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ON THE INTEGRALS OF UNBOUNDED FUNCTIONS WITH APPLICATIONS

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Rezumat. *Subiectul in aceasta lucrare este orientat pe studiul posibilei evaluări a integralelor funcţiilor reale nemărginite cu variabile reale. Autorul introduce noţiunea de integrală Riemann generalizată a funcţiilor nemărginite, unde diviziunile finite ale domeniului de definiţie pentru integrala Riemann sunt substituite de diviziuni numărabile şi unde sumele finite integrale pentru integrala Riemann sunt substituite de serii integrale. Interferenta dintre noţiunile de integrală Riemann generalizată a funcţiilor nemărginite şi integrala Lebesgue a funcţiilor nemărginite, analizată în aceasta lucrare, este benefică pentru ambele noţiuni.*

Abstract. *The subject in this paper is focused on the study of a possible evaluation of real unbounded function integrals with real variable. The author introduces the notion of the generalized Riemann integral of unbounded functions where the finite divisions of definition domain for Riemann integral are substituted by the numerable divisions and where the integral finite sums for Riemann integral are substituted by the integral series. The interference between the notions of unbounded functions generalized Riemann integral and of unbounded functions Lebesgue integral, analyzed in this paper, is beneficial for both notions.*

Keywords: Unbounded functions, Integrable functions, Integral series, Mechanical modelling

1. Introduction notions

In this paper we are limited to a function of a single variable $f(x)$, defined on the interval $[a,b]$, with the values in the real numbers set R.

Definition 1

A point $x_0 \in [a,b]$ is a singular (unbounded) point of the function $f \geq 0$ if there exist a sequence $\{x_n\}_{n\in N} \subset [a,b]$, $x_n \to x_0$, such that $f(x_n) \to \infty$.

Definition 2

Let us consider a set $A \subset R$ and $x_0 \in R$. The point x_0 is named an accumulation point of the set A if there exist a sequence $\{x_n\}_{n\in N} \subset A$, $x_n \to x_0$.

Definition 3

The set $A' \subset R$ of the accumulation points of the set $A \subset R$ is named the derivative set of the set A or that $A' \subset R$ is a derivate of the set $A \subset R$.

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Definition 4

A set $A \subset R$ is named reducible set if from them is deduced a empty set after a finite number of derived operations of the set A.

Definition 5

The union between the set A and the set of accumulation points of the set A is named closure of the set A and is denoted by $\bar{A}$.

Definition 6

The set A is named closed if $A = \bar{A}$ or, with other words, if the set A contain their accumulation points.

Definition 7

The set $A \subset R$ is named perfect set if coincide with their derivate, thus $A = A'$

Observation: The empty set is considered a particular case of perfect set.

We formulate, without proof [1], the following:

Theorem 1

Any closed set $A \subset R$ is a union between a reducible set and a perfect set (in particular empty set).

The procedure for generation of the Cantor's set by the following infinite process is reminded:

The interval $[a,b]$ is divided in three equal intervals and the middle interval without extremities is eliminated. The two remains intervals are submitted at the same procedure and so on. The set deduced after this infinite process is named the Cantor's set. This set, included in the interval $[a,b]$, is a perfect set and is not numerable.

I suggest the set of the above type as possible division in definition of integrable unbounded functions.

2. Infinite divisions, associate series and integrable functions

Definition 8

A division of the interval $[a,b]$ is a reducible set $d \subset [a,b]$ that include the extremities of the interval as accumulations points of the division points set [2]. The norm of the division is a maximum length of the subintervals defined by the division and is denoted by $\nu(d)$.

An example of a function unbounded on the interval $[0,1]$, a step function denoted by f_M, for which we can associate an area between their diagram and abscise axis [2], is defined below.

