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Diagonal operators and coupled fixed points via weakly Picard operator technique*

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Abstract

In this note, using the weakly Picard operator technique we will present some qualitative results for operators generated by an operator defined on a Cartesian product of a metric space with itself. The work is based on a recent paper of the authors.

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keywords: diagonal operator, fixed point, coupled fixed point, weakly Picard operator, qualitative properties, research directions.

1 Introduction

Let X be a nonempty set and $T : X \times X \to X$ be an operator. By definition, the operator $U_T : X \to X$, defined by

$$U_T(x) := T(x, x),\ x \in X$$

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is called the diagonal operator generated by the operator T.

On the other hand, an operator $V : X \to X$ is said to be a diagonal operator if there exists $T : X \times X \to X$ such that $V = U_T$.

For the historical roots of the theory of diagonal operators see [7].

If X is a nonempty set and $T : X \times X \to X$ is a given operator, the problem to find $(x, y) \in X \times X$ satisfying

$$\begin{cases} x = T(x, y) \\ y = T(y, x) \end{cases} \tag{1}$$

is said to be a coupled fixed point problem for T.

Of course, a solution of the coupled fixed point problem (1) is a fixed point of the operator $S_T : X \times X \to X \times X$ given by

$$S_T(x, y) := (T(x, y), T(y, x)).$$

In particular, it is also of interest to find the so-called fixed point for T, which is a coupled fixed point $(x, y) \in X \times X$ with $x = y$, i.e., an element $x \in X$ with $x = T(x, x)$. Notice that, in this case, $x \in X$ is a fixed point of the diagonal operator U_T generated by T.

In this paper, using the weakly Picard operator technique we will present fixed point and coupled fixed point results for several classes of operators generated by an operator $T : X \times X \to X$ on a metric space (X, d).

2 Main results

Let X be a nonempty set and $V : X \to X$ be an operator and denote by $\Delta(X)$ the diagonal of $X \times X$. We also denote by $\mathcal{P}(X)$ the family of all subsets of X and by $Fix(V)$ the fixed point set of V, i.e., $Fix(V) := \{x \in X \mid x = V(x)\}$.

For an operator $T : X \times X \to X$ we consider the following operators generated by T.

I. The operator $S_T : X \times X \to X \times X$ defined by

$$S_T(x, y) := (T(x, y), T(y, x)).$$

Notice again that a fixed point $(x, y) \in X \times X$ of S_T is a coupled fixed point of T. We also point out that

$$(x, x) \in Fix(S_T) \Longrightarrow x \in Fix(U_T).$$

II. The operator $D_T : X \times X \to X \times X$ defined by $D_T(x, y) := (y, T(x, y))$.

We have that: $(x, y) \in Fix(D_T) \Longrightarrow x = y$ and $x \in Fix(U_T)$.

III. The operator $M_T : X \to \mathcal{P}(X)$ defined by

$$M_T(x) := Fix(T(\cdot, x).$$

It is clear that $Fix(U_T) = Fix(M_T)$.

Let $s(X) := \{(x_n)_{n\in N} | x_n \in X,\ n \in N\}$. Let $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ an operator. By definition, the triple $(X, c(X), Lim)$ is called an L-space (Fréchet [4]) if the following axioms are satisfied:

(i) If $x_n = x,$ for all $n \in N$, then $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$.

(ii) If $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$, then for all subsequences, $(x_{n_i})_{i\in N}$, of $(x_n)_{n\in N}$ we have that $(x_{n_i})_{i\in N} \in c(X)$ and $Lim(x_{n_i})_{i\in N} = x$.

By definition, an element of $c(X)$ is a convergent sequence, while $x := Lim(x_n)_{n\in N}$ is the limit of this sequence and we write $x_n \to x$ as $n \to +\infty$.

An L-space will be denoted by $(X, \to)$.

Recall now the following important abstract concept.

Definition 1. *(I.A. Rus [12]) Let $(X, \to)$ be an L-space. Then $V : X \to X$ is called, by definition, a Picard operator if:*

(i) $Fix(V) = \{x^*\}$;

(ii) $(V^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

Definition 2. *(I.A. Rus [10]) Let $(X, \to)$ be an L-space. Then $V : X \to X$ is called, by definition, a weakly Picard operator if, for all $x \in X$, we have that $(V^n(x))_{n\in\mathbb{N}} \to x^*(x) \in Fix(V)$ as $n \to \infty$.*

If A is weakly Picard operator then we consider the operator $V^\infty : X \to X$ given by

$$V^\infty (x) := \lim_{n\to+\infty} V^n (x) .$$

In particular, since any metric space (X, d) is an L-space with the convergence defined by te metric strucure, the above concepts can be considered in this context too. In this case, several classical results in fixed point theory can be easily transcribed in terms of (weakly) Picard operators. For example, any contraction mapping on a complete metric space is a Picard operator.

Moreover, a Picard operator V for which there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is increasing, continuous in 0 and satisfying $\psi(0) = 0$, such that

$$d(x, x^*) \leq \psi(d(x, V(x))),\ \text{for all } x \in X,$$

is called a ψ-Picard operator. In particular, if $\psi(t) := Ct$ (where $C > 0$), then V is called a C-Picard operator. For example, any contraction mapping with constant $\alpha \in (0,1)$ on a complete metric space is a $\frac{1}{1-\alpha}$-Picard operator.

For related results concerning the weakly Picard operator theory see [10]-[12].

The following result gives the relations between the fixed point and coupled fixed points of the operator T and the diagonal operator U_T.

