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Toeplitz operators on $L^2_a(\mathbb{C}_+)^*$

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Abstract

In this paper we show that if $a \in \mathbb{D}$ and R and S are two bounded linear operators from $L^2_a(\mathbb{C}_+)$ into itself, such that $R\mathcal{T}_{\psi_1}S = \mathcal{T}_{\psi_1 \circ t_a}$ for all $\psi_1 \in L^\infty(\mathbb{C}_+)$, then there exist $\alpha, \beta \in \mathbb{C}$ such that $R = \alpha V_a, S = \beta V_a$ and $\alpha\beta = 1$. Here $t_a(s) = \frac{-ids+(1-c)}{(1+c)s+id}$ and $V_a f = (f \circ t_a)l_a, f \in L^2_a(\mathbb{C}_+)$ where $l_a(s) = \frac{1-|a|^2}{((1+c)s+id)^2}$. Let $\mathcal{A} = \left\{\mathcal{T}_\phi : \phi \in C_c^\infty(\mathbb{C}_+)\right\}$ and $\mathfrak{T} = Cl\ \mathcal{A}$, where $Cl\ \mathcal{A}$ denotes the closure of $\mathcal{A}$ in $\mathcal{L}(L^2_a(\mathbb{C}_+))$. Let $\mathfrak{A}$ be the smallest closed algebra generated by the Toeplitz operators $\mathcal{T}_\phi, \phi \in L^\infty(\mathbb{C}_+)$ in $\mathcal{L}(L^2_a(\mathbb{C}_+))$. In this paper we further show that the set of all Toeplitz operators defined on the Bergman space $L^2_a(\mathbb{C}_+)$ of the right half plane with essentially bounded symbols is dense in the strong operator topology in the space of all bounded linear operators. As applications of these results we establish that there is no bounded projection from $\mathcal{L}(L^2_a(\mathbb{C}_+))$ onto $\mathfrak{T}$ and if $\Phi_0 : \mathfrak{A} \to \mathcal{L}(L^2_a(\mathbb{C}_+))$ is a linear isometry such that, for each pair of vectors $f, g \in L^2_a(\mathbb{C}_+)$,

$$\sup\{|\langle \Phi_0(A)f, g\rangle| : A \in \mathfrak{A} \text{ and } ||A|| = 1\} = ||f||||g||,$$

then their exits a unique extension of Φ_0 to a linear isometry, Φ, mapping $\mathcal{L}(L^2_a(\mathbb{C}_+))$ into itself.

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1 Introduction

Let $\mathbb{C}_+ = \{s = x + iy \in \mathbb{C} : \text{Re}s > 0\}$ be the right half plane. Let $d\widetilde{A}(s) = dxdy$ be the area measure on $\mathbb{C}_+$. Let $L^2(\mathbb{C}_+, d\widetilde{A})$ be the space of complex-valued, absolutely square-integrable, measurable functions on $\mathbb{C}_+$ with respect to the area measure. The space $L^2(\mathbb{C}_+, d\widetilde{A})$ is a Hilbert space with the inner product defined by

$$\langle f, g\rangle = \int_{\mathbb{C}_+} f(s)\overline{g(s)}d\widetilde{A}(s),$$

and the coressponding norm is defined by

$$||f||_2 = \langle f, f\rangle^{\frac{1}{2}} = \left[\int_{\mathbb{C}_+} |f(s)|^2 d\widetilde{A}(s)\right]^{\frac{1}{2}} < \infty.$$

The Bergman space of the right half plane denoted as $L^2_a(\mathbb{C}_+)$ is the closed subspace of $L^2(\mathbb{C}_+, d\widetilde{A})$ consisting of those functions in $L^2(\mathbb{C}_+, d\widetilde{A})$ that are analytic. The functions $H(s, w) = \frac{1}{(s+\overline{w})^2}, s \in \mathbb{C}_+, w \in \mathbb{C}_+$ are the reproducing kernels [4] for $L^2_a(\mathbb{C}_+)$. Let $\mathbf{h}_w(s) = \frac{H(s,w)}{\sqrt{H(w,w)}} = \frac{2\text{Re}w}{(s+\overline{w})^2}$. The functions $\mathbf{h}_w, w \in \mathbb{C}_+$ are the normalized reproducing kernels for $L^2_a(\mathbb{C}_+)$. The sequence of vectors $\{\epsilon_n(s)\}_{n=0}^{\infty} = \left\{\frac{2}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{1-s}{1+s}\right)^n \frac{1}{(1+s)^2}\right\}_{n=0}^{\infty}$ forms an orthonormal basis for $L^2_a(\mathbb{C}_+)$. Let $L^\infty(\mathbb{C}_+)$ be the space of complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{C}_+$. Define for $f \in L^\infty(\mathbb{C}_+)$,

$$||f||_\infty = \text{ess}\sup_{s\in\mathbb{C}_+} |f(s)| < \infty.$$

The space $L^\infty(\mathbb{C}_+)$ is a Banach space with respect to the essential supremum norm. For $\phi \in L^\infty(\mathbb{C}_+)$, we define the multiplication operator $\mathcal{M}_\phi$ from $L^2(\mathbb{C}_+, d\widetilde{A})$ into $L^2(\mathbb{C}_+, d\widetilde{A})$ by $(\mathcal{M}_\phi f)(s) = \phi(s)f(s)$; the Toeplitz operator $\mathcal{T}_\phi$ from $L^2_a(\mathbb{C}_+)$ into $L^2_a(\mathbb{C}_+)$ by $\mathcal{T}_\phi f = P_+(\phi f)$, where P_+ denote the orthogonal projection from $L^2(\mathbb{C}_+, d\widetilde{A})$ onto $L^2_a(\mathbb{C}_+)$. The Toeplitz operator $\mathcal{T}_\phi$ is bounded and $||\mathcal{T}_\phi|| \leq ||\phi||_\infty$. For more details see [6] and [7]. The big Hankel operator $\mathcal{H}_\phi$ from $L^2_a(\mathbb{C}_+)$ into $(L^2_a(\mathbb{C}_+))^\perp$ is defined by $\mathcal{H}_\phi f = (I - P_+)(\phi f), f \in L^2_a(\mathbb{C}_+)$. The little Hankel operator $\mathbf{\Gamma}_\phi$ is the mapping from $L^2_a(\mathbb{C}_+)$ into $L^2_a(\mathbb{C}_+)$ defined by $\mathbf{\Gamma}_\phi f = P_+\mathcal{M}_\phi(\mathcal{J}f)$ where $\mathcal{J}$ is the mapping from $L^2(\mathbb{C}_+, d\widetilde{A})$ into $L^2(\mathbb{C}_+, d\widetilde{A})$ such that $\mathcal{J}f(s) = f(\overline{s})$.