The function f_M has the constant value $\sqrt{2}$ on the interval $[0,1/2)$ denoted by $[x_0^1, x_1^1)$, has the value $(\sqrt{2})^2$ on the interval $[1/2,(1/2+(1/2)^2))$ denoted by $[x_1^1, x_2^1)$, …, has the value $(\sqrt{2})^{n+1}$ on the interval $[\sum_{i=1}^{n} 1/2^i, \sum_{i=1}^{n+1} 1/2^i)$ denoted by $[x_n^1, x_{n+1}^1)$ etc. Function f_M is a step function, unbounded on $[0,1]$ and thus a function non integrable Riemann on the interval $[0,1]$. An infinite division of the interval $[0,1]$ may be the set compound from the values $x_0^1, x_1^1, x_2^1, \ldots, x_n^1, x_{n+1}^1, \ldots$ with a right extremity of the interval $[0,1]$ as accumulation point of the division.

The integral sum associate of this division, for $x_i^1 \le \xi_i < x_{i+1}^1$, is of the form:

$$\sum_{i=0}^{\infty} f_M(\xi_i)(x_{i+1}^1 - x_i^1) = \sqrt{2}/2 + (\sqrt{2})^2/2^2 + \ldots + (\sqrt{2})^{n+1}/2^{n+1} + \ldots = \sqrt{2}+1 \qquad (1)$$

We consider a positive unbound function f with a finite number of singularities on the definition interval $[a,b]$, to simplify the talk. We will define an infinite generalization Riemann divisions of the interval that contains the singularities of the function as accumulation points of the division.

Let $f : [a,b] \to R$ be a positive unbound function with a finite number of singularities. The values of function $f(x)$ in the points of singularities are supposed of null values that is not a restriction in our study.

Definition 9

A division associate to function f is a reducible set $d \subset [a,b]$ that include the singularities points of the function as accumulation points of the division points set and also include the extremities of the interval $[a,b]$ as accumulations points of the division, if they are not singularities points of the function (for the symmetry of the notations). We consider thus the accumulation points of the division as the set of points of the form $\{x^0, x^1, x^2, \ldots, x^{p-1}, x^p\}$, where x^0, x^p are extremities of the definition interval. The norm of the division is a superior length of the subintervals defined by the division and is denoted by $\nu(d)$.

Definition 10

The associate generalization Riemann integral series $\sigma_R(f,d)$ of function f and division $d \subset [a, b]$ with the accumulation points set $\{x^0, x^1, x^2, \ldots$ $\ldots, x^{p-1}, x^p\} \subset d$ for $x_i^j \le \xi_i^j < x_{i+1}^j$, $1 \le j \le p$, $i \in Z$ and $[x_i^j, x_{i+1}^j] \subset (x^{j-1}, x^j)$, is as follows.

$$\sigma_R(f,d) = \sum_{-\infty}^{\infty} \sum_{j=1}^{p} f(\xi_i^j)(x_{i+1}^j - x_i^j) \qquad (2)$$

The order of the subintervals between two successive accumulation points is defined by the index i and position of a subinterval of the definition interval $[a,b]$ between two successive accumulation points of the division points set is identified by index j. The values ξ_i^j are named the intermediary values of the division d.

Lemma 1

Let $f : [a,b] \rightarrow R$ be a positive function with a singularity point $x_0 \in [a,b]$. Then there exist a sequence $\{d_n\}_{n \in N}$ of divisions associate to function f with $\nu(d_n) \rightarrow 0$ and a selection of the intermediary values, such that the integral series attached, to have infinite limit.

Proof:

The function f is supposed unbounded at the left of point x_0 such that there exist an increasing sequence $x_n \rightarrow x_0$ with $f(x_n) \rightarrow \infty$ strict increasing. The sequence $\{x_n\}_{n \in N}$ is used for defining the sequence $\{d_n\}_{n \in N}$ from the text of the lemma. The property specified is a consequence of a possibility to select a division with the norm as small as we want and the integral series attached, to be as much as we want.