Theorem 1. *Let $(X, \to)$ be an L-space and $T : X \times X \to X$ be an operator. We suppose that S_T is a weakly Picard operator. Then we have:*

(a) U_T is a weakly Picard operator;

(b) $S_T^\infty(x,x) = (U_T^\infty(x), U_T^\infty(x)),\ \forall\ x \in X$;

(c) if S_T is a Picard operator, then:

(1) $Fix(S_T) = \{(x^*, x^*)\}$;

(2) U_T is a Picard operator and $Fix(U_T) = \{x^*\}$.

Now we consider instead of the L-space $(X, \to)$ a metric space (X, d). We can consider, on $X \times X$, the following metrics:

$$d_M((x_1,y_1),(x_2,y_2)) := d(x_1,x_2) + d(y_1,y_2), \tag{2}$$

and

$$d_C((x_1,y_1),(x_2,y_2)) := \max(d(x_1,x_2), d(y_1,y_2)). \tag{3}$$

For example, working in terms of the metric d_M, we have the following result.

Theorem 2. *([8]) Let $(X, \leq)$ be a partially ordered set and let $d : X \times X \to \mathbb{R}_+$ be a complete metric on X. Let $T : X \times X \to X$ be an operator with closed graph which has the mixed monotone property (i.e., T is increasing in the first variable and decreasing in its second one) on $X \times X$. Assume that the following conditions are satisfied:*

(i) there exists $k \in (0,1)$ such that

$$d(T(x,y),T(u,v)) + d(T(y,x),T(v,u)) \leq k[d(x,u) + d(y,v)],\ \forall x \leq u, y \geq v;$$

(ii) there exist $x_0, y_0 \in X$ such that $x_0 \leq T(x_0,y_0)$ and $y_0 \geq T(y_0,x_0)$;

Then, the following conclusions hold:

(a) there exists $(x^*, y^*) \in X \times X$ a coupled fixed point for T such that the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in X defined, for $n \in \mathbb{N}$, by

$$\begin{cases} x_{n+1} = T(x_n, y_n) \\ y_{n+1} = T(y_n, x_n), \end{cases} \tag{4}$$

have the property that $(x_n)_{n\in\mathbb{N}} \to x^*$, $(y_n)_{n\in\mathbb{N}} \to y^*$ as $n \to \infty$. Moreover, for every pair $(x, y) \in X \times X$ with $x \le x_0$ and $y \ge y_0$ (or reversely), we have that $(T^n(x, y))_{n\in\mathbb{N}}$ converges to x^* and $(T^n(y, x))_{n\in\mathbb{N}}$ converges to y^*.

(b) for all $n \in \mathbb{N}^*$, the following estimation holds

$$d(T^n(x_0, y_0), x^*)+d(T^n(y_0, x_0), y^*)\le\frac{sk^n}{1-sk}\cdot[d(x_0, T(x_0, y_0))+d(y_0, T(y_0, x_0))].$$

(c) If additionally, for any $(x, y), (u, v) \in X \times X$ there exists $(p, q) \in X \times X$ such that

$$(x \le p,\ y \ge q) \text{ or } (p \le x,\ q \ge y)$$

$$(u \le p,\ v \ge q) \text{ or } (p \le u,\ q \ge v)$$

then there is a unique coupled fixed point $(x^*, y^*) \in X \times X$ for T with $x^* = y^*$.

Remark 1. *Similar results can be given if, instead of d_M one use on $X \times X$ the metric d_C. See [9].*

If (X, d) is a metric space, then the following qualitative properties concerning the behavior of an operator $V : X \to X$ are well-known in fixed point theory.

(i) *the fixed point equation*

$$x = V(x),\ x \in X$$

is called *well-posed* if $Fix(V) = \{x^*_V\}$ and for any sequence $(x_n) \subset X$ such that

$$d(x_n, V(x_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$x_n \to x^*_V \text{ as } n \to \infty.$$

(ii) the operator V has the *Ostrowski property* (or the operator V has the limit shadowing property) if $Fix(V) = \{x^*\}$ and for any sequence $(x_n) \subset X$ such that

$$(d(x_{n+1}, V(x_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$x_n \to x^* \text{ as } n \to \infty.$$

(iii) *the fixed point equation*

$$x = V(x),\ x \in X$$

is *generalized Ulam-Hyers stable* if there exists an increasing function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$, continuous in 0 with $\eta(0) = 0$, and for each $\varepsilon > 0$ and for each solution y^* of the inequality

$$d(y, V(y)) \leq \varepsilon$$

there exists a solution x^* of the fixed point equation with

$$d(x^*, y^*) \leq \eta(\varepsilon).$$

Concerning the coupled fixed point problem and the fixed points of the diagonal operator, we have the following result.

Theorem 3. *Let (X, d) be a metric space and $T : X \times X \to X$ be an operator. Then:*

(a) If the fixed point equations for S_T is well-posed and $Fix(S_T)=\{(x^*, x^*)\}$, then the fixed point equations for U_T is well-posed;

(b) If the operator S_T has the Ostrowski property and $Fix(S_T) = \{(x^*, x^*)\}$, then the operator U_T has the Ostrowski property;

(c) If the fixed point equations for S_T is generalized Ulam-Hyers stable and all the fixed points of S_T are of the form (x^*, x^*), then the fixed point equations for U_T is generalized Ulam-Hyers stable.

Remark 2. *For other considerations on the above problems see [7] and the references therein.*

Remark 3. *For other results concerning the operators D_T and M_T see [7].*

We conclude our note with the following open question.

Open problem. *Let $(X, \leq)$ be an ordered set and $V : X \to X$. In which conditions there exists an operator $T : X \times X \to X$ such that:*

(i) $T(\cdot, x)$ is increasing,

(ii) $T(x, \cdot)$ is decreasing,

(iii) $V = U_T$?

Construct a coupled fixed point theory for U_T.

For related results see [1], [2], [3], [5], [6], [13], [14], [15].

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