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $dA = rdr\frac{d\theta}{2\pi}$ be the normalized Lebesgue area measure so that the measure of $\mathbb{D}$ equals to

1. Let $L_a^2(\mathbb{D})$ be the Bergman space, the Hilbert space of functions, analytic on $\mathbb{D}$ and square integrable with respect to the measure dA. It is well known that $L_a^2(\mathbb{D})$ is a closed subspace [5] of the Hilbert space $L^2(\mathbb{D}, dA)$ with the set of functions $\{\sqrt{n+1}z^n : n \geq 0\}$ as an orthonormal basis. Let P be the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$. Let $L^\infty(\mathbb{D})$ be the space of complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{D}$ with the essential supremum norm. For $\phi \in L^\infty(\mathbb{D})$, the multiplication operator M_ϕ from $L^2(\mathbb{D}, dA)$ into $L^2(\mathbb{D}, dA)$ is defined by $M_\phi f = \phi f$, the Toeplitz operator T_ϕ from $L_a^2(\mathbb{D})$ into itself is defined by $T_\phi(f) = P(\phi f)$ for $f \in L_a^2(\mathbb{D})$. The little Hankel operator Γ_ϕ from $L_a^2(\mathbb{D})$ into $L_a^2(\mathbb{D})$ is defined by $\Gamma_\phi f = PM_\phi(Jf)$ where J is the mapping from $L^2(\mathbb{D}, dA)$ into itself such that $Jf(z) = f(\overline{z})$.

Since the point evaluation at $z \in \mathbb{D}$, is a bounded functional, there is a function K_z in $L_a^2(\mathbb{D})$ such that $f(z) = \langle f, K_z \rangle$ for all f in $L_a^2(\mathbb{D})$. Let $K(z,w)$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by $K(z,w) = \overline{K_z(w)}$. The function $K(z,w) = \frac{1}{(1-z\overline{w})^2}, z, w \in \mathbb{D}$ and is called the Bergman reproducing kernel [10]. For $a \in \mathbb{D}$, let $k_a(z) = \frac{K(z,a)}{\sqrt{K(a,a)}} = \frac{1-|a|^2}{(1-\overline{a}z)^2}$. The function $k_a, a \in \mathbb{D}$ is called the normalized reproducing kernel for $L_a^2(\mathbb{D})$.

Define $M : \mathbb{C}_+ \to \mathbb{D}$ by $Ms = \frac{1-s}{1+s}$. Then M is one-one, onto and $M^{-1} : \mathbb{D} \to \mathbb{C}_+$ is given by $M^{-1}(z) = \frac{1-z}{1+z}$. Thus M is its self-inverse. Let $W : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{C}_+)$ be defined by $Wg(s) = \frac{2}{\sqrt{\pi}} g(Ms) \frac{1}{(1+s)^2}$. Then $W^{-1} : L_a^2(\mathbb{C}_+) \to L_a^2(\mathbb{D})$ is given by $W^{-1}G(z) = 2\sqrt{\pi} G(Mz) \frac{1}{(1+z)^2}$, where $Mz = \frac{1-z}{1+z}$. If $a \in \mathbb{D}$ and $a = c + id$, $c, d \in \mathbb{R}$, then $t_a(s) = \frac{-ids+(1-c)}{(1+c)s+id}$ is an automorphism from $\mathbb{C}_+$ onto $\mathbb{C}_+$ and $(t_a \circ t_a)(s) = s$. Let $l_a(s) = \frac{1-|a|^2}{((1+c)s+id)^2}$. It is not difficult to see that $t_a'(s) = -l_a(s)$. For $a \in \mathbb{D}$ and $f \in L_a^2(\mathbb{C}_+)$, define V_a from $L_a^2(\mathbb{C}_+)$ into itself by $V_a f = (f \circ t_a) l_a$. For $a \in \mathbb{D}$, it is not difficult to see that $(i) V_a l_a = 1$, (ii) $V_a^{-1} = V_a, V_a^2 = I$, (iii) V_a is a self-adjoint, unitary operator and $V_a P_+ = P_+ V_a$. Let $\mathcal{L}(H)$ be the set of all bounded linear operators from the Hilbert space H into itself. Let $I_{\mathcal{L}(H)}$ denote the identity operator in $\mathcal{L}(H)$ and $\mathcal{LC}(H)$ denote the space of all compact operators in $\mathcal{L}(H)$.

Let $\mathbb{U}$ be an open, connected, nonempty and proper subset of $\mathbb{C}$. Let $C_c^\infty(\mathbb{U})$ be the set of all infinitely differentiable functions on U whose support is a compact subset of $\mathbb{U}$. The space $C_c^\infty(\mathbb{U})$ is w^*-dense in $L^\infty(\mathbb{U})$.

Definition 1. *The weak topology on* $\mathcal{L}(H)$ *is the topology generated by the* *open neighbourhood base*

$$\mathcal{N}\left(A;(x_i)_{i=1}^n,(y_i)_{i=1}^n,\epsilon\right)=\left\{B\in\mathcal{L}(H):\left|\sum_{i=1}^n\langle y_i,(A-B)x_i\rangle\right|<\epsilon\right\}$$

for $A\in\mathcal{L}(H),\epsilon>0$ *and any finite sets of vectors* $(x_i)_{i=1}^n$ *and* $(y_i)_{i=1}^n$ *in* H*. If* (B_α) *is a net in* $\mathcal{L}(H)$*, then* $B_\alpha\longrightarrow A$ *weakly if* $\langle y,(B_\alpha-A)x\rangle\longrightarrow 0$ *for each* $x,y\in H$*.*

Definition 2. *The ultraweak topology on* $\mathcal{L}(H)$ *is the topology generated by the open neighbourhood base*

$$\mathcal{N}\left(A;(x_i)_{i=1}^\infty,(y_i)_{i=1}^\infty,\epsilon\right)=\left\{B\in\mathcal{L}(H):\left|\sum_{i=1}^\infty\langle y_i,(A-B)x_i\rangle\right|<\epsilon\right\}$$

for $A\in\mathcal{L}(H),\epsilon>0$ *and sequences* $(x_i)_{i=1}^\infty$ *and* $(y_i)_{i=1}^\infty$ *in* H *with* $\sum_{i=1}^\infty||x_i||^2<\infty$ *and* $\sum_{i=1}^\infty||y_i||^2<\infty$*. If* (B_α) *is a net in* $\mathcal{L}(H)$ *then* $B_\alpha\longrightarrow A$ *ultraweakly if* $\sum_{i=1}^\infty\langle y_i,(B_\alpha-A)x_i\rangle\longrightarrow 0$ *for each pair of sequences* $(x_i)_{i=1}^\infty$ *and* $(y_i)_{i=1}^\infty$ *in* H *satisfying* $\sum_{i=1}^\infty||x_i||^2<\infty$ *and* $\sum_{i=1}^\infty||y_i||^2<\infty$*.*

If H is finite dimensional, then the norm topology and the ultraweak topology coincide. The strong topology and the ultraweak topology are both finer than the weak topology, but in general, the ultraweak and strong topologies are not comparable. But the ultraweak topology is stronger than the weak topology, i.e., if $B_\alpha\longrightarrow A$ ultraweakly, then $B_\alpha\longrightarrow A$ weakly. But if (B_α) is a bounded net in $\mathcal{L}(H)$, i.e., there is some $M\geq 0$ such that $||B_\alpha||\leq M$ for all α; then $B_\alpha\longrightarrow A$ ultraweakly if and only if $B_\alpha\longrightarrow A$ weakly.

