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FROUDE VERSUS FROUDE IN FISH LADDER DESIGN

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Rezumat. *Modelarea hidrodinamică se bazează pe teoria similitudinii și analiza dimensională. Complexitatea curgerilor în configurații geometrice și medii de lucru diferite necesită folosirea criteriilor de similitudine specifice atât în construcția unui model fizic cât și a unui model de simulare numerică. De aceea, exprimarea unui criteriu de similitudine este importantă. În modelarea curgerilor cu suprafață liberă se folosește criteriul de similitudine Froude. Prin definiție, ca raport între forța de inerție și forța masică de greutate, numărul Froude este dependent de pătratul vitezei. Și totuși sunt situații în care se preferă expresia numărului Froude ca dependență liniară de viteză. Lucrarea arată că modul de exprimare a numărului Froude este important mai ales atunci când dimensionăm o scară de pești.*

Abstract. *Hydrodynamic modelling is based on the similitude theory and dimensional analysis. The complexity of flows in different geometric environments and fluids requires the use of the similarity criteria in both the construction of a physical model and a numerical simulation model. That is why expressing a criterion of similarity is important. In modelling open channel flows, the Froude similarity criterion is used. By definition, as a ratio between the force of the inertia and the weight mass, the Froude number is dependent on the square of the velocity. And yet there are situations in which the Froude number expression is preferred as linear speed dependence. The paper shows that the way of expressing the Froude number is especially important when we scale and design a fish ladder or fish passage.*

Keywords: the Froude number, hydrodynamic modelling, fish ladder.

1. Introduction

The meaning of “Froude versus Froude” is based on the definition of the Froude number and the various formulas used for the mathematical expression of the Fr (Froude number).

The Froude number is a dimensionless variable which is frequently used in the modelling of hydraulic phenomena encountered not only in free-flowing flows, as we are accustomed to in hydraulics or hydropower, but also in other areas of research.

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Thus, the aim of this paper is to present the different expressions of the Froude number, correlated with the application that uses it.

On the other hand, it shows the influence of the Froude expression in various formulas used in the design of open channels, where it is no longer indifferent whether the variation of the Froude number is linear or parabolic, as in hydraulic modelling. The sizing example refers to the hydraulic jump and implicitly to the dimensioning of fish ladders.

William Froude (1810-1879) was an English engineer and naval architect who studied ships' stability and hull ship shape optimization. His theoretical and experimental researches allowed him to establish a formula bearing his name, through which the test results of the model (the small-scale size of the prototype) could be used to predict the behaviour of the full-sized ships (the prototype). The method is used even today in hydraulic and aerodynamic modelling [1, 2].

The Froude number, Fr, is defined as the ratio between the inertial force (F_i) and the gravity (F_g). The expression of the inertial force is approximated as:

$$F_i = \rho \frac{dV}{dx} \frac{dx}{dt} L^3 \propto \rho V^2 L^2 , \tag{1}$$

where: ρ is the fluid density, V the characteristic velocity, L the characteristic length, x the linear dimension and t is the time.

Also, the gravity force is:

$$F_g = \rho L^3 g \tag{2}$$

with g the acceleration of the gravity.

Hence, the Froude number is given by definition as:

$$\mathrm{Fr} = \frac{V^2}{gL} . \tag{3}$$

"Often" the Froude number is used as the ratio:

$$\mathrm{Fr} = \frac{V}{\sqrt{gL}} , \tag{4}$$

which is the square root of the Froude given in (3).

But the Froude is a dimensionless variable which is frequently used in the study of hydraulic phenomena of a free surface flow.

In this case, for the specific energy in a cross-section, the Froude number will represent the ration between the double of the kinetic specific energy of the flow and the potential specific energy for a medium depth, h_m:

$$\mathrm{Fr} = \frac{\alpha V^2}{g h_m} , \tag{5}$$

where: α is the Coriolis coefficient and depends on the velocity distribution in the channel cross-section [3]. Generally, α is greater than unity, but can be equal to unity when the flow is uniform across the section [4]. In these circumstances, the Froude number expressed as in equation (3) is the same as in equation (5) if h_m is the characteristic length as L, too.

