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Approximate viability for nonlinear evolution inclusions with application to controllability*

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Abstract

We investigate approximate viability for a graph with respect to fully nonlinear quasi–autonomous evolution inclusions. As application, an approximate null controllability result is given.

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1 Introduction and preliminary results

Let X be a real Banach space and let $A : D(A) \subset X \rightsquigarrow X$ be an m–dissipative operator, generating the nonlinear semigroup of contractions $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \geq 0\}$. Let $F : I \times X \rightsquigarrow X$ be a multi–function with nonempty values, where $I \subset \mathbb{R}$ is a nonempty and open from the right interval. Consider the Cauchy problem

$$\begin{cases} y'(t) \in Ay(t) + F(t, y(t)), \\ y(\tau) = \xi \in \overline{D(A)}. \end{cases} \tag{1}$$

By an integral solution of (1) on $[\tau, T] \subset I$ we mean a continuous function $y : [\tau, T] \to \overline{D(A)}$ which is an integral solution of

$$\begin{cases} y'(t) \in Ay(t) + f(t), \\ y(\tau) = \xi \in \overline{D(A)}, \end{cases} \tag{2}$$

for some $f \in L^1(\tau, T; X)$ satisfying $f(t) \in F(t, y(t))$ a.e. for $t \in [\tau, T]$.
We recall that a continuous function $y : [\tau, T] \to \overline{D(A)}$ is called an integral solution of (2) if

$$\|y(t) - u\| \leq \|\xi - u\| + \int_\tau^t [y(s) - u, f(s) - v]_+ ds,$$

for every $u \in D(A)$, $v \in Au$ and $t \in [\tau, T]$. Here and thereafter $[\cdot, \cdot]_+$ denotes the right directional derivative of the norm. Concerning properties of $[\cdot, \cdot]_+$ see, e.g., [13] Section 1.2. We point out that in order to stress the dependence of integral solution of (2) on τ, ξ and $f(\cdot)$, we shall denote it by $y(\cdot, \tau, \xi, f)$.

It is well known (see, e.g., [2]) that for every $f \in L^1(\tau, T; X)$ and $\xi \in \overline{D(A)}$ the Cauchy problem (2) has a unique integral solution. Moreover, if $y_1(\cdot) = y(\cdot, \tau, \xi, f)$ and $y_2(\cdot) = y(\cdot, \tau, \eta, g)$ on $[\tau, T]$ for some $f, g \in L^1(\tau, T; X)$ and $\xi, \eta \in \overline{D(A)}$, then

$$\|y_1(t) - y_2(t)\| \leq \|\xi - \eta\| + \int_\tau^t \|f(s) - g(s)\| ds, \tag{3}$$

and

$$\|y_1(t) - y_2(t)\| \leq \|\xi - \eta\| + \int_\tau^t [y_1(s) - y_2(s), f(s) - g(s)]_+ ds, \tag{4}$$

for every $t \in [\tau, T]$.

Remark 1. *Let us point out an important feature concerning inequalities* (3) *and* (4)*. In fact, if* $0 \in D(A)$*,* $0 \in A0$ *and* $g \equiv 0$ *then* $y_2(t) = y(t, \tau, 0, 0) = 0$*, for every* $t \in [\tau, T]$*. This means that the following inequalities hold:*

$$\|y_1(t)\| \leq \|\xi\| + \int_\tau^t \|f(s)\| ds,$$

and

$$\|y_1(t)\| \leq \|\xi\| + \int_\tau^t [y_1(s), f(s)]_+ ds,$$

for every $t \in [\tau, T]$*, where* $y_1(\cdot) = y(\cdot, \tau, \xi, f)$ *for some* $f \in L^1(\tau, T; X)$ *and* $\xi \in \overline{D(A)}$*.*

Let us denote by $\mathbb{B}$ the closed unit ball of X. Let $\varepsilon > 0$. We introduce the notion of ε–solution of (1) that we are dealing with in this paper.

Definition 1. A function $y : [\tau, T] \to \overline{D(A)}$ is said to be an ε–solution of (1) on $[\tau, T] \subset I$ if it is a solution of

$$\begin{cases} y'(t) \in Ay(t) + F(t, y(t) + \varepsilon\mathbb{B}), \\ y(\tau) = \xi \in \overline{D(A)}, \end{cases} \tag{5}$$

on $[\tau, T]$.

Let $K : I \rightsquigarrow \overline{D(A)}$ be a given multi–function with graph $\mathcal{K}$, i.e.,

$$\mathcal{K} = \{(t, x);\ t \in I,\ x \in K(t)\}.$$

Using the definition of ε–solution given above, we introduce the concept of approximate viability for $\mathcal{K}$ with respect to (1).

Definition 2. We say that $\mathcal{K}$ is approximate viable (globally approximate viable) with respect to (1) if for any $(\tau, \xi) \in \mathcal{K}$ there exists $T > \tau$ with $[\tau, T] \subset I$ (for all $T > \tau$ with $[\tau, T] \subset I$) and for any $\varepsilon > 0$ there exists an ε–solution $y : [\tau, T] \to \overline{D(A)}$ of (1) on $[\tau, T]$ satisfying $\operatorname{dist}(y(t); K(t)) \leq \varepsilon$, for all $t \in [\tau, T]$.

Here and thereafter we use the distance between two subsets C and D in X given by

$$\operatorname{dist}(C, D) = \inf_{x \in C, y \in D} \|x - y\|.$$

The starting point of this paper lies in [10], where characterizations of approximate viability for a nonempty subset K of $\overline{D(A)}$ with respect to the fully nonlinear autonomous differential inclusion $y'(t) \in Ay(t) + F(y(t))$ are given.

Here, we allow K to be the graph, $\mathcal{K}$, of a given multi–function and we provide necessary and sufficient conditions in order that $\mathcal{K}$ be approximate viable with respect to the fully nonlinear quasi–autonomous differential inclusion $y'(t) \in Ay(t) + F(t, y(t))$.

