda) = \lambda^3 - 2\lambda^2 - 1$$

I have used the LinearAlgebra Packet from Maple 10 software, in order to find the eigenvalues of matrix A. They are as follows:

$$\lambda_1 = \frac{1}{6}\left(116+12\sqrt{93}\right)^{\frac{1}{3}} + \frac{2}{3}\frac{1}{\left(116+12\sqrt{93}\right)^{\frac{1}{3}}} + \frac{1}{3}$$

$$\lambda_2 = -\frac{1}{12}\left(116+12\sqrt{93}\right)^{\frac{1}{3}} - \frac{1}{3}\frac{1}{\left(116+12\sqrt{93}\right)^{\frac{1}{3}}} + \frac{1}{3} + \frac{1}{2} I\sqrt{3}\left(\frac{1}{6}\left(116+12\sqrt{93}\right)^{\frac{1}{3}} - \frac{2}{3}\frac{1}{\left(116+12\sqrt{93}\right)^{\frac{1}{3}}}\right)$$

$$\lambda_3 = -\frac{1}{12}\left(116+12\sqrt{93}\right)^{\frac{1}{3}} - \frac{1}{3}\frac{1}{\left(116+12\sqrt{93}\right)^{\frac{1}{3}}} + \frac{1}{3} - \frac{1}{2} I\sqrt{3}\left(\frac{1}{6}\left(116+12\sqrt{93}\right)^{\frac{1}{3}} - \frac{2}{3}\frac{1}{\left(116+12\sqrt{93}\right)^{\frac{1}{3}}}\right)$$

As you can easily see, the first eigenvalue is positive, so the system is UNSTABLE.

2.4. THE NON-DIRECTED GRAPHS' METHOD

Another idea I have tried was to obtain a new matrix, more meaningful. For this fact I built a nondirected graph based on the graphic which represents the direct dependence of the velocity v, with respect to the angle of the front wheel, denoted by φ, which could increase from 0 to at most 3π/2 when the car robot is moving backward.

Figure 1 shows the nondirected graph associated to the following events:

1. the car robot is moving forward with positive velocity;

2. it is disturbed by a noise, so it goes backward, with negative velocity;

3. while it is moving backward, the front wheel starts to develop an angle φ; in 2 seconds the angle has π/2 radians and the direction of rotation of the car robot motor is reversing;

4. the car robot is moving forward, with positive velocity; from now it can go to the first step as nothing happened, or it could become unstable;

5. the last step represents the instability of the robot, when φ is too large to be regain by the front wheel.

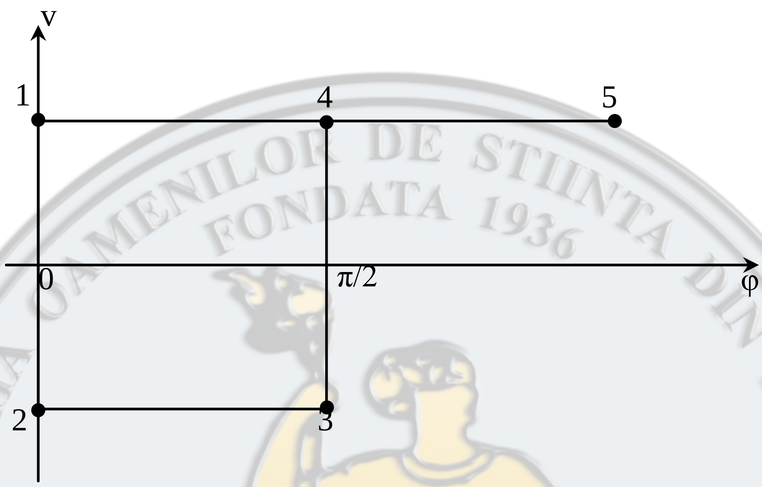


Figure 1. The nondirected graph

The adjacent matrix[1] associated to the nondirected graph described above is:

$$B := \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

I have used the eigenvalues method to study its stability and I found that:

$$Eigenvalues(B) := \begin{bmatrix} 0 \\ -\frac{1}{2}\sqrt{10+2\sqrt{17}} \\ \frac{1}{2}\sqrt{10+2\sqrt{17}} \\ -\frac{1}{2}\sqrt{10-2\sqrt{17}} \\ \frac{1}{2}\sqrt{10-2\sqrt{17}} \end{bmatrix}$$

B has at least one positive eigenvalue, hence the system in UNSTABLE.