Let α be the value of the norm of a fixed division d. We modify the division, in the neighbourhood of the point x_0, with conservation of the norm division or with a decreasing of them such that the value of the integral series attached to modified division to be as much as we want. Because $x_n \rightarrow x_0$ and $f(x_n) \rightarrow \infty$, for fixed $x_p^d \in d$ with $x_0 - x_p^d \le \alpha$, it follows that $x_n - x_p^d \rightarrow \; \rightarrow x_0 - x_p^d \ne 0$ with $x_n - x_p^d < \alpha$ and $f(x_n)(x_n - x_p^d) \rightarrow \infty$.

We can search x_n with $x_p^d < x_n < x_0$ such that $f(x_n)(x_n - x_p^d)$ in the modified division to be as much as is desired.

The possibility described by the above lemma must be eliminated in the definition of the integral of an unbound function f.

Lemma 2

Let $f : [a,b] \rightarrow R$ be a positive function with a singularity point $x_0 \in [a,b]$. Then there exist a division d associate to function f and a selection of the intermediary values, such that the integral series attached, to be divergent.

Proof:

The function f is supposed unbounded at the left of point x_0 such that there exist an increasing sequence $x_n \to x_0$ with $f(x_n) \to \infty$ strict increasing. The sequence $\{x_n\}_{n\in N}$ is used for defining the division d from the text of the lemma. We modify the arbitrary initial division d^*, where $x_0 \in d^*$, in the neighbourhood of the point x_0, at the left of x_0, such that the value of terms of the integral series attached to modified division to be a divergent sequence. Because $x_n \to x_0$ and $f(x_n) \to \infty$, for fixed $x_p^d \in d^*$ with $0 < x_0 - x_p^d \le b-a$, it follows that sequence $\{x_n - x_p^d\}_{n\in N}$ is increasing, $x_n - x_p^d \to x_0 - x_p^d$ with $x_n - x_p^d < b-a$ such that the sequence is bounded and is not convergent to zero but $f(x_n)(x_n - x_p^d) \to \infty$. We can search x_n with $x_p^d < x_n < x_0$ such that $f(x_n)(x_n - x_p^d)$ in the modified division to be as much as is desired. We repeat the procedure for the next term of the division d^* modified such that the value of this term of the integral series to be as much as is desired in comparison with previous term of the integral series and so on. After this procedure we deduce the division d specified in the lemma.

Theorem 2

A positive function $f:[a,b] \to R$ is bounded if for every division with a finite number of accumulation points and for every selection of intermediary points the integral series attached is convergent.

Proof:

Let $f:[a,b] \to R$ be a positive function. We suppose that for every selection of intermediary points the integral series attached is convergent. If $f:[a,b] \to R$ is unbounded then from lemma 2 there exist a division d associate to function f and a selection of the intermediary values, such that the integral series attached to have infinite limit. This is a contradiction as a consequence of supposition that function f is unbounded. It permits us to say that function f is bounded.

Definition 11

The norm attached to integral series $\sigma_R(f,d)$ of function f and division $d \subset [a,b]$ is defined as superior of the terms $f(\xi_i^j)(x_{i+1}^j - x_i^j)$, referred to indexes i, j and to arbitrary intermediary values ξ_i^j. This norm is denoted by $\nu(\sigma_R(f,d))$.

Let $f:[a,b] \to R$ be a positive unbound function with a finite number of singularities and a division with the accumulation points set of the form $\{x^0, x^1, x^2, ..., x^{p-1}, x^p\}$, where x^0, x^p are extremities of the definition interval and $x^1, x^2, ..., x^{p-1}$ are the singularities points of the function f.

An arbitrary sequence $\{d_n\}_{n\in N}$ of divisions for the interval $[a,b]$ and attached sequence of integral series $\{\sigma_R(f,d_n)\}_{n\in N}$ with $\nu(\sigma_R(f,d_n)) \to 0$ are used for definition of the generalized Riemann integral of unbound function f.