Theorem 1.1. *(Von Neumann's Density Theorem) Let* $\mathcal{R}$ *be any self-adjoint algebra in* $\mathcal{L}(H)$ *containing the identity. Then the ultraweak, strong and weak closures of* $\mathcal{R}$ *in* $\mathcal{L}(H)$ *are all the same.*

Proof. For proof see [9]. $\square$

The layout of this paper is as follows. In §2, we prove certain algebraic properties of Toeplitz and Hankel operators defined on the Bergman space of the right half plane and show that the set of all Toeplitz operators defined on the Bergman space $L^2_a(\mathbb{C}_+)$ of the right half plane with essentially bounded symbols is dense in the strong operator topology in the space of all bounded linear operators. Let $\mathcal{A} = \left\{\mathcal{T}_\phi : \phi \in C_c^\infty(\mathbb{C}_+)\right\}$ and $\mathfrak{T} = Cl\ \mathcal{A}$, where $Cl\ \mathcal{A}$ denotes the closure of $\mathcal{A}$ in $\mathcal{L}(L^2_a(\mathbb{C}_+))$. Let $\mathfrak{A}$ be the smallest closed algebra generated by the Toeplitz operators $\mathcal{T}_\phi, \phi \in L^\infty(\mathbb{C}_+)$ in $\mathcal{L}(L^2_a(\mathbb{C}_+))$. In §3, we show that there is no bounded projection from $\mathcal{L}(L^2_a(\mathbb{C}_+))$ onto $\mathfrak{T}$ and if $a \in \mathbb{D}$ and R and S are two bounded linear operators from $L^2_a(\mathbb{C}_+)$ into itself, such that $R\mathcal{T}_{\psi_1}S = \mathcal{T}_{\psi_1 \circ t_a}$ for all $\psi_1 \in L^\infty(\mathbb{C}_+)$, then there exist $\alpha, \beta \in \mathbb{C}$ such that $R = \alpha V_a, S = \beta V_a$ and $\alpha\beta = 1$. Further, using Von Neumann's density theorem we establish that if $\Phi_0 : \mathfrak{A} \to \mathcal{L}(L^2_a(\mathbb{C}_+))$ is a linear isometry such that, for each pair of vectors $f, g \in L^2_a(\mathbb{C}_+)$,

$$\sup\{|\langle \Phi_0(A)f, g\rangle| : A \in \mathfrak{A} \text{ and } ||A|| = 1\} = ||f||\ ||g||,$$

then their exits a unique extension of Φ_0 to a linear isometry, Φ, mapping $\mathcal{L}(L^2_a(\mathbb{C}_+))$ into itself.

2 Toeplitz and Hankel operators

In this section we prove certain algebraic properties of Toeplitz and Hankel operators defined on the Bergman space of the right half plane. We show that the set of all Toeplitz operators defined on the Bergman space $L^2_a(\mathbb{C}_+)$ of the right half plane with essentially bounded symbols is dense in the strong operator topology in $\mathcal{L}(L^2_a(\mathbb{C}_+))$.

Lemma 2.1. *If* $a \in \mathbb{D}$ *and* $\phi \in L^\infty(\mathbb{C}_+)$, *the following hold:*

(i) $V_a \mathcal{T}_\phi V_a = \mathcal{T}_{\phi \circ t_a}$;

(ii) $V_a \mathcal{H}_\phi V_a = \mathcal{H}_{\phi \circ t_a}$.

Proof. Let $f \in L^2_a(\mathbb{C}_+)$. Since $(l_a \circ t_a)(s) l_a(s) = s$, we obtain

$$\begin{aligned} V_a \mathcal{T}_\phi V_a f &= V_a \mathcal{T}_\phi[(f \circ t_a) l_a] \\ &= V_a P_+[\phi (f \circ t_a) l_a] \\ &= P_+ V_a[\phi (f \circ t_a) l_a] \\ &= P_+[(\phi \circ t_a) f (l_a \circ t_a) l_a] \\ &= P_+[(\phi \circ t_a) f] \\ &= \mathcal{T}_{\phi \circ t_a} f. \end{aligned}$$

This proves (i). Now to establish (ii), let $f \in L^2_a(\mathbb{C}_+)$. Then

$$\begin{aligned}
V_a \mathcal{H}_\phi V_a f &= V_a \mathcal{H}_\phi [(f \circ t_a) l_a] \\
&= V_a (I - P_+)[\phi (f \circ t_a) l_a] \\
&= (I - P_+) V_a [\phi (f \circ t_a) l_a] \\
&= (I - P_+)[(\phi \circ t_a) f (l_a \circ t_a) l_a] \\
&= (I - P_+)[(\phi \circ t_a) f] \\
&= \mathcal{H}_{\phi \circ t_a} f.
\end{aligned}$$

□

Lemma 2.2. *For each $s \in \mathbb{C}_+$, the linear functional $f \mapsto f(s)$ on $L^2_a(\mathbb{C}_+)$ is bounded. Consequently, $f(s) = \langle f, \underline{H}_s \rangle$ for some $\underline{H}_s \in L^2_a(\mathbb{C}_+)$. Further, $||\underline{H}_s|| \leq \frac{1}{\sqrt{\pi} r}$ and $r = dist(s, i\mathbb{R})$.*

Proof. Let $r = dist(s, i\mathbb{R})$ and

$$\gamma_s(s_1) = \begin{cases} 0, & \text{if } |s_1 - s| \geq r \\ \frac{1}{\pi r^2}, & \text{if } |s_1 - s| < r. \end{cases}$$