2. Hydraulic Modelling and the Froude Number

It is well known that in hydraulics, as in many others fields of physical sciences, the experiment needs to be modelled in the laboratory, when the phenomena are complex and depend on many variables and the parameters dimensionless numbers are used. For example, the Navier-Stokes equations can be normalized using a standard procedure [5] and the dimensionless parameters of the Navier-Stokes equations are revealed (the Euler number, the Froude number, the Reynolds number and the Strouhal number).

The model-prototype similarity of free-surface flows phenomena is obtained using the Froude similitude because the gravity effects are predominant. Considering the Froude scaling ($Fr_\mathrm{p} = Fr_\mathrm{m}$) by using equation (3) or (4) the same result will be obtained [5]:

$$\frac{V_p}{V_m} = \sqrt{\bar{\lambda}} \quad \text{or} \quad \left(\frac{V_p}{V_m}\right)^2 = \bar{\lambda} \tag{6}$$

where V_p and V_m are the prototype speed and the model speed, and the geometrical scale is $\bar{\lambda} = \frac{L_p}{L_m}$. The subscript letters p and m are for the prototype (full-scale) and model; the significance of L is the same as in equation (1).

The scaling factors using the Froude similarity are given in Table 1.

Table 1. Froude scaling factors [6, 7]

No.	*Property*	*Scaling factor*
1	Velocity (V)	$\sqrt{\bar{\lambda}}$
2	Time (t)	$\sqrt{\bar{\lambda}}$
3	Volumetric flow (Q)	$\bar{\lambda}^{5/2}$
4	Area (A)	$\bar{\lambda}^{2}$

In free-surface flows, the Froude similarity is used “when friction losses are small and the flow is highly turbulent” [6]. Spillways, flow over weirs, flow past bridge piers, waves, hydraulic jump phenomena, ship models are some examples of the Froude number modelling.

In this paragraph, relation (6) shows us that the way we express the Froude does not matter in the similitude procedure.

3. The Froude Number Expression in Different Research Topics

The Froude number is expressed by different formulas depending on the field of research and subject (e.g. shallow waves, wind engineering, geophysical mass flows, agitated tanks, meteorology, hydrology, anthropology and the examples may continue). Some of these examples are briefly presented below.

In the shallow water theory, the Froude number is expressed as the ration between the stream velocity, *V*, and the wave velocity, *c*, or, as it is frequently named, the wave celerity:

$$\mathrm{Fr} = \frac{V}{c}. \tag{7}$$

The wave celerity is given by $c = \frac{L}{T}$ and from the small-amplitude wave theory [8]:

$$c = \frac{gT}{2\pi} \tanh\left(\frac{2\pi h}{L}\right), \tag{8}$$

where *T* is the wave period, *L* the wavelength, *H* the height of the wave and *h* the water depth,

Fig.1. Shallow water with $h/L \leq 0.04$, the celerity is determined by the depth not by the wave period, $c = \sqrt{gh}$, as in the case of deep water waves with $h/L \geq 0.5$ with $c = \frac{gT}{2\pi}$ [8].

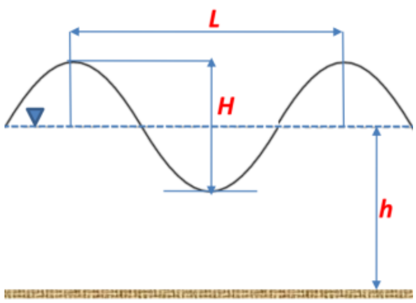


Fig. 1. Sketch of a sinusoidal wave.

In the case of lee waves, the Froude number calculated with (7) cannot always reflect physical phenomena. Lee waves are generated by the flow of mesoscale currents in the ocean or in the atmosphere when the mass of fluid passes over a rough topography [9, 10]. The phenomenon of lee waves depends on the Brünt-Väisäla (buoyancy) frequency and the note with N and is recommended [11] to calculate the Froude number for the ocean lee waves as:

$$\mathrm{Fr}_{lee} = \frac{Nh_0}{V}, \tag{9}$$

where: N is the buoyancy frequency, h_0 the height of the background topography, V the background velocity.

In meteorology, the Froude number is calculated as:

$$\mathrm{Fr} = \frac{V}{Nh_0}, \tag{10}$$

where: N is the buoyancy frequency, h_0 the obstacle height (e.g. mountain), V the wind velocity. The same relation for the Froude number is used in the case of stratified flows over obstacles [12].