Definition 12

A positive function $f:[a,b] \to R$ with singularities $x^1, ..., x^{p-1}$ is generalized Riemann integrable on $[a,b]$ if for every sequence of divisions $\{d_n\}_{n\in N}$ and attached sequence of integral series $\{\sigma_R(f,d_n)\}_{n\in N}$ with $\nu(\sigma_R(f,d_n)) \to 0$ the sequence $\{\sigma_R(f,d_n)\}_{n\in N}$ has a finite limit.

From definition 12 we can formulate the following property:

Theorem 3

A positive function $f:[a,b] \to R$ with singularities $x^1, ..., x^{p-1}$ is generalized Riemann integrable on $[a,b]$ if and only if for every sequence of divisions $\{d_n\}_{n\in N}$ and attached sequence of integral series $\{\sigma_R(f,d_n)\}_{n\in N}$ with $\nu(\sigma_R(f,d_n)) \to 0$ the sequence $\{\sigma_R(f,d_n)\}_{n\in N}$ has a same finite I limit.

Definition 13

A function $f:[a,b] \to R$ with a finite number of singularities, where $f = f^+ - f^-$ and $f^+ \geq 0$, $f^- \geq 0$, is generalized Riemann integrable on $[a,b]$ if functions f^+ and f^- are generalized Riemann integrable on $[a,b]$.

Theorem 4

Let function $f:[a,b] \to R$ be a generalized Riemann integrable on the interval $[a,b]$ then function f is Riemann integrable on any subinterval $[c,d] \subset [a,b]$ where the function is bounded.

Proof:

We suppose that the function f is positive and is not integrable Riemann on the subinterval $[c,d] \subset [a,b]$ where the function is bounded. There is a sequence $\left\{d_n^{'}\right\}_{n\in N}$ of associate divisions to function f on the subinterval $[c,d]$ with $\nu(d_n^{'}) \to 0$ and a selection of the intermediary values, such that the integral series attached, to have infinite limit. But this sequence of divisions and integral series on the interval $[c,d]$ can be completed up to a sequence of divisions and integral series on the interval $[a,b]$ such that the integral series attached, to have infinite limit on $[a,b]$. There is a contradiction with hypothesis that function $f:[a,b]\to R$ is generalized Riemann integrable on the interval $[a,b]$. It follows that function f is integrable Riemann on the subinterval $[c,d]\subset[a,b]$ where the function is bounded.

Theorem 5

Let function $f:[a,b]\to R$ be with a finite number of singularities, generalized Riemann integrable on $[a,b]$ then the discontinuity points set of the function f is of null Lebesgue measure.

Proof:

We consider a division d of the interval $[a,b]$, such that the interval $[a,b]$ is a union between the finite set of unbound points of the function f and a numerable set of subintervals of the interval $[a,b]$ on which the function f is Riemann integrable and thus a numerable set of subintervals of the interval $[a,b]$ on which the function f has the discontinuity points set of null Lebesgue measure. But a numerable union of sets of null Lebesgue measure is of null Lebesgue measure. From this the theorem is verified.

The following theorem [3] is formulated without proof.

Theorem 6

Let function $f:[a,b]\to R$ be with the discontinuity points set of null Lebesgue measure then function f is measurable Lebesgue.

Some considerations on the unbound function $f:[a,b]\to R$ with a finite number of singularities and generalized Riemann integrable on $[a,b]$ are described as follows. Let d be a generalized Riemann division associate to function f with accumulation points set $\{x^0,x^1,x^2,...,x^{p-1},x^p\}\subset d$. The function f is bounded on the each subinterval $[x_i^d,x_{i+1}^d]$ of the division d.