Then $\displaystyle\int_{\mathbb{C}_+} |\gamma_s(s_1)|^2 d\widetilde{A}(s_1) = (\pi r^2) \left(\frac{1}{\pi r^2} \right)^2 = \frac{1}{\pi r^2} < \infty$. So $\gamma_s \in L^2_a(\mathbb{C}_+)$. Hence, for arbitrary $f \in L^2_a(\mathbb{C}_+)$,

$$\begin{aligned}
\langle f, P_+ \gamma_s \rangle_{L^2_a(\mathbb{C}_+)} &= \langle f, \gamma_s \rangle_{L^2_a(\mathbb{C}_+)} \\
&= \frac{1}{\pi r^2} \int_{|s_1 - s| < r} f(s_1) d\widetilde{A}(s_1),
\end{aligned}$$

which equals $f(s)$ by the mean value theorem. Thus, we may take $\underline{H}_s = P_+ \gamma_s$. Finally,

$$||\underline{H}_s||_2^2 \leq ||\gamma_s||_2^2 = \frac{1}{\pi r^2}.$$

□

Lemma 2.3. *Assume that $\phi \in L^\infty(\mathbb{C}_+)$ and that the support supp ϕ is a compact subset of $\mathbb{C}_+$. Then $\mathcal{M}_\phi |_{L^2_a(\mathbb{C}_+)}$ is a compact operator. Further, the operators $\mathcal{T}_\phi$ and $\mathcal{H}_\phi$ are also compact.*

Proof. Suppose *supp* $\phi = \mathbb{M}$ is a compact subset of $\mathbb{C}_+$, and let $r = dist(\mathbb{M}, i\mathbb{R})$. Then $r > 0$. Suppose the sequence $\{f_n\}$ in $L_a^2(\mathbb{C}_+)$ converges weakly to 0. Then the sequence $\{f_n\}$ must be bounded. Assume $||f_n||_2 \leq C$ for all n. Then

$$|f_n(s)| = |\langle f_n, \underline{H}_s \rangle| \leq ||f_n||_2 \; ||\underline{H}_s||_2 \leq \frac{C}{\sqrt{\pi} r}$$

for all $s \in \mathbb{M}$. Hence $|\phi(s) f_n(s)| \leq \frac{C||\phi||_\infty}{\sqrt{\pi} r}$ for all $s \in \mathbb{C}_+$. Also, $f_n \longrightarrow 0$ weakly implies $f_n(s) = \langle f_n, \underline{H}_s \rangle \longrightarrow 0$ for all $s \in \mathbb{C}_+$. Thus, we may apply the Lebesgue dominated convergence theorem to conclude that

$$\begin{aligned} ||\phi f_n||_2^2 &= \int_{\mathbb{C}_+} |\phi(s) f_n(s)|^2 d\widetilde{A}(s) \\ &= \int_{\mathbb{M}} |\phi(s) f_n(s)|^2 d\widetilde{A}(s) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus the operator $\mathcal{M}_\phi \mid_{L_a^2(\mathbb{C}_+)}$ maps weakly convergent sequences into norm convergent sequences and therefore must be compact. Since $\mathcal{T}_\phi = P_+ \mathcal{M}_\phi$ and $\mathcal{H}_\phi = (I - P_+)\mathcal{M}_\phi$, hence $\mathcal{T}_\phi$ and $\mathcal{H}_\phi$ are also compact. □

Lemma 2.4. *Let* $G(s) \in L^\infty(\mathbb{C}_+)$. *Then the little Hankel operator* $\mathbf{\Gamma}_G$ *determined on* $L_a^2(\mathbb{C}_+)$ *by* G *is equivalent to the little Hankel operator* Γ_ϕ *determined on* $L_a^2(\mathbb{D})$ *by the function* $\phi(z) = \left(\frac{1+\bar{z}}{1+z}\right)^2 G(Mz)$.

Proof. Notice that the sequence of vectors $\left\{\sqrt{n+1} z^n\right\}_{n=0}^\infty$ forms an orthonormal basis for $L_a^2(\mathbb{D})$. Then

$$\begin{aligned} \mathbf{\Gamma}_G(W(\sqrt{n+1} z^n)) &= P_+ \left(G \mathcal{J} \left(\frac{2}{\sqrt{\pi}} \left(\frac{1-s}{1+s} \right)^n \frac{1}{(1+s)^2} \sqrt{n+1} \right) \right) \\ &= W P W^{-1} \left(G(s) \frac{2}{\sqrt{\pi}} \left(\frac{1-\overline{s}}{1+\overline{s}} \right)^n \frac{1}{(1+\overline{s})^2} \sqrt{n+1} \right) \\ &= W \Gamma_{\left(\frac{1+\bar{z}}{1+z}\right)^2 G(Mz)} \left(\sqrt{n+1} z^n \right) \text{ for all } n \geq 0. \end{aligned}$$

Thus $\mathbf{\Gamma}_G$ is unitarily equivalent to Γ_ϕ where $\phi(z) = \left(\frac{1+\bar{z}}{1+z}\right)^2 G(Mz)$. The result follows.

□

Lemma 2.5. *Let $G(s) \in L^\infty(\mathbb{C}_+)$. Then the Toeplitz operator $\mathcal{T}_G$ defined on $L^2_a(\mathbb{C}_+)$ with symbol G is unitarily equivalent to the Toeplitz operator T_ϕ defined on $L^2_a(\mathbb{D})$ with symbol $\phi(z) = G\left(\frac{1-z}{1+z}\right)$. If further, $G(Mz) = G(M|z|)$ where $Mz = \frac{1-z}{1+z}$ then the Toeplitz operator $\mathcal{T}_G$ is a diagonal operator.*

Proof. The operator W maps $\sqrt{n+1}z^n$ to the function $\frac{2}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{1-s}{1+s}\right)^n\frac{1}{(1+s)^2}$ which belongs to $L^2_a(\mathbb{C}_+)$. The Toeplitz operator $\mathcal{T}_G$ maps this vector to $P_+\left(G(s)\frac{2}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{1-s}{1+s}\right)^n\frac{1}{(1+s)^2}\right)$ which is equal to

$$
\begin{aligned}
WPW^{-1}\left(G(s)\frac{2}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{1-s}{1+s}\right)^n\frac{1}{(1+s)^2}\right) &= WP\left(G\left(\frac{1-z}{1+z}\right)z^n\sqrt{n+1}\right)\\
&= WT_\phi(z^n\sqrt{n+1}),
\end{aligned}
$$

where $\phi(z) = G\left(\frac{1-z}{1+z}\right)$. Therefore $\mathcal{T}_G$ is unitarily equivalent to T_ϕ. Notice that if $G(Mz) = G(M|z|)$, then $\phi(z) = \phi(|z|)$. That is, ϕ is radial. We shall show now that if $\phi \in L^\infty(\mathbb{D})$ and ϕ is radial then T_ϕ is a diagonal operator in $\mathcal{L}(L^2_a(\mathbb{D}))$. This can be verified as follows: passing to polar coordinates, we have

$$
\begin{aligned}
\langle T_\phi z^n, z^m\rangle &= \int_{\mathbb{D}} \phi(z)z^n\overline{z}^m dA(z)\\
&= \int_0^1 \frac{1}{2\pi}\int_0^{2\pi} \phi(r)r^{n+m}e^{(n-m)it}2rdtdr.
\end{aligned}
$$