It is important to note that the usual trick used to pass from the quasi–autonomous case to the autonomous case, i.e., to consider the Banach space $Y = \mathbb{R} \times X$, the operator $\mathcal{A} = (0, A)$, the multi–function $G : Y \rightsquigarrow Y$ defined by $G(t, y) = (1, F(t, y))$ and the Cauchy problem

$$\begin{cases} x'(t) \in \mathcal{A}x(t) + G(x(t)), \\ x(\tau) = (z_0, \xi) \in \overline{D(\mathcal{A})}, \end{cases} \tag{6}$$

does not work here. In fact, if the function $(z(\cdot), y(\cdot))$ is an ε–solution of (6) on $[\tau, T]$, we do not get that $y(\cdot)$ is an ε–solution of (1) on $[\tau, T]$, due to the variation $t - \tau + z_0 + \varepsilon\mathbb{B}_{\mathbb{R}}$, which appears in the time variable in $F(\cdot, \cdot)$ in the Cauchy problem (5). We denoted by $\mathbb{B}_{\mathbb{R}}$ the unit ball of $\mathbb{R}$.

Our approach here consists in adapting the tangency condition used in [15] (see also [8] and [7]) to our setting. By using this tangency condition, we provide in Section 2 some sufficient and necessary conditions for a graph $\mathcal{K}$ to be approximate viable with respect to (1). As application, we investigate in Section 3 approximate null controllability for fully nonlinear evolution inclusions.

Now let us present some preliminary results. As we have mentioned above, we aim to adapt the tangency condition used in [15] to our setting. Let $(\tau, \xi) \in \mathcal{K}$ and let $\mathcal{S}_{[\tau,\tau+h]}F(\cdot, \xi)$ be the set of all integrable selections of the multi–function $F(\cdot, \xi)$ defined on $[\tau, \tau+h]$ for some $h > 0$ with $[\tau, \tau+h] \subset I$. We say that $F(\cdot, \cdot)$ satisfies the tangency condition if,

(TC) for every $(\tau, \xi) \in \mathcal{K}$,

$$\liminf_{h \to 0^+} \frac{1}{h} \operatorname{dist}\left(\left\{y(\tau + h, \tau, \xi, f),\ f \in \mathcal{S}_{[\tau,\tau+h]}F(\cdot, \xi)\right\}; K(\tau + h)\right) = 0. \tag{7}$$

Remark 2. *Notice that if $F(\cdot, \xi)$ takes nonempty closed values and is measurable and integrally bounded, then $S_{[\tau,\tau+h]}F(\cdot, \xi)$ is nonempty* (see, e.g., [1, Theorem 8.1.3])*. We recall that $F(\cdot, \xi)$ is integrally bounded if $F(t, \xi) \subset l(t)\mathbb{B}$, for a.e. $t \in I$, for some $l \in L^1(I, \mathbb{R}_+)$.*

We point out that in order to study viability for a graph $\mathcal{K}$ with respect to (1), appropriate tangency conditions have been used in many papers; see, e.g., [16], [17], [3] and [15]. Let us recall the following tangency condition which was used in [15];

(TC1) for every $(\tau,\xi) \in \mathcal{K}$,

$$\liminf_{h\to 0^+} \frac{1}{h}\mathrm{dist}\left(\left\{y(\tau+h,\tau,\xi,f),\ f\in \mathcal{S}_{[\tau,\tau+h]}F(\tau,\xi)\right\}; K(\tau+h)\right) = 0. \quad (8)$$

The next proposition which follows directly from (7) will be very useful in the sequel.

Proposition 1. *Let $(\tau,\xi)\in\mathcal{K}$. The following conditions are equivalent:*

(i) *the relation* (7) *holds true;*

(ii) *there exist two sequences, $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$ and $(f_n)_n$ such that $f_n \in S_{[\tau,\tau+h_n]}F(\cdot,\xi)$ for each $n\in\mathbb{N}^\star$, satisfying*

$$\liminf_{n\to+\infty}\frac{1}{h_n}\mathrm{dist}\big(y(\tau+h_n,\tau,\xi,f_n); K(\tau+h_n)\big) = 0;$$

(iii) *there exist three sequences, $(h_n)_n$ in $\mathbb{R}_+$ with $h_n\downarrow 0$, $(f_n)_n$ such that $f_n\in S_{[\tau,\tau+h_n]}F(\cdot,\xi)$ for each $n\in\mathbb{N}^\star$ and $(p_n)_n$ in X with $\lim_{n\to+\infty} p_n = 0$, satisfying*

$$y(\tau+h_n,\tau,\xi,f_n)+h_np_n\in K(\tau+h_n),$$

for $n = 1, 2, ...$

Next, we shall clarify the relationship between **(TC)** and **(TC1)**.

Proposition 2. *Let X be a separable Banach space and let $(\tau,\xi)\in\mathcal{K}$. Assume that $F(\cdot,\xi)$ is a nonempty and closed valued multi–function. Then the following assertions hold true.*

(i) *If $F(\cdot,\xi)$ is $\varepsilon-\delta$ lower semicontinuous at τ and* (8) *holds true, then* (7) *holds.*

(ii) *If $F(\cdot,\xi)$ is $\varepsilon-\delta$ upper semicontinuous at τ and* (7) *holds true, then* (8) *holds.*

Before proceeding to the proof, we shall recall a well known result concerning measurable multi–functions, that will play crucial role in what follows (see, e.g., [12, Lemma 1.3]).

Lemma 1. *Let X be a separable Banach space, $U : [\tau, T] \rightsquigarrow X$ a measurable multi–function with nonempty and closed values and $g : [\tau, T] \to X$, $k : [\tau, T] \to \mathbb{R}_+$ measurable functions. Assume that*

$$W(t) := U(t) \cap (g(t) + k(t)\mathbb{B}) \neq \emptyset,$$

for a.e. $t \in [\tau, T]$. Then there exists a measurable function $w : [\tau, T] \to X$ such that $w(t) \in W(t)$ for a.e. $t \in [\tau, T]$.