We can observe that relation (10) is in accordance with (7), while (10) is (9) inverted because "for lee waves generated by abyssal hills at the bottom of the ocean" the Froude number calculated with (10) and, having the parameters significance from (9), is very small [11].

Therefore, the Froude number expression takes into account the peculiarities of the studied phenomenon, keeping the physical principle of the dimensionless quantity, in the case of the sedimentary bedforms (in channel morphology or costal engineering) which depend on the flow regime. In the study of sediments flow, the densimetric Froude number is used and expressed as:

$$\mathrm{Fr} = \frac{V}{\sqrt{\dfrac{g\,h\,\Delta\rho}{\rho}}}, \tag{11}$$

where: V is the mean velocity, h the water depth, $\Delta\rho$ is the bulk density contrast and ρ the density.

The density contrast is used for the two-phase flow with different densities (e.g. water-sand [13, 14], or water-mist [15] where the physical phenomenon depends on the Froude number).

Even in biology the concept of the "Froude's Law" was applied for the first time by D'Arcy Wentworth Thompson in his *magnus opus* "On Grows and Form" [16] where he noticed the correspondence of animal speed with dimensional size, $V \propto \sqrt{\lambda}$, see Table 1.

In anthropology [17], the Froude number is used in animal gait study. Robert McNeil Alexander was the one who applied the Froude number to animal locomotion. He used the Froude number as in relation (3) where L is the hip height above the ground. Also, he demonstrated that between animals with dynamic similarity there is a geometrical similarity for the same Froude number. In Figure 2, we can observe that, if the Froude number is the unity (and Fr^2 =1), the rhinos will have the relative stride length in the vicinity of humans and camels [18].

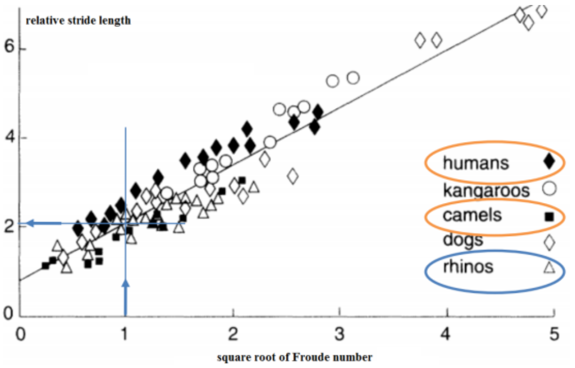


Fig. 2. Alexander's graph of relative stride length vs. square root of Froude; (relative stride length = stride length/leg length) [18].

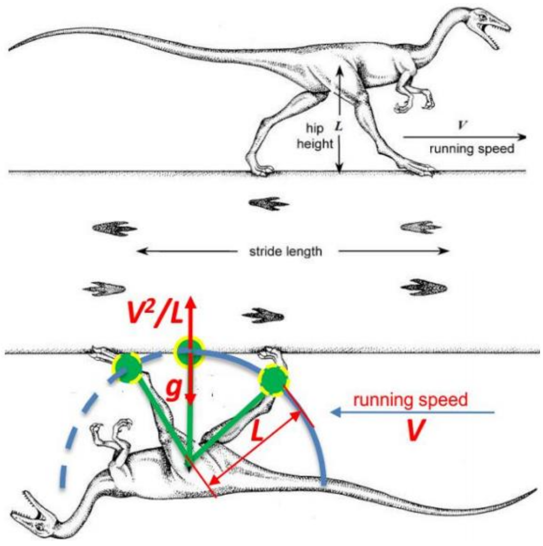


Fig. 3. Inverse pendulum sketch [17, 19].

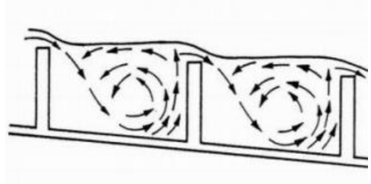
The Froude number applied in anthropology is, in fact, the ratio between the kinetic energy and the potential energy of the same mass of the body *m* which oscillates at the end of the rigid segment representing the leg of length *L*. In fact, it is the model of the inverse pendulum, Fig. 3 [19]. The Froude number equal to 1 (Fr = 1) is the transition between walking and running.

We should note that the Froude number expressed as in relation (4) is named the dimensionless speed and is marked with β [17].