Thus

$$
\langle T_\phi z^n, z^m\rangle = \begin{cases} 0, & n \neq m;\\ \displaystyle\int_0^1 F(r^2)r^{2n}2rdr, & n = m.\end{cases}
$$

But $\int_0^1 F(r^2)r^{2n}2rdr = \int_0^1 F(t)t^n dt$ where $F(t) = \phi\left(t^{\frac{1}{2}}\right), t \in [0,1)$. Thus the matrix of T_ϕ with respect to the orthonormal basis $\{e_n(z)\}_{n=0}^\infty$ where $e_n(z) = \sqrt{n+1}z^n, n = 0,1,2,\cdots$, is a diagonal matrix with diagonal entries

$$
a_n(\phi) = \int_0^1 F(t)(n+1)t^n dt
$$

where $F(t) = \phi\left(t^{\frac{1}{2}}\right), t \in [0,1)$. This implies $\mathcal{T}_G$ is a diagonal operator in $\mathcal{L}(L^2_a(\mathbb{C}_+))$ since $\mathcal{T}_G$ is unitarily equivalent to T_ϕ. $\square$

Theorem 2.1. *The set $\{\mathcal{T}_\psi : \psi \in L^\infty(\mathbb{C}_+)\}$ is dense in $\mathcal{L}(L_a^2(\mathbb{C}_+))$ in the strong operator topology.*

Proof. Let $T \in \mathcal{L}(L_a^2(\mathbb{C}_+))$. Then $S = W^{-1}TW \in \mathcal{L}(L_a^2(\mathbb{D}))$. Let $\underline{f_i}, \underline{g}_i \in L_a^2(\mathbb{D}), i = 1, 2, \cdots, N$. We shall show that there exists $\phi \in L^\infty(\mathbb{D})$ such that $\langle S\underline{f}_i, \underline{g}_i\rangle = \langle T_\phi \underline{f_i}, \underline{g_i}\rangle, i = 1, 2, \cdots, N$. Let $f_1, f_2, \cdots, f_n$ be a basis of the finite-dimensional subspace of $L_a^2(\mathbb{D})$ generated by $\underline{f}_1, \cdots, \underline{f}_N$. Let $g_1, g_2, \cdots, g_n$ be a basis of the finite-dimensional subspace of $L_a^2(\mathbb{D})$ generated by $\underline{g}_1, \underline{g}_2, \cdots \underline{g}_N$. Clearly, it is sufficient to find $\phi \in L^\infty(\mathbb{D})$ such that $\langle Sf_i, g_i\rangle = \langle T_\phi f_i, g_i\rangle$ for all $i = 1, \cdots, n$, and $j = 1, \cdots, m$. Define an operator $\bigwedge$ from $L^\infty(\mathbb{D})$ into $\mathbb{C}^{n\times m}$ by $(\bigwedge \phi)_{i,j} = \langle T_\phi f_i, g_j\rangle$. Suppose $u = \left(u_{ij}\right)_{1\le i\le n}^{1\le j\le m} \in \mathbb{C}^{n\times m}$ and u is orthogonal to the range of $\bigwedge$. Then $\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\bigwedge \phi\right)_{i,j}\overline{u_{ij}} = 0$ for all $\phi \in L^\infty(\mathbb{D})$. This implies that

$$\int_{\mathbb{D}} \phi(z)\sum_{i=1}^{n}\sum_{j=1}^{m}\overline{u_{ij}}f_i(z)\overline{g_j(z)}dA(z) = 0$$

for all $\phi \in L^\infty(\mathbb{D})$.

Hence

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\overline{u_{ij}}f_i(z)\overline{g_j(z)} = 0 \tag{1}$$

almost everywhere in $\mathbb{D}$. Since the left side of (1) is continuous on $\mathbb{D}$, this equality holds in fact on the whole of $\mathbb{D}$. Thus the function

$$F(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{m}\overline{u_{ij}}f_i(x)\overline{g_j(\overline{y})},$$

which is analytic in $\mathbb{D}\times\mathbb{D}$, equals zero whenever $x = \overline{y}$. By [8], this implies that $F \equiv 0$ on $\mathbb{D}\times\mathbb{D}$. Because the functions $f_i, i = 1, 2, \cdots, n$ are linearly independent we have $\sum_{j=1}^{m} u_{ij}g_j(\overline{y}) = 0$ for all $y \in \mathbb{D}, i = 1, 2, \cdots, n$; but $g_j, j = 1, 2, \cdots, m$, are also linearly independent, and so $u_{ij} = 0$ for all i, j. That is, $u \equiv 0$. Thus the range of $\bigwedge$ is equal to $\mathbb{C}^{n\times m}$. Thus it follows that the collection $\mathcal{B} = \{T_\phi : \phi \in L^\infty(\mathbb{D})\}$ is dense in $\mathcal{L}(L_a^2(\mathbb{D}))$ in the weak operator topology. As $\mathcal{B}$ is a subspace, i.e. a convex set, its weak operator topology and strong operator topology closures coincide. Now let $T \in \mathcal{L}(L_a^2(\mathbb{C}_+))$. This implies $W^{-1}TW \in \mathcal{L}(L_a^2(\mathbb{D}))$. Since $\mathcal{B}$ is dense in

$\mathcal{L}(L^2_a(\mathbb{D}))$ in strong operator topology hence, there exists a sequence $\{\phi_N\} \in L^\infty(\mathbb{D})$ such that $T_{\phi_N} \longrightarrow W^{-1}TW$ in strong operator topology. Thus $\mathcal{T}_{\phi_N \circ M} = WT_{\phi_N}W^{-1} \longrightarrow T$ in strong operator topology as $N \to \infty$ and $\phi_N \circ M \in L^\infty(\mathbb{C}_+)$.