Proof of Proposition 2. Let us, for example, prove **(i)**. The proof of **(ii)** follows the same arguments. Let $(\tau, \xi) \in \mathcal{K}$ and assume that (8) holds true. According to Proposition 1, there exist $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$, $(f_n)_n$ such that $f_n \in S_{[\tau,\tau+h_n]}F(\tau, \xi)$ for each $n \in \mathbb{N}^\star$ and $(p_n)_n$ in X with $\lim_{n\to+\infty} p_n = 0$, such that

$$y(\tau + h_n, \tau, \xi, f_n) + h_n p_n \in K(\tau + h_n), \tag{9}$$

for all $n = 1, 2, ...$ Since $F(\cdot, \xi)$ is $\varepsilon - \delta$ lower semicontinuous at τ, then there exists $\delta_n > 0$ such that

$$F(\tau, \xi) \subset F(s, \xi) + \frac{1}{n}\mathbb{B},$$

for all $n = 1, 2, ...$, and all $s \in [\tau, \tau+\delta_n]$. Let $k_n \in \mathbb{N}^*$ be such that $h_{k_n} < \delta_n$. Then

$$f_{k_n}(s) \in F(s, \xi) + \frac{1}{n}\mathbb{B},$$

for all $s \in [\tau, \tau + h_{k_n}]$. Therefore,

$$F(s, \xi) \cap \left(f_{k_n}(s) + \frac{1}{n}\mathbb{B} \right) \neq \emptyset,$$

for all $s \in [\tau, \tau+h_{k_n}]$. By Lemma 1, for all $n = 1, 2, ...$ there exist measurable functions $g_{k_n(\cdot)}$ and $b_{k_n}(\cdot)$ such that $g_{k_n}(s) \in F(s, \xi)$, $b_{k_n}(s) \in \mathbb{B}$ and

$$g_{k_n}(s) = f_{k_n}(s) + \frac{1}{n} b_{k_n}(s), \tag{10}$$

for a.e. $s \in [\tau, \tau + h_{k_n}]$. Since

$$y(\tau + h_{k_n}, \tau, \xi, g_{k_n}) = y(\tau + h_{k_n}, \tau, \xi, f_{k_n}) + h_{k_n} p_{k_n} + h_{k_n} q_{k_n}$$

for every $n = 1, 2...$, where

$$q_{k_n} = \frac{1}{h_{k_n}} \big(y(\tau + h_{k_n}, \tau, \xi, g_{k_n}) - y(\tau + h_{k_n}, \tau, \xi, f_{k_n}) \big) - p_{k_n},$$

then, from (9),

$$y(\tau + h_{k_n}, \tau, \xi, g_{k_n}) \in K(\tau + h_{k_n}) + h_{k_n} q_{k_n},$$

for every $n = 1, 2....$ By using (3) and (10) we get $\lim_n q_{k_n} = 0$. In view of **(iii)** in Proposition 1, we deduce that (7) holds. The proof is therefore complete. $\square$

Concerning the multi–function $F(\cdot, \cdot)$, we have:

Definition 3. (i) We say that $F(\cdot, \cdot)$ is integrally bounded if, for each $(\tau, \xi) \in I \times X$, there exist $l \in L^1(I, \mathbb{R}_+)$ and $\rho_1 > 0$ such that $F(t, x) \subset l(t)\mathbb{B}$, for a.e. $t \in I$ and all $x \in B(\xi, \rho_1)$, where $B(\xi, \rho_1)$ is the closed ball with center ξ and radius ρ_1.

(ii) We say that $F(\cdot, \cdot)$ satisfies a sublinear growth condition if there exits $c \in L^1(I, \mathbb{R}_+)$ such that

$$F(t, x) \subset c(t)(1 + \|x\|)\mathbb{B}, \tag{11}$$

for a.e. $t \in I$ and for every $x \in X$.

Concerning the graph $\mathcal{K}$, we have:

Definition 4. (i) The graph $\mathcal{K}$ is said to be locally closed from the left if for each $(\tau, \xi) \in \mathcal{K}$, there exist $\overline{T} > \tau$ and $\rho_2 > 0$ such that for each $(\tau_n, \xi_n) \in ([\tau, \overline{T}] \times B(\xi, \rho_2)) \bigcap \mathcal{K}$ with $(\tau_n)_n$ nondecreasing, $\lim_n \tau_n = \overline{\tau}$ and $\lim_n \xi_n = \overline{\xi}$, we have $(\overline{\tau}, \overline{\xi}) \in \mathcal{K}$.

(ii) The graph $\mathcal{K}$ is said to be X–closed if for each $(\tau_n, \xi_n) \in \mathcal{K}$ with $\lim_n \tau_n = \overline{\tau}$, $\overline{\tau} \in I$ and $\lim_n \xi_n = \overline{\xi}$, we have $(\overline{\tau}, \overline{\xi}) \in \mathcal{K}$.

2 Sufficient and necessary conditions for approximate viability

We first state the standing hypothesis concerning the multi–function $K(\cdot)$.

(K) The multi–function $K : I \rightsquigarrow \overline{D(A)}$ has nonempty closed values and is uniformly continuous, that is, for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$K(t) \subset K(s) + \varepsilon\mathbb{B},$$

whenever $|t - s| \leq \delta$.

The following result gives sufficient conditions for a graph $\mathcal{K}$ to be approximate viable with respect to (1).

Theorem 1. *Let X be a Banach space, $A : D(A) \subset X \rightsquigarrow X$ an m–dissipative operator, $K : I \rightsquigarrow \overline{D(A)}$ a multi–function satisfying* **(K)** *and $F(\cdot,\cdot)$ an integrally bounded multi–function with nonempty and closed values. If* **(TC)** *holds true, then $\mathcal{K}$ is approximate viable with respect to* (1).