[1]The adjacent matrix associated to a nondirected graph has 0 on the main diagonal, and it is symmetrical with respect to the first diagonal

2.5. THE DIRECTED GRAPHS' METHOD

The last approach of my problem is the usage of directed graphs to say if the system is stable or not.

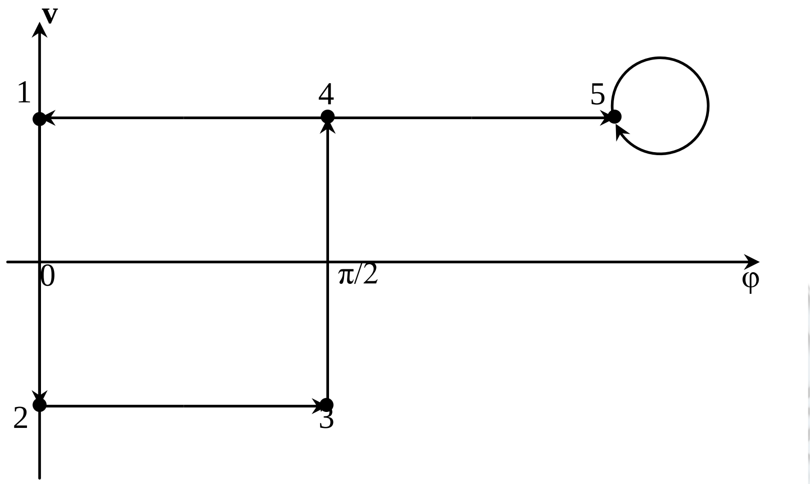


Figure 2. The directed graph

The directed graph from figure 2, represents exactly the same steps as the nondirected one. However, you can easily see a difference in step 5: if the car robot is passing to step 5, it cannot come back, but it will remain felt down.

The adjacent matrix associated to this directed graph is denoted by C and it has the following values:

$$C := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with the following eigenvalues:

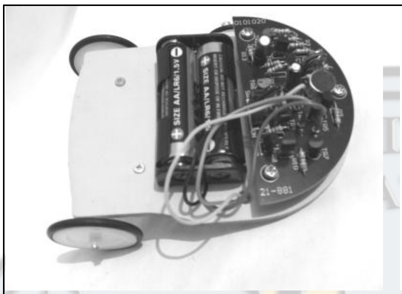
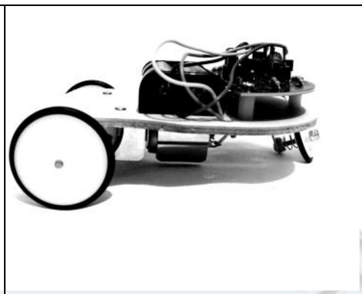
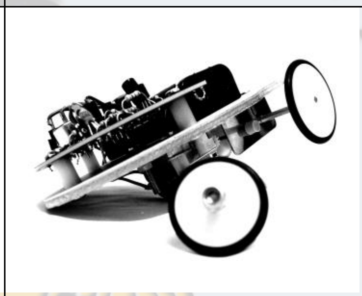
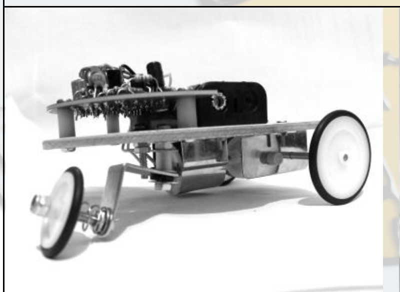
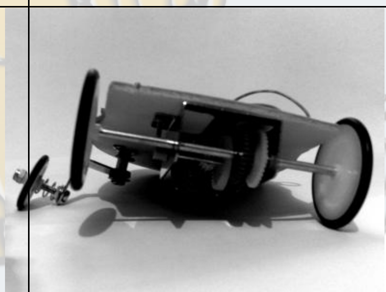
$$Eigenvalues(C) := \begin{bmatrix} -1 \\ I \\ -I \\ 1 \\ 1 \end{bmatrix}$$

Hence, the system is again UNSTABLE.

3. Results

I stated a problem about the forced instability of an electric car-robot, and I tried to prove this from Mathematical point of view.

There will be few pictures with the car robot while going it down.

	
Top view	Lateral view
	
Front view	The car robot felt down
	
Lateral front view	Back view

Conclusions

All the methods use were successfully applied, except first one because of the lack of a rigorous meaning of the system described.

To sum up, modifying a car robot, for your personal enjoyment, you may obtain an unstable one.

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