□

Remark 2.1. *Notice that Theorem 2.1 remains in force if we replace $L^\infty(\mathbb{C}_+)$ by $C^\infty_c(\mathbb{C}_+)$ which is a w^*-dense subset of $L^\infty(\mathbb{C}_+)$.*

3 The algebra of Toeplitz operators

Let $\mathcal{A} = \{\mathcal{T}_\phi : \phi \in C^\infty_c(\mathbb{C}_+)\}$. Then $\mathcal{A} \subset \mathcal{L}(L^2_a(\mathbb{C}_+))$ and let $\mathfrak{T} = Cl\ \mathcal{A}$, where $Cl\ \mathcal{A}$ denotes the closure of $\mathcal{A}$ in $\mathcal{L}(L^2_a(\mathbb{C}_+))$. Let $\mathfrak{A}$ be the smallest closed algebra generated by the Toeplitz operators $\mathcal{T}_\phi, \phi \in L^\infty(\mathbb{C}_+)$ in $\mathcal{L}(L^2_a(\mathbb{C}_+))$. The algebra $\mathfrak{A}$ is called the Toeplitz algebra.
In this section we show that there is no bounded projection from $\mathcal{L}(L^2_a(\mathbb{C}_+))$ onto $\mathfrak{T}$ and if $a \in \mathbb{D}$ and R and S are two bounded linear operators from $L^2_a(\mathbb{C}_+)$ into itself, such that $R\mathcal{T}_{\psi_1}S = \mathcal{T}_{\psi_1 \circ t_a}$ for all $\psi_1 \in L^\infty(\mathbb{C}_+)$, then there exist $\alpha, \beta \in \mathbb{C}$ such that $R = \alpha V_a, S = \beta V_a$ and $\alpha\beta = 1$. We also show that if $\Phi_0 : \mathfrak{A} \to \mathcal{L}(L^2_a(\mathbb{C}_+))$ is a linear isometry and if for each pair of vectors $f, g \in L^2_a(\mathbb{C}_+)$,

$$\sup\{|\langle \Phi_0(A)f, g\rangle| : A \in \mathfrak{A} \text{ and } ||A|| = 1\} = ||f||\ ||g||,$$

then their exits a unique extension of Φ_0 to a linear isometry, Φ, mapping $\mathcal{L}(L^2_a(\mathbb{C}_+))$ into itself.

Lemma 3.1. *There is no bounded projection from $\mathcal{L}(L^2_a(\mathbb{C}_+))$ onto $\mathfrak{T}$.*

Proof. From Lemma 2.3, it follows that $\mathcal{T}_\phi \in \mathcal{LC}(L^2_a(\mathbb{C}_+))$ for $\phi \in C^\infty_c(\mathbb{C}_+)$. Hence $Cl\ \mathcal{A} \subset \mathcal{LC}(L^2_a(\mathbb{C}_+))$. We shall now show that $\mathcal{LC}(L^2_a(\mathbb{C}_+)) \subset \mathfrak{T}$. Let $\mathrm{Tr}(L^2_a(\mathbb{C}_+))$ denote the space of all trace class operators on $L^2_a(\mathbb{C}_+)$ equipped with the trace norm $||\ .\ ||_{Tr}$. It is well-known that dual of $\mathcal{LC}(L^2_a(\mathbb{C}_+)) \simeq \mathrm{Tr}(L^2_a(\mathbb{C}_+))$ and the pairing is given by $(K, T) \mapsto Tr(KT) = Tr(TK)$; where $Tr(A)$ denotes trace of A where $A \in \mathcal{L}(L^2_a(\mathbb{C}_+))$. Suppose $\mathfrak{T} = Cl\ \mathcal{A}$ is a proper subset of $\mathcal{LC}(L^2_a(\mathbb{C}_+))$. By the Hahn Banach theorem, there exists $T \in \mathrm{Tr}(L^2_a(\mathbb{C}_+)), T \neq 0$ such that $Tr(T\mathcal{T}_\phi) = 0$ for all $\phi \in C^\infty_c(\mathbb{C}_+)$. Let A, B be two Hilbert -Schmidt operators such that $T = AB^*$. Let $f_n = A\epsilon_n, g_n = B\epsilon_n$ where $\{\epsilon_n\}_{n=0}^\infty$ is an orthonormal basis for $L^2_a(\mathbb{C}_+)$. Then

$$Tr(T\mathcal{T}_\phi) = Tr(B^*\mathcal{T}_\phi A) = \sum_{n=0}^{\infty} \langle B^*\mathcal{T}_\phi A\epsilon_n, \epsilon_n\rangle,$$

and so the last condition may be rewritten as

$$\sum_{n=0}^{\infty}\langle \mathcal{T}_\phi f_n, g_n\rangle = 0 \text{ for all } \phi \in C_c^\infty(\mathbb{C}_+).$$

That is,

$$\sum_{n=0}^{\infty}\int_{\mathbb{C}_+}\phi(s)f_n(s)\overline{g_n(s)}d\widetilde{A}(s) = 0 \tag{2}$$

for all $\phi \in C_c^\infty(\mathbb{C}_+)$. Because supp ϕ is compact and $\sum_{n=0}^{\infty}||f_n||^2 < \infty$ and $\sum_{n=0}^{\infty}||g_n||^2 < \infty$, we may interchange the integration and summation signs in (2) to obtain

$$\int_{\mathbb{C}_+}\phi(s)F(s,\overline{s})d\widetilde{A}(s) = 0$$

for all $\phi \in C_c^\infty(\mathbb{C}_+)$, where $F(x,y) = \sum_{n=0}^{\infty} f_n(x)\overline{g_n(\overline{y})} = Tr(TG_{x,y})$ and $G(x,y) = \langle ., \underline{H}_x\rangle \underline{H}_{\overline{y}}$.

It then follows that $F(s,\overline{s}) = 0$ for almost all $s \in \mathbb{C}_+$; in other words, the function $F(x,y)$, analytic in $\mathbb{C}_+ \times \mathbb{C}_+$, vanishes when $x = \overline{y}$. By the uniqueness principle [8], we obtain $F = 0$ everywhere on $\mathbb{C}_+ \times \mathbb{C}_+$, i.e., $Tr(TG_{x,y}) = 0$ for all $x, y \in \mathbb{C}_+$. Since the reproducing kernels $\underline{H}_s, s \in \mathbb{C}_+$ span $L_a^2(\mathbb{C}_+)$ hence $Tr(TK) = 0$ for all rank one operators K; by linearity and continuity, $Tr(TK) = 0$ for all compact K. Hence $T = 0$. This is a contradiction. Thus $Cl\ \mathcal{A} = \mathfrak{T} = \mathcal{LC}(L_a^2(\mathbb{C}_+))$. Since there is no bounded projection [5] from $\mathcal{L}(L_a^2(\mathbb{C}_+))$ onto $\mathcal{LC}(L_a^2(\mathbb{C}_+))$ and $\mathcal{LC}(L_a^2(\mathbb{C}_+)) = \mathfrak{T}$, hence the theorem follows. □