The next lemma is the main step through the proof of Theorem 1.

Lemma 2. *Let X be a Banach space, $A : D(A) \subset X \rightsquigarrow X$ an m–dissipative operator, $K : I \rightsquigarrow \overline{D(A)}$ a multi–function with a locally closed from the left graph $\mathcal{K}$ and $F(\cdot,\cdot)$ an integrally bounded multi–function with nonempty and closed values. Let $(\tau,\xi) \in \mathcal{K}$ and let $l(\cdot)$ and ρ_1 be as in* Definition 3*. Let $\overline{T}$ and $\rho_2 > 0$ be as in* Definition 4 *and take $\rho = \min\{\rho_1,\rho_2\}$. If* **(TC)** *holds true, then there exists $T \in (\tau,\overline{T})$ such that for every $\varepsilon \in (0,1)$ there exist a nondecreasing function $\sigma : [\tau,T] \to [\tau,T]$, a measurable function $f \in [\tau,T] \to X$ and a continuous function $v : [\tau,T] \to X$ such that*

(i) *$t - \varepsilon \le \sigma(t) \le t$, for all $t \in [\tau,T]$ and $\sigma(T) = T$;*

(ii) *$v(\sigma(t)) \in K(\sigma(t)) \cap B(\xi,\rho)$, for all $t \in [\tau,T]$;*

(iii) *$f(t) \in F(t, v(\sigma(t)))$, for a.e. $t \in [\tau,T]$ and $\|f(t)\| \le l(t)$ for a.e. $t \in [\tau,T]$;*

(iv) *$v(\tau) = \xi$ and $\big\|v(t) - y\big(t,\sigma(s),v(\sigma(s)),f\big)\big\| \le (t-\sigma(s))\varepsilon$ for all t, $s \in [\tau,T]$, $\tau \le s \le t \le T$;*

(v) *$\|v(t) - v(\sigma(t))\| \le \varepsilon$, for all $t \in [\tau,T]$.*

We point out that the proof of the above lemma is purely technical and follows the same arguments as those used in the proof of [15, Lemma 1] (see also [7, Lemma 11.3.1]). However, it is important to note that if we use the tangency condition **(TC1)** instead of **(TC)** in the above lemma, then we get in **(iii)** in Lemma 2, $f(t) \in F(\sigma(t), v(\sigma(t)))$, for a.e. $t \in [\tau,T]$. Actually, this $\sigma(t)$–variation, which appears in the first variable in $F(\cdot,\cdot)$, is a drawback for the existence of ε–solution of the Cauchy problem (1).

Next, and before proceeding to the proof of Theorem 1, we note that, under hypotheses of Lemma 2, if we assume further that $F(\cdot,\cdot)$ satisfies the sublinear growth condition (11) and the graph $\mathcal{K}$ is X–closed, then we get the following refined version of Lemma 2.

Lemma 3. *Let X be a Banach space, $A : D(A) \subset X \rightsquigarrow X$ an m–dissipative operator, $K : I \rightsquigarrow \overline{D(A)}$ a multi–function with a X–closed graph $\mathcal{K}$ and $F(\cdot,\cdot)$ a multi–function with nonempty closed values satisfying the sublinear growth condition* (11). *If* **(TC)** *holds true, then for every $(\tau,\xi) \in \mathcal{K}$, $T > \tau$ with $[\tau,T] \subset I$ and $\varepsilon \in (0,1)$ there exist a nondecreasing function $\sigma : [\tau,T] \to [\tau,T]$, a measurable function $f : [\tau,T] \to X$ and a continuous function $v : [\tau,T] \to X$ such that*

(i) $t-\varepsilon \le \sigma(t) \le t$, *for all $t \in [\tau,T]$ and $\sigma(T) = T$;*

(ii) $v(\sigma(t)) \in K(\sigma(t))$, *for all $t \in [\tau,T]$;*

(iii) $f(t) \in F(t, v(\sigma(t)))$, *for a.e. $t \in [\tau,T]$;*

(iv) $v(\tau) = \xi$ *and* $\big\|v(t) - y\big(t,\sigma(s),v(\sigma(s)),f\big)\big\| \le (t-\sigma(s))\varepsilon$ *for all t, $s \in [\tau,T]$, $\tau \le s \le t \le T$;*

(v) $\|v(t) - v(\sigma(t))\| \le \varepsilon$, *for all $t \in [\tau,T]$.*

Let us now pass to the proof of Theorem 1.

Proof of Theorem 1. Let $(\tau,\xi) \in \mathcal{K}$ and $T > \tau$, $\rho > 0$ as in Lemma 2. Let $\varepsilon > 0$ and let δ be such that

$$K(t) \subset K(s) + \frac{\varepsilon}{2}\mathbb{B}, \tag{12}$$

whenever $|t-s| \le \delta$. Take $\varepsilon' \in (0,1)$ and

$$0 < \varepsilon' \le \min\left\{\frac{\varepsilon}{2(T-\tau+1)}, \delta\right\}.$$