The following notations are used:

$$m_i=\inf\{f(x);\ x_i^d<x<x_{i+1}^d\},\ \ M_i=\sup\{f(x);\ x_i^d<x<x_{i+1}^d\} \qquad (3)$$

$$\varphi(d,x)=\begin{cases} m_i & \text{for } x\in(x_i^d,x_{i+1}^d) \text{ with } \{x_i^d,x_{i+1}^d\}\in d \\ 0 & \text{in the rest}\end{cases} \qquad (4)$$

$$\psi(d,x)=\begin{cases} M_i & \text{for } x\in(x_i^d, x_{i+1}^d) \text{ with } \{x_i^d, x_{i+1}^d\}\in d \\ 0 & \text{in the rest} \end{cases} \tag{5}$$

The function f is generalized Riemann integrable on $[a,b]$ and thus the discontinuity points set of the function f is of null Lebesgue measure such that the functions $\varphi(d,x)$ and $\psi(d,x)$ defined for $x\in[a,b]$, using the function f, have the discontinuity points set of null Lebesgue measure and are Lebesgue measurable on $[a,b]$.

The Darboux series $s(d,x)$ and $S(d,x)$ attached to function f and division d are integral series, supposed with finite values, attached to division d and functions $\varphi(d,x)$ and $\psi(d,x)$. The following inequalities are produced:

$$s(d,x)\le\sigma_R(f,d)\le S(d,x) \tag{6}$$

Without restriction of the generality, we suppose in the following that the sequence of divisions $\{d_n\}_{n\in N}$ has the property

$$d_1\subset d_2\subset\ldots\subset d_n\subset d_{n+1}\subset\ldots \tag{7}$$

Conditions from relation (7) involve the inequalities:

$$s(d_1,x)\le s(d_2,x)\le\ldots\le s(d_n,x)\le\ldots\le \underline{I}\le\overline{I}\le\ldots\le S(d_n,x)\le\ldots\le S(d_2,x)\le S(d_1,x) \tag{8}$$

The $\{s(d_n,x)\}_{n\in N}$ sequence, supposed with the finite values, increasing and bounded, converges to finite $\underline{I}$ value, but the $\{S(d_n,x)\}_{n\in N}$ sequence, decreasing and bounded, converges to finite $\overline{I}$ value.

The following criterion about existence of generalized Riemann integral is formulated.

Theorem 7

A positive function $f:[a,b]\to R$ with singularities $x^1,\ldots,x^{p-1}$ is generalized Riemann integrable on $[a,b]$ if and only if for every sequence of divisions $\{d_n\}_{n\in N}$ and attached sequence of finite integral series $\{s(d_n,x)\}_{n\in N}$, $\{S(d_n,x)\}_{n\in N}$ with $\nu(s(d_n,x))\to 0$, $\nu(S(d_n,x))\to 0$, sequence $\{S(d_n,x)-s(d_n,x)\}\to 0$.

Proof:

First suppose that function $f:[a,b]\to R$ is generalized Riemann integrable on $[a,b]$ and suppose a sequence of divisions $\{d_n\}_{n\in N}$ with the property (7) and with attached sequence of integral series $\{s(d_n,x)\}_{n\in N}$, $\{S(d_n,x)\}_{n\in N}$, where $\nu(s(d_n, x)) \to 0$, $\nu(S(d_n, x)) \to 0$.

Then the sequences $\{s(d_n, x)\}_{n\in N}$, $\{S(d_n, x)\}_{n\in N}$ have the same finite limit I because function f is generalized Riemann integrable on $[a,b]$. Thus the sequence specified in the text of the theorem, $\{S(d_n, x) - s(d_n, x)\}_{n\in N}$, converges to zero. Inverse, we suppose that for every sequence of divisions $\{d_n\}_{n\in N}$ with the property (7) and attached sequence of finite integral series $\{s(d_n, x)\}_{n\in N}$, $\{S(d_n, x)\}_{n\in N}$ with $\nu(s(d_n, x)) \to 0$, $\nu(S(d_n,\ x)) \to 0$, sequence $\{S(d_n, x) - s(d_n, x)\}_{n\in N}$ converges to zero. Conditions from relation (8) involve that sequence $\{s(d_n, x)\}_{n\in N}$ converges to $\underline{I}$ and the sequence $\{S(d_n, x)\}_{n\in N}$ converges to $\overline{I}$. The sequence $\{S(d_n, x) - s(d_n, x)\}_{n\in N}$ converges to zero such that $\underline{I} = \overline{I}$. From the relation (6) it follows that $\sigma_R(f, d_n) \to I = \underline{I} = \overline{I}$.