Theorem 3.1. *Let $a \in \mathbb{D}$. If R and S are two bounded linear operators from $L_a^2(\mathbb{C}_+)$ into itself, such that $R\mathcal{T}_{\psi_1}S = \mathcal{T}_{\psi_1\circ t_a}$ for all $\psi_1 \in L^\infty(\mathbb{C}_+)$, then there exist $\alpha, \beta \in \mathbb{C}$ such that $R = \alpha V_a, S = \beta V_a$ and $\alpha\beta = 1$.*

Proof. Suppose $R\mathcal{T}_{\psi_1}S = \mathcal{T}_{\psi_1\circ t_a}$ for all $\psi_1 \in L^\infty(\mathbb{C}_+)$. Then by Lemma 2.1, $R\mathcal{T}_{\psi_1}S = V_a\mathcal{T}_{\psi_1}V_a$. Since $V_a^2 = I$ and $V_a^* = V_a$, we obtain

$$V_aR\mathcal{T}_{\psi_1}SV_a = \mathcal{T}_{\psi_1} \tag{3}$$

for all $\psi_1 \in L^\infty(\mathbb{C}_+)$. Let $\underline{M}_a = V_a R$ and $SV_a = \underline{G}_a$. Then from (3), we obtain $\underline{M}_a \mathcal{T}_{\psi_1} \underline{G}_a = \mathcal{T}_{\psi_1}$ for all $\psi_1 \in L^\infty(\mathbb{C}_+)$. Hence

$$(W^{-1}\underline{M}_a W)(W^{-1}\mathcal{T}_{\psi_1} W)(W^{-1}\underline{G}_a W) = W^{-1}\mathcal{T}_{\psi_1} W \tag{4}$$

for all $\psi_1 \in L^\infty(\mathbb{C}_+)$. Let $W^{-1}\underline{M}_a W = M_a$ and $W^{-1}\underline{G}_a W = G_a$. From Lemma 2.5 and from (4), it follows that $M_a T_{\psi_1 \circ M} G_a = T_{\psi_1 \circ M}$ for all $\psi_1 \in L^\infty(\mathbb{C}_+)$ where $Ms = \frac{1-s}{1+s}$. That is, $M_a T_\psi G_a = T_\psi$ for all $\psi \in L^\infty(\mathbb{D})$. We shall show that G_a commutes with the Bergman shift operator T_z defined by $T_z f = zf$ on $L^2_a(\mathbb{D})$. Suppose on the contrary that there is a nonzero vector f in $\mathrm{Ran}(G_a T_z - T_z G_a)$. Now $M_a T_\psi G_a T_z = T_\psi T_z = T_{\psi z} = M_a T_{\psi z} G_a = M_a T_\psi T_z G_a$. Hence, $M_a T_\psi (G_a T_z - T_z G_a) = 0$ for all $\psi \in L^\infty(\mathbb{D})$. So Ker $M_a \supseteq E$ where $E = \{T_\psi f : \psi \in L^\infty(\mathbb{D})\}$. Suppose there is a vector $g \in L^2_a(\mathbb{D})$ such that g is orthogonal to the set E. Then

$$\begin{aligned}\int_{\mathbb{D}} g(z)\overline{\psi(z) f(z)}\, dA(z) &= \langle g, \psi f\rangle \\ &= \langle g, P(\psi f)\rangle \\ &= \langle g, T_\psi f\rangle = 0,\end{aligned}$$

for all $\psi \in L^\infty(\mathbb{D})$. As $\overline{f} g \in L^1(\mathbb{D})$, we obtain $\overline{f} g = 0$. This implies at least one of the analytic functions f, g is identically zero. But we have assumed that $f \neq 0$. Thus g must be the zero function. It follows therefore that $\overline{E} = L^2_a(\mathbb{D})$. Now since $E \subseteq \mathrm{Ker} M_a$, we have $M_a = 0$. Hence $T_\psi = M_a T_\psi G_a = 0$ for all $\psi \in L^\infty(\mathbb{D})$ which is not true. Thus G_a commutes with T_z. Now let $G_a 1 = g \in L^2_a(\mathbb{D})$. Then $G_a z^n = G_a T_z^n 1 = T_z^n G_a 1 = z^n g$ for all $n \geq 0$ and therefore, $G_a p = gp$ for all polynomials $p(z)$. Since polynomials are dense [10] in $L^2_a(\mathbb{D})$, for $f \in L^2_a(\mathbb{D})$, take a sequence $\{p_n\}$ of polynomials converging to f in $L^2_a(\mathbb{D}, dA)$. This implies $G_a p_n \to G_a f$ in norm and $\langle p_n, k_z\rangle \to \langle f, k_z\rangle$ for all $z \in \mathbb{D}$. That is, $p_n(z) \to f(z)$ and $(G_a p_n)(z) \to (G_a f)(z)$ for all $z \in \mathbb{D}$. On the other hand, $(G_a p_n)(z) = (p_n g)(z) = p_n(z) g(z) \to f(z) g(z)$, for all $z \in \mathbb{D}$. Thus $G_a f = gf$ for all $f \in L^2_a(\mathbb{D})$; i.e., G_a is the operator of multiplication by $g \in L^2_a(\mathbb{D})$. Now $M_a T_\psi G_a = T_\psi$ for all $\psi \in L^\infty(\mathbb{D})$ implies $G_a^* T_{\overline{\psi}} M_a^* = T_{\overline{\psi}}$ for all $\psi \in L^\infty(\mathbb{D})$; thus, we can deduce in the same way that M_a^* is the operator of multiplication by some $\phi \in L^2_a(\mathbb{D})$. Hence $M_a = T_{\overline{\phi}}$ and we have $T_{\overline{\phi}} T_\psi T_g = T_\psi$ for all $\psi \in L^\infty(\mathbb{D})$. Taking $\psi \equiv 1$, we obtain $T_{\overline{\phi} g} = T_1 = I_{\mathcal{L}(L^2_a(\mathbb{D}))}$. For non-negative integers m and r, the functions z^m and z^r belong to $L^2_a(\mathbb{D})$. Since $T_{\overline{\phi} g} = I_{\mathcal{L}(L^2_a(\mathbb{D}))}$, we get

$$\begin{aligned}\int_{\mathbb{D}} z^m\ \overline{z}^r\ \overline{\phi(z)} g(z) dA(z) &= \langle \overline{\phi} g z^m, z^r \rangle \\ &= \langle z^m, z^r \rangle \\ &= \int_{\mathbb{D}} z^m \overline{z}^r dA(z).\end{aligned}$$