Notice that if the multi–function $K(\cdot)$ satisfies hypothesis **(K)**, then the graph $\mathcal{K}$ is locally closed from the left. We apply Lemma 2 for ε'. There exist σ, f and v such that **(i)**$\sim$**(v)** in Lemma 2 hold true. Let $y(t) = y(t,\tau,\xi,f)$ for every $t \in [\tau,T]$. We claim that $y(\cdot)$ is an ε–solution of (1) on $[\tau,T]$. Indeed, from **(iii)** and **(v)** we have

$$f(t) \in F(t, v(t) + \varepsilon'\mathbb{B}),$$

for a.e. $t \in [\tau,T]$. Using **(iv)** with $s = \tau$, we get

$$\|v(t) - y(t)\| \le (t-\tau)\varepsilon' \le (T-\tau)\varepsilon', \tag{13}$$

for all $t \in [\tau, T]$. Consequently,

$$f(t) \in F\big(t, y(t) + (T - \tau)\varepsilon'\mathbb{B} + \varepsilon'\mathbb{B}\big)$$

for a.e. $t \in [\tau, T]$. From the choice of ε' we deduce that

$$f(t) \in F(t, y(t) + \varepsilon\mathbb{B}),$$

for a.e. $t \in [\tau, T]$. Accordingly, $y(\cdot)$ is an ε–solution of (1) on $[\tau, T]$ as claimed. To finish the proof let us check that

$$\operatorname{dist}(y(t); K(t)) \leq \varepsilon,$$

for every $t \in [\tau, T]$. Indeed, from **(i)**, **(ii)**, **(v)**, (13), (12) and the choice of ε' we get

$$\begin{aligned} \operatorname{dist}(y(t); K(t)) &\leq \operatorname{dist}(y(t); K(\sigma(t))) + \operatorname{dist}(K(\sigma(t)); K(t)) \\ &\leq \|y(t) - v(\sigma(t))\| + \operatorname{dist}(K(\sigma(t)); K(t)) \\ &\leq \|y(t) - v(t)\| + \|v(t) - v(\sigma(t))\| + \frac{\varepsilon}{2} \\ &\leq (T - \tau)\varepsilon' + \varepsilon' + \frac{\varepsilon}{2} \leq \varepsilon, \end{aligned}$$

for every $t \in [\tau, T]$. □

Next we shall show that, under the hypotheses of Theorem 1, if we assume further that $F(\cdot, \cdot)$ satisfies the sublinear growth condition (11), then $\mathcal{K}$ is globally approximate viable with respect to (1). More precisely, we have the following result.

Theorem 2. *Let X be a Banach space, $A : D(A) \subset X \rightsquigarrow X$ an m–dissipative operator, $K : I \rightsquigarrow \overline{D(A)}$ a multi–function satisfying* **(K)** *and $F(\cdot, \cdot)$ a multi–function with nonempty and closed values satisfying the sublinear growth condition* (11). *If* **(TC)** *holds true, then $\mathcal{K}$ is globally approximate viable with respect to* (1).

Proof. The proof follows the same arguments as those used in the proof of Theorem 1, using this time Lemma 3 instead of Lemma 2. □

Now we investigate necessary conditions of approximate viability for a graph $\mathcal{K}$ with respect to (1).

Let us first recall that in [10, Theorem 3.3] the author proved that if X is a separable Banach space and $F : X \rightsquigarrow X$ is $\varepsilon - \delta$ upper semicontinuous satisfying a sublinear growth condition, then approximate viability of the set $K \subset X$ with respect to $x'(t) \in Ax(t) + F(x(t))$ implies the following tangency condition

$$\liminf_{h\to 0^+} \frac{1}{h} \mathrm{dist}\left(\left\{y(\tau+h,\tau,\xi,f),\ f\in \mathcal{S}_{[\tau,\tau+h]}F(\xi)\right\};K\right) = 0,$$

for each $\xi \in K$, where $\mathcal{S}_{[\tau,\tau+h]}F(\xi)$ is the set of all integrable selections of the multi–function $s \mapsto F(\xi)$ defined on $[\tau,\tau+h]$ for some $h > 0$ with $[\tau,\tau+h] \subset I$.

The same result was established in the semilinear autonomous case when $A : D(A) \subset X \to X$ generates a C_0–semigroup (see [14, Theorem 4.1]) and in the autonomous case with $A \equiv 0$ (see [11, Theorem 3.2] and [6, Theorem 3]).

Here, we prove that approximate viability for a graph $\mathcal{K}$ with respect to (1) implies also a tangency condition, under some Carathéodory conditions on $F(\cdot,\cdot)$. More precisely, let us first state the following Carathéodory assumptions on the multi–function $F(\cdot,\cdot)$.

(F1) For each $x \in X$ the multi–function $F(\cdot,x)$ is measurable.

(F2) There exists $k \in L^1(I;\mathbb{R}_+)$ such that for each $(\tau,\xi) \in \mathcal{K}$ there exist a nondecreasing function $w : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous at 0 with $w(0) = 0$ and a bounded open set $\Omega \subset X$ containing ξ such that

$$F(t,x) \subset F(t,\xi) + k(t)w(\|x-\xi\|)\mathbb{B},$$

for each $x \in \Omega$ and for a.e. $t \in I$.

The following result provides necessary conditions of approximate viability for a graph $\mathcal{K}$ with respect to (1).

Theorem 3. *Let X be a separable Banach space, $A : D(A) \subset X \rightsquigarrow X$ an m–dissipative operator such that $0 \in D(A)$ and $0 \in A0$. Let $K : I \rightsquigarrow \overline{D(A)}$ be a multi–function with the graph $\mathcal{K}$ and let $F(\cdot,\cdot)$ be a multi–function with nonempty and closed values satisfying* (**F1**), (**F2**) *and the sublinear growth condition* (11). *If $\mathcal{K}$ is approximate viable with respect to* (1), *then* (7) *holds true for a.e. $\tau \in I$ and for every $\xi \in K(\tau)$.*