The following property, without proof, is formulated.

Theorem 8

Let $f: [a,b] \to R$ be a positive unbound function with a finite number of singularities and d a division of the interval $[a,b]$ on x axes. Let $\varphi(d, x)$ and $\psi(d, x)$ be step functions defined on the $[a,b]$ interval, using f function. There exist a division d^* on axes y associate to division d on axes x such that $\sigma_R(\varphi, d) = \sigma_L(\varphi, d^*)$, $\sigma_R(\psi, d) = \sigma_L(\psi, d^*)$, with σ_R, σ_L the generalized Riemann series respectively the Lebesgue series being denoted.

For the Lebesgue and generalized Riemann integral we formulate:

Theorem 9

Let $f:[a,b] \to R$ be a positive function with a finite number of singularities. The function f is Lebesgue integrable on $[a,b]$ if f is generalized Riemann integrable on $[a,b]$ and these integrals are identically.

Proof:

Function $f: [a,b] \to R$ is positive, unbound, with a finite number of singularities and d a division of the interval $[a,b]$ on x axes with the accumulation points set of the form $\{x^0, x^1, x^2, ..., x^{p-1}, x^p\}$, where x_0, x_p are extremities of the definition interval. The step functions $\varphi(d, x)$ and $\psi(d, x)$ are defined by the d division. The following inequalities, for $x \in (x_i^d, x_{i+1}^d)$ and $\{x_i^d, x_{i+1}^d\} \in d \subset [a,b]$, are used:

$$\varphi(d, x) \le f(x) \le \psi(d, x) \tag{9}$$

Let $\{d_n\}_{n\in N}$ be a sequence of divisions attached to function f, with the generalized Riemann series $\{\sigma_R(f,d_n)\}_{n\in N}$ and with norm $\nu(\sigma_R(f,d_n))\to$ $\to 0,\ n\to\infty$.

Analogue inequalities as above, for $x\in(x_i^{d_n},x_{i+1}^{d_n})$ and $\{x_i^{d_n},x_{i+1}^{d_n}\}\in\ \in d_n\subset[a,b]$ can be written.

$$\varphi(d_n,x)\le f(x)\le\psi(d_n,x) \tag{10}$$

The sequence $\{d_n\}_{n\in N}$ on the x axes and the associate sequence $\{d_n^*\}_{n\in N}$ on the y axes, using theorem 8, assure the relations:

$$\sigma_R(\varphi_n,d_n)=\sigma_L(\varphi_n,d_n^*)\le\ \sigma_R(f,d_n)\le\sigma_R(\psi_n,d_n)=\sigma_L(\psi_n,d_n^*)$$

$$\sigma_R(\varphi_n,d_n)=\sigma_L(\varphi_n,d_n^*)\le\ \sigma_L(f,d_n^*)\le\sigma_R(\psi_n,d_n)=\sigma_L(\psi_n,d_n^*) \tag{11}$$

$$\varphi_n=\varphi(d_n,x),\ \ \psi_n=\psi(d_n,x)$$

The hypothesis that there is a generalized Riemann integral of function f on $[a,b]$ involve that

$$\sigma_R(\varphi_n,d_n)\to I,\ \ \sigma_R(\psi_n,d_n)\to I,\ \sigma_R(f,d_n)\to I \tag{12}$$

The (11) and (12) relations permit us to conclude that $\sigma_L(f,d_n^*)\to I$ and thus function f is Lebesgue integrable and the two integrals, generalized Riemann and Lebesgue, are identically.