This implies that the measure $d\gamma(z) = (\overline{\phi}(z)q(z) - 1)dA(z)$ on $\mathbb{D}$ is annihilated by all monomials $z^m\overline{z}^r, m, r \geq 0$. By linearity and Stone-Weierstrass theorem it is annihilated by all functions in $C(\overline{\mathbb{D}})$, and so is the zero measure and necessarily $\overline{\phi}q = 1$ on $\mathbb{D}$. But this means that the functions $\overline{\phi} = \frac{1}{q}$ is both analytic and anti-analytic, and so must be constant. Hence $M_a = T_{\overline{\phi}} = \alpha I_{\mathcal{L}(L_a^2(\mathbb{D}))}$ and $G_a = T_q = \frac{1}{\alpha} I_{\mathcal{L}(L_a^2(\mathbb{D}))} = \beta I_{\mathcal{L}(L_a^2(\mathbb{D}))}$ where $\beta = \frac{1}{\alpha}, \alpha, \beta \in \mathbb{C}$. Thus $M_{\underline{a}} = \alpha I_{\mathcal{L}(L_a^2(\mathbb{C}_+))}$ and $\underline{G}_a = \beta I_{\mathcal{L}(L_a^2(\mathbb{C}_+))}$ where $\alpha\beta = 1$ and $R = \alpha V_a$ and $S = \beta V_a$. This completes the proof. □

Theorem 3.2. *Let* $\Phi_0 : \mathfrak{A} \to \mathcal{L}(L_a^2(\mathbb{C}_+))$ *be a linear isometry such that, for each pair of vectors* $f, g \in L_a^2(\mathbb{C}_+)$,

$$\sup\{|\langle \Phi_0(A)f, g\rangle| : A \in \mathfrak{A} \textit{ and } ||A|| = 1\} = ||f||\ ||g||. \tag{5}$$

Then their exits a unique extension of Φ_0 *to a linear isometry,* Φ, *mapping* $\mathcal{L}(L_a^2(\mathbb{C}_+))$ *into itself.*

Proof. Let $f, g \in L_a^2(\mathbb{C}_+)$. Define linear functional $w_{f,g}^0$ on $\mathfrak{A}$ by $w_{f,g}^0(A) = \langle \Phi_0(A)f, g\rangle$. From (5), we obtain $||w_{f,g}^0|| = ||f||\ ||g||$. By the Hahn-Banach theorem [2], there exists an extension, $w_{f,g}$ of $w_{f,g}^0$ to $\mathcal{L}(L_a^2(\mathbb{C}_+))$ such that $||w_{f,g}|| = ||f||\ ||g||$. Further, by [3], it follows that $w_{f,g}$ has a unique decopmosition $w_{f,g} = L_1 + L_2$, where L_1 is ultraweakly continuous; $L_2\big|_{\mathcal{LC}(L_a^2(\mathbb{C}_+))} = 0$; and $||w_{f,g}|| = ||L_1|| + ||L_2||$. Since $\big|\ ||w_{f,g}||\ \big| = ||\ w_{f,g}\big|_{\mathcal{LC}(L_a^2(\mathbb{C}_+))}|| = ||L_1||$, we obtain $L_2 = 0$. That is, $w_{f,g}$ is ultrawekly continuous. Now, to prove the uniqueness of extensions, let Φ_1 and Φ_2 be two linear isometries that are extensions of Φ_0 to $\mathcal{L}(L_a^2(\mathbb{C}_+))$. Then the linear functionals $T \to \langle \Phi_1(T)f, g\rangle$ and $T \to \langle \Phi_2(T)f, g\rangle$ are both ultraweakly continuous and agree on $\mathfrak{A}$. From Theorem 2.1 and by Theorem 1.1, it follows that the Toeplitz algebra $\mathfrak{A}$ is ultraweakly dense in $\mathcal{L}(L_a^2(\mathbb{C}_+))$. Thus we obtain $\langle \Phi_1(T)f, g\rangle = \langle \Phi_2(T)f, g\rangle$ for all $f, g \in L_a^2(\mathbb{C}_+)$. Hence $\Phi_1 = \Phi_2$. Notice that the linear functional $w_{f,g}$ is ultraweakly continuous. Now for each $T \in \mathcal{L}(L_a^2(\mathbb{C}_+))$, define $\Omega_T : L_a^2(\mathbb{C}_+) \times L_a^2(\mathbb{C}_+) \to \mathbb{C}$ as $\Omega_T(f, g) = w_{f,g}(T)$. The map Ω_T is a bilinear form on $L_a^2(\mathbb{C}_+)$ bounded by $||T||$. Thus by [1], there exists an operator $\Phi(T)$ in

$\mathcal{L}(L^2_a(\mathbb{C}_+))$ such that $\Omega_T(f,g) = w_{f,g}(T) = \langle \Phi(T)f, g\rangle$, for all $f, g \in L^2_a(\mathbb{C}_+)$, and $||\Phi(T)|| \leq ||T||$. The map Φ is linear and that $\Phi\big|_{\mathfrak{A}} = \Phi_0$. We shall now prove that Φ is an isometry. Let P_n denote the orthogonal projection from $L^2_a(\mathbb{C}_+)$ onto the span of $\{\epsilon_1, \epsilon_2, \cdots \epsilon_n\}$ where $\{\epsilon_i\}_{i=0}^{\infty}$ forms an orthonormal basis for $L^2_a(\mathbb{C}_+)$. It is easy to see that $P_n \to \alpha I_{\mathcal{L}(L^2_a(\mathbb{C}_+))}$ in the strong operator topology. Let $T \in \mathcal{L}(L^2_a(\mathbb{C}_+))$ and let $T_n = TP_n$ for each $n \in \mathbb{N}$. Then $T_n \in \mathfrak{A}$ for all $n \in \mathbb{N}$ as the operators T_n are of finite-rank and therefore compact and $\mathcal{LC}(L^2_a(\mathbb{C}_+)) \subset \mathfrak{A}$. Again, $T_n \to T$ in the strong operator topology as $P_n \to \alpha I_{\mathcal{L}(L^2_a(\mathbb{C}_+))}$ in the strong operator topology. Further, for $f, g \in L^2_a(\mathbb{C}_+)$,

$$\langle \Phi(T)f, g\rangle = w_{f,g}(T) = \lim_{n\to\infty} w_{f,g}(T_n) = \lim_{n\to\infty} \langle \Phi_0(T_n)f, g\rangle.$$

Hence, given $\epsilon > 0$, there exist unit vectors f and g in $L^2_a(\mathbb{C}_+)$ such that $|\langle \Phi(T)f, g\rangle| > ||\Phi_0(T_n)|| - \epsilon = ||T_n|| - \epsilon$. Thus, $||\Phi(T)|| \geq ||T_n||$, for all n. Since $||T_n|| \to ||T||$, we have $||\Phi(T)|| = ||T||$, for all $T \in \mathcal{L}(L^2_a(\mathbb{C}_+))$ and Φ is an isometry. □

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