Proof. It is well known that if $k \in L^1(I,\mathbb{R}_+)$ then there exists a negligible set $Z \subset I$ such that

$$\lim_{t\to 0^+} \frac{1}{t}\int_\tau^{\tau+t} |k(s)-k(\tau)|ds = 0, \tag{14}$$

for every $\tau \in I \setminus Z$. Let $\tau \in I \setminus Z$, $\xi \in K(\tau)$ and $T > \tau$ be as in Definition 2. Take $(\varepsilon_n) \subset (0,1)$ such that $\varepsilon_n \downarrow 0$ and $\sqrt{\varepsilon_n} < T - \tau$ for every $n \in \mathbb{N}^*$. Since $\mathcal{K}$ is approximate viable with respect to (1), there exists a sequence of ε_n–solutions $y_n : [\tau, T] \to X$ of (1) satisfying

$$\operatorname{dist}(y_n(t); K(t)) \leq \varepsilon_n, \tag{15}$$

for each $t \in [\tau, T]$, where $y_n(\cdot) = y(\cdot, \tau, \xi, f_n)$ and $(f_n) \subset L^1(\tau, T; X)$ satisfying

$$f_n(s) \in F(s, y_n(s) + \varepsilon_n \mathbb{B}), \tag{16}$$

for every $n \in \mathbb{N}^*$ and for a.e. $s \in [\tau, T]$. Using (16) and (11), we get

$$\|f_n(s)\| \leq c(s)(2 + \|y_n(s)\|), \tag{17}$$

for every $n \in \mathbb{N}^*$ and for a.e. $s \in [\tau, T]$. From Remark 1 and the above inequality one has

$$\|y_n(t)\| \leq \|\xi\| + \int_\tau^t \|f_n(s)\| ds \leq \|\xi\| + \int_\tau^t c(s)(2 + \|y_n(s)\|) ds,$$

for every $n \in \mathbb{N}^*$ and every $t \in [\tau, T]$. Applying Gronwall's inequality, we get

$$\|y_n(t)\| \leq M,$$

for every $n \in \mathbb{N}^*$ and every $t \in [\tau, T]$, where $M = (\|\xi\| + 2C)e^C$ and $C = \int_\tau^T c(s) ds$. It follows from (17) that

$$\|f_n(s)\| \leq c(s)(2 + M), \tag{18}$$

for every $n \in \mathbb{N}^*$ and for a.e. $s \in [\tau, T]$. Using (3), we get

$$\begin{aligned} \|y_n(t) - \xi\| &\leq \int_\tau^t \|f_n(s)\| ds + \|y(t, \tau, \xi, 0) - \xi\| \\ &\leq \int_\tau^t c(s)(2 + M) ds + \|y(t, \tau, \xi, 0) - \xi\|, \end{aligned}$$

for every $n \in \mathbb{N}^*$ and every $t \in [\tau, T]$. Let $t_n = \sqrt{\varepsilon_n}$. Then

$$\|y_n(t) - \xi\| \leq \delta_n,$$

for every $n \in \mathbb{N}^*$ and every $t \in [\tau, \tau + t_n]$, where

$$\delta_n = \int_\tau^{\tau + t_n} c(s)(2 + M) ds + \sup_{\tau \leq t \leq \tau + t_n} \|y(t, \tau, \xi, 0) - \xi\|.$$

Hence $y_n(t) \in \xi + \delta_n \mathbb{B}$, for every $n \in \mathbb{N}^*$ and every $t \in [\tau, \tau + t_n]$. Therefore, from (16), we infer that

$$f_n(s) \in F(s, \xi + (\delta_n + \varepsilon_n)\mathbb{B})$$

for every $n \in \mathbb{N}^*$ and a.e. $s \in [\tau, \tau + t_n]$. Using hypothesis **(F2)** and taking into account that $\lim_n \delta_n = 0$, we get for n sufficiently large that

$$f_n(s) \in F(s, \xi + (\delta_n + \varepsilon_n)\mathbb{B}) \subset F(s, \xi) + k(s) w_n \mathbb{B},$$

for a.e. $s \in [\tau, \tau + t_n]$, where $w_n = w(\delta_n + \varepsilon_n)$. Thus,

$$F(s, \xi) \cap (f_n(s) + k(s) w_n \mathbb{B}) \neq \emptyset,$$

for a.e. $s \in [\tau, \tau + t_n]$. By virtue of Lemma 1, we deduce that there exist measurable functions $g_n(\cdot)$ and $b_n(\cdot)$ such that $g_n(s) \in F(s, \xi)$, $b_n(s) \in \mathbb{B}$ and

$$g_n(s) = f_n(s) + k(s) w_n b_n(s), \tag{19}$$

for a.e. $s \in [\tau, \tau + t_n]$. Therefore, combining (19), (15), (3) and making use of the choice of t_n, for n sufficiently large we get

$$\begin{aligned}
&\frac{1}{t_n} \operatorname{dist}\big(y(\tau + t_n, \tau, \xi, g_n); K(\tau + t_n)\big) \\
&\le \frac{1}{t_n} \big\| y(\tau + t_n, \tau, \xi, g_n) - y(\tau + t_n, \tau, \xi, f_n) \big\| + \frac{1}{t_n} \operatorname{dist}\big(y(\tau + t_n, \tau, \xi, f_n); K(\tau + t_n)\big) \\
&\le \frac{1}{t_n} \int_\tau^{\tau + t_n} k(s) w_n ds + \frac{\varepsilon_n}{t_n} \le w_n \frac{1}{t_n} \int_\tau^{\tau + t_n} |k(s) - k(\tau)| ds + w_n k(\tau) + \sqrt{\varepsilon_n}.
\end{aligned}$$

Finally, from (14), taking into account that $\lim_{n \to +\infty} w_n = 0$ and considering the continuity of $w(\cdot)$ at 0 and $w(0) = 0$, we deduce that

$$\liminf_{n \to +\infty} \frac{1}{t_n} \operatorname{dist}\big(y(\tau + t_n, \tau, \xi, g_n); K(\tau + t_n)\big) = 0.$$

From Proposition 1 **(ii)**, we conclude that (7) holds true. The proof is therefore complete. □

3 Application to approximate null controllability

We consider the state evolution inclusion

$$\begin{cases} x'(t) \in Ax(t) + f(t, x(t)) + u(t), \\ x(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{20}$$

where $A : D(A) \subset X \rightsquigarrow X$ is an m–dissipative operator, X is a Banach space, $u(\cdot)$ is a measurable control taking values in $\mathbb{B}$ and $f : \mathbb{R}_+ \times X \to X$ satisfies

(f1) $f(\cdot, x)$ is continuous for each $x \in X$;

(f2) $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$ and $\|f(t, x) - f(t, y)\| \le L\|x - y\|$, for every $x, y \in X$, $t \in \mathbb{R}_+$ and for some $L > 0$.