3. The possible structure of unbounded points of the function

Definition 14

Let X be a topological space and $Y\subset X$. The set Y is named dense in X if $\bar{Y}=X$ or, with other words, if any point of X is accumulation point of Y.

In the case $X=[a.b]$, where the interval $[a,b]$ is endowed with the restriction of the topology of set R defined with the open intervals, the set of rational numbers from $[a,b]$ is dense, for example, in topological space $[a,b]$.

Theorem 10

Let $f:[a,b]\to R$ be a function with the set U of unbounded points. If set U is dense in $[a,b]$ then the function f is unbounded in any point from $[a,b]$.

Proof:

We consider a point $x \in [a,b]$, $x \notin U$. Let $\{I_n\}_{n \in N}$ a sequence of open subintervals centred in x such that:

$$I_1 \supset I_2 \supset ... \supset I_n \supset I_{n+1} \supset ... \quad (13)$$

We suppose that length of I_n converges to zero. Because the set U is dense in $[a,b]$ there is a sequence $\{x_n\}_{n \in N} \subset U$ such that $x_n \to x$. We can extract a subsequence of $\{x_n\}_{n \in N}$ and subsequence of $\{I_n\}_{n \in N}$ denoted with the same names such that:

$$x_1 \in I_1,\ x_1 \notin I_2, ..., x_n \in I_n,\ x_n \notin I_{n+1}, ... \quad (14)$$

Because $x_1 \in U$ there exist a sequence $\{x_{1n}\}_{n \in N} \subset [a,b],\ x_{1n} \to x_1$ such that $f(x_{1n}) \to \infty$. We can choose the value $y_1 \in [a,b]$ such that $y_1 \in I_1,\ y_1 \notin I_2$ with $f(y_1) > 1$ and similarly the value $y_n \in [a,b]$ such that $y_n \in I_n,\ y_n \notin I_{n+1}$ with $f(y_n) > n$ and so on. The sequence $\{y_n\}_{n \in N} \subset [a,b]$ converges to x and $f(y_n) \to \infty$.

We perform the property that point x is unbounded point of the function f. For other details we recommend [3],[4].

4. Applications

The applications of the Riemann integral for bounded functions are numerous. The problem of existence of unbounded function integral is also put in the mechanical modelling [2]. We describe below an example in this sense.

Let $f(x):[a,b] \to R$ be a real function defined by.

$$f(x) = \begin{cases} \dfrac{1}{(b-x)^{\alpha}} & \text{for} \quad a \le x < b \\ 0 & \text{for} \quad x = b \end{cases} \quad (15)$$

We use in (15) the real constant α. The existence of integral of this function is studied. The function f is Riemann integrable on each subinterval $[a,x] \subset [a,b]$ and the values of the integral have the expression $1/(\alpha-1)[1/(b-x)^{\alpha-1} - \ -1/(b-a)^{\alpha-1}\ ]$. The existence of integral of unbound function f on the interval $[a,b]$ is studied, in this particular case, by analysing the limit of integral of function f on $[a,x]$ for $x \to b$. The function f is generalized Riemann integrable on $[a,b]$ if $\alpha < 1$ and the values of integral have the expression $1/(1-\alpha)/(b-a)^{\alpha-1}$.

This function has an infinite value of generalized Riemann integral on interval $[a,b]$ for $\alpha \geq 1$. The above application suggests us other possible cases, more complicated, in the mechanical modelling and not only, that involve the notion of generalised Riemann integral for unbounded functions.

Conclusions

The problem of integral existence of real unbounded functions with real variable is analysed, for the case of finite number of singularities.

The generalized Riemann integral is defined by using the divisions compounded by a numerable number of points and by using the integral series associated to division and to function analysed.

The study of existence of generalized Riemann integral of unbounded function and the study of possible structure of the singularities points of the unbounded function are correlated.

The interference between the notions of generalized Riemann integral and of Lebesgue integral of unbounded functions is beneficial for both notions.

A question arise of this study about how much an integrable function can be unbounded.

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