We will prove an approximate null controllability result for (20), that is, for each $\varepsilon > 0$ find a control $u(\cdot)$ and a solution $x(\cdot)$ of (20) issuing from the initial point x_0 and satisfying $\|x(T)\| \le \varepsilon$ in some time T.
Our main result in this section is the following.

Theorem 4. *Let X be a Banach space, $A : D(A) \subset X \rightsquigarrow X$ an m–dissipative operator with $0 \in D(A)$, $0 \in A0$ and let $f : \mathbb{R}_+ \times X \to X$ satisfy* **(f1)** *and* **(f2)**. *Then, for any $x_0 \in \overline{D(A)}$ with $0 < \|x_0\| < \dfrac{1}{2L}$, there exists $T > 0$ such that for any $\varepsilon > 0$ there exist a measurable control $u(\cdot)$ with $u(t) \in \mathbb{B}$ for a.e. $t \in [0, T]$ and a solution $x : [0, T] \to X$ of (20) satisfying $\|x(T)\| \le \varepsilon$.*

Proof. Consider the Banach space $Y = X \times \mathbb{R}$, the operator $\mathcal{A} = (A, 0)$ and the cylindrical domain $\mathcal{K} = \mathbb{R}_+ \times K$, where

$$K = \{(x, z) \in \overline{D(A)} \times \mathbb{R}; \|x\| \le |z|\}.$$

Define $G : \mathbb{R}_+ \times Y \rightsquigarrow Y$ by

$$G(t, y) = (f(t, x) + \frac{1}{2}\mathbb{B}) \times \{Lz - \frac{1}{2}\},$$

where $y = (x, z)$. We can easily verify that $G(\cdot, \cdot)$ has nonempty closed values and satisfies a sublinear growth condition. Consider the following Cauchy problem

$$\begin{cases} y'(t) \in \mathcal{A}y(t) + G(t, y(t)), \\ y(0) = \xi \in \overline{D(\mathcal{A})}. \end{cases} \tag{21}$$

In order to apply our main result on approximate viability (Theorem 2), we have to check that the tangency condition **(TC)** holds with $F(\cdot, \cdot)$ replaced by $G(\cdot, \cdot)$. To this end, let $(\tau, y) \in \mathcal{K}$ be fixed but arbitrary and let $y = (x, z) \in K$. We assume first that $x \neq 0$. Let us show that there exist the sequences $(h_n) \subset \mathbb{R}_+$, $h_n \downarrow 0^+$, $(p_n) \subset \mathbb{R}$ with $p_n \to 0$ and (g_n) with $g_n(t) \in f(t,x) + \frac{1}{2}\mathbb{B}$ for a.e. $t \in [\tau, \tau + h_n]$ such that

$$\big(y(\tau + h_n, \tau, x, g_n), z + h_n(Lz - \frac{1}{2}) + h_n p_n\big) \in K,$$

for every $n \in \mathbb{N}^*$. Let $h > 0$ be arbitrary. We define $g : [\tau, \tau + h] \to X$ by $g(t) = f(t,x) - \frac{x}{2\|x\|}$. From Remark 1, the following inequality holds

$$\|y(\tau + h, \tau, x, g)\| \leq \|x\| + \int_\tau^{\tau+h} [y(s, \tau, x, g), g(s)]_+ ds. \tag{22}$$

By the upper semicontinuity of $[\cdot, \cdot]_+$ we get

$$\begin{aligned} &\limsup_{h \to 0^+} \frac{1}{h} \int_\tau^{\tau+h} [y(s, \tau, x, g), g(s)]_+ ds \leq \\ &[y(\tau, \tau, x, g), g(\tau)]_+ = [x, g(\tau)]_+ = [x, f(\tau, x) - \frac{x}{2\|x\|}]_+ \\ \leq \quad &[x, f(\tau, x)]_+ + [x, -\frac{x}{2\|x\|}]_+ \leq \|f(\tau, x)\| - \frac{1}{2} \leq L\|x\| - \frac{1}{2}. \end{aligned}$$

Hence, there exist the sequences $(h_n) \subset \mathbb{R}_+$ with $h_n \downarrow 0^+$ and $(p_n) \subset \mathbb{R}$ with $p_n \to 0$ such that

$$\int_\tau^{\tau+h_n} [y(s, \tau, x, g), g(s)]_+ ds \leq h_n(L\|x\| - \frac{1}{2}) + h_n p_n,$$

for all $n \in \mathbb{N}^*$. Let $g_n : [\tau, \tau + h_n] \to X$ be such that $g_n(t) = g(t)$ for all $t \in [\tau, \tau + h_n]$. From (22) and the above inequality we get

$$\begin{aligned} \|y(\tau + h_n, \tau, x, g_n)\| &\leq \|x\| + h_n(L\|x\| - \frac{1}{2}) + h_n p_n \\ &\leq |z| + h_n(L|z| - \frac{1}{2}) + h_n p_n \\ &\leq \big|z + h_n(Lz - \frac{1}{2}) + h_n p_n\big|. \end{aligned}$$

Then (7) holds true. If $x = 0$, we check easily that (7) holds. By virtue of Theorem 2 we deduce that $\mathcal{K}$ is globally approximate viable with respect to (21). Let $x_0 \in \overline{D(A)}$ be such that $0 < \|x_0\| < \frac{1}{2L}$ and let

$$T = \frac{1}{L} \log \frac{1}{1 - 2L\|x_0\|}.$$

For $\varepsilon > 0$ take $\varepsilon' > 0$ such that

$$\varepsilon' \leq \min\left\{\frac{1}{2L}, \frac{\varepsilon}{e^{LT}LT+1}\right\}.$$

Since $\mathcal{K}$ is globally approximate viable with respect to (21), there exists $x : [0,T] \to X$ a solution of

$$\begin{cases} x'(t) \in Ax(t) + f(t, x(t) + \varepsilon'\mathbb{B}) + \dfrac{1}{2}\mathbb{B}, \\ x(0) = x_0, \end{cases} \tag{23}$$

on $[0,T]$ and $z : [0,T] \to \mathbb{R}_+$ a solution of

$$\begin{cases} z'(t) \in L(z(t) + \varepsilon'\mathbb{B}_{\mathbb{R}}) - \dfrac{1}{2}, \\ z(0) = \|x_0\|, \end{cases} \tag{24}$$

on $[0,T]$ such that

$$\operatorname{dist}((x(t), z(t)), K) \leq \frac{\varepsilon'}{2} < \varepsilon', \tag{25}$$

for every $t \in [0,T]$. We recall that $\mathbb{B}_{\mathbb{R}}$ is the unit ball of $\mathbb{R}$. From (23), we deduce that $x(\cdot)$ is a solution of

$$\begin{cases} x'(t) \in Ax(t) + g(t), \\ x(0) = x_0, \end{cases}$$

on $[0,T]$, for some $g \in L^1(0,T;X)$ with $g(t) \in f(t, x(t) + \varepsilon'\mathbb{B}) + \frac{1}{2}\mathbb{B}$ a.e. for $t \in [0,T]$. Taking into account that $f(\cdot,\cdot)$ satisfies **(f2)** and from the choice of ε', we deduce that $g(t) \in f(t, x(t)) + \mathbb{B}$ for a.e. $t \in [0,T]$. Hence, there exists a measurable control $u(\cdot)$ with $u(t) \in \mathbb{B}$ for a.e. $t \in [0,T]$ such that $x(\cdot)$ is a solution of (20) on $[0,T]$. To finish the proof, let us check that $\|x(T)\| \leq \varepsilon$. Indeed, using (25), one proves that

$$\|x(t)\| \leq |z(t)| + \varepsilon',$$

for every $t \in [0,T]$. From (24), by a simple calculation, one gets

$$|z(t)| \leq \left|e^{Lt}(\|x_0\| - \frac{1}{2L}) + \frac{1}{2L}\right| + e^{Lt}L\varepsilon' t,$$

for every $t \in [0,T]$. Thus, using the choice of T, we find

$$|z(T)| \leq e^{LT}L\varepsilon' T.$$

Finally, from the choice of ε', we deduce that

$$\|x(T)\| \leq e^{LT} L\varepsilon' T + \varepsilon' \leq \varepsilon.$$

□

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References

[1] J. -P. Aubin, H. Frankowska: Set-valued analysis, Birkhäuser Inc MA, Boston, 1990.

[2] P. Bénilan: Solutions integrales d'équations d'évolution dans un espace de Banach, C. R. Acd. Sci. Paris, A-B 274 : $A47 - -A50$, 1972.

[3] M. Burlică: Viability for multi–valued semilinear reaction–diffusion systems, Ann. Acad. Rom. Sci. Ser. Math. Appl., 2 : $3 - -24$, 2010.

[4] O. Cârjă: On the minimal time function for distributed control systems in Banach spaces, J. Optim. Theory App., 44 : $397 - -406$, 1984.

[5] O. Cârjă, M.D.P. Monteiro Marques: Weak tangency, weak invariance and Carathéodory mappings, J. Dyn. Control Syst., 8 : $445 - -461$, 2002.

[6] O. Cârjă, A. Lazu: Approximate weak invariance for differential inclusions in Banach spaces, J. Dyn. Control Syst., 18 : $215 - -227$, 2012.

[7] O. Cârjă, M. Necula, I.I. Vrabie: Viability, invariance and applications, North-Holland Math. Stud., 207, Elsevier, Amsterdam, 2007.

[8] O. Cârjă, M. Necula, I.I. Vrabie: Necessary and sufficient conditions for viability for nonlinear evolution inclusions, Set-Valued Anal., 16 : $701 - -731$, 2008.

[9] O. Cârjă, M. Necula, I.I. Vrabie: Necessary and sufficient conditions for viability for semilinear differential inclusions, Trans. Amer. Math. Soc., 361 : $343 - -390$, 2009.

[10] I. Căpraru: Approximate weak invariance and relaxation for fully nonlinear differential inclusions, Mediterr. J. Math., 10 : $201 - -212$, 2013.

[11] F.H. Clarke, Y.S. Ledyaev, M.L. Radulescu: Approximate invariance and differential inclusions in Hilbert spaces, J. Dyn. Control Syst., $3 : 493 - -518$, 1997.

[12] H. Frankowska: A priori estimates for operational differential inclusions, J. Differ. Equ., $84 : 100 - -128$, 1990.

[13] V. Lakshmikantham, S. Leela: Nonlinear differential equations in abstract spaces, Pergamon Press, Oxford, 1981.

[14] A.I. Lazu, V. Postolache: Approximate weak invariance for semilinear differential inclusions in Banach spaces, Cent. Eur. J. Math., $9 : 1143 - -1155$, 2011.

[15] M. Necula, M. Popescu, I. I. Vrabie: Nonlinear evolution equations on locally closed graphs, Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM, $104 : 97 - -114$, 2010.

[16] M. Popescu: A viability result for evolution equations on locally closed graphs, Ann. Acad. Rom. Sci. Ser. Math. Appl., $1 : 62 - -82$, 2009.

[17] M. Popescu: A viability result for differential inclusions on graphs, Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), $55 : 383 - -402$, 2009.