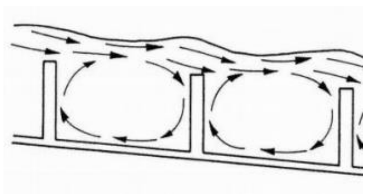
4. The Fish Ladder Design and the Froude

A fish ladder (or fishway, or fish passage) is a river construction consisting of a series of pools in steps and with water flowing from pool to pool, Fig. 4. The aim of a fish ladder is to facilitate and restore the migration of fish and biologic continuity along the river when the river cross-section is blocked by other human utilities (e.g. hydropower plant). The fish ascend the fish ladder by jumping or swimming from pool to pool in both directions along the river [20]. In fish ladders, the classification of the two types is distinguished: the weir type fish ladder (the pools are separated by weirs) and the orifice type fish ladder (the pools are separated by walls having a submerged orifice on the bottom side of the wall) [20, 21].

Fig. 4. Weir type fish ladder at Lopwell Dam, https://upload.wikimedia.org/.



a)



b)

Fig. 5. Plunging flow (a); Streaming flow (b).

A fish ladder is a dissipater of energy and good hydraulic conditions in the benefit of fish are obtained by means of a plunging flow regime [20, 22]. In this case the surface layers in the pool are uniformly mixing and the quality of water is better than that of a streaming flow regime, Fig. 5.

The flow on the fish ladder is under the effect of gravity and the hydraulic model associated will conform to the Froude similarity with the Froude number expressed by relations (3) or (4). If the Froude number is equal to the unity, the flow is in a critical flow state; if the Froude number is less than the unity, the flow is subcritical and if the Froude number is more than the unity, the flow is in supercritical state.

In supercritical flows, the inertial forces become dominant and the flow is rapidly varied [1, 23]. For such a flow regime, the hydraulic jump phenomenon is easily obtained. A hydraulic jump occurs when a supercritical flow meets a subcritical flow. The downstream water level will rise as the result of the transition from an unstable flow to a stable one.

The classification of the hydraulic jump is based on the Froude number calculated with the parameters of the initial supercritical stream (see section 1, Fig. 6).

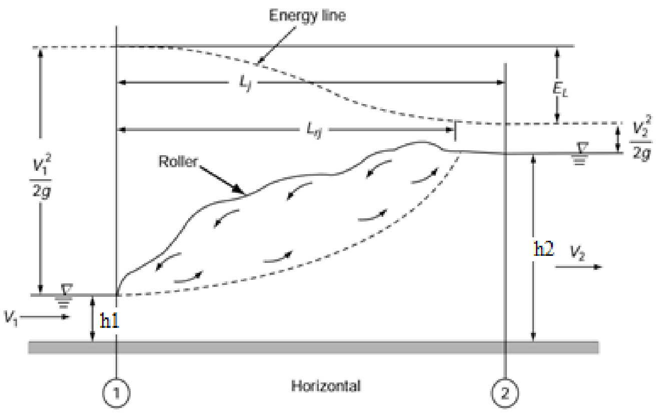


Fig. 6. Hydraulic jump, sketch representation [23].

Table 2. Hydraulic jump classification

		$Fr_1 = \frac{V_1}{\sqrt{g y_1}}$	$Fr_1 = \frac{V_1^2}{g h_1}$
1.	Critical Regime	1	1
2.	Undular Jump	$1.0 < Fr_1 \leq 1.7$	$1.0 < Fr_1 \leq 2.89$
3.	Weak Jump	$1.7 < Fr_1 \leq 2.5$	$2.89 < Fr_1 \leq 6.25$
4.	Oscillating Jump	$2.5 < Fr_1 \leq 4.5$	$6.25 < Fr_1 \leq 20.25$
5.	'Steady' Jump	$4.5 < Fr_1 \leq 9.0$	$20.25 < Fr_1 \leq 81.0$
6.	Strong or Choppy Jump	$Fr_1 > 9.0$	$Fr_1 > 81$

In Table 2, we give the Froude number limits for each type of hydraulic jump, highlighting that the expression of the Froude number formula is very important. "Often", relation (4) is given in most cases of bibliographic references, but in the table, we present the limits of the Froude number calculated with (3), too.

Based on relations (3) and (4), the Froude number was plotted when the characteristic length is considered constant and the velocity varies, Fig. 7, and in Figure 8 the velocity is constant and *L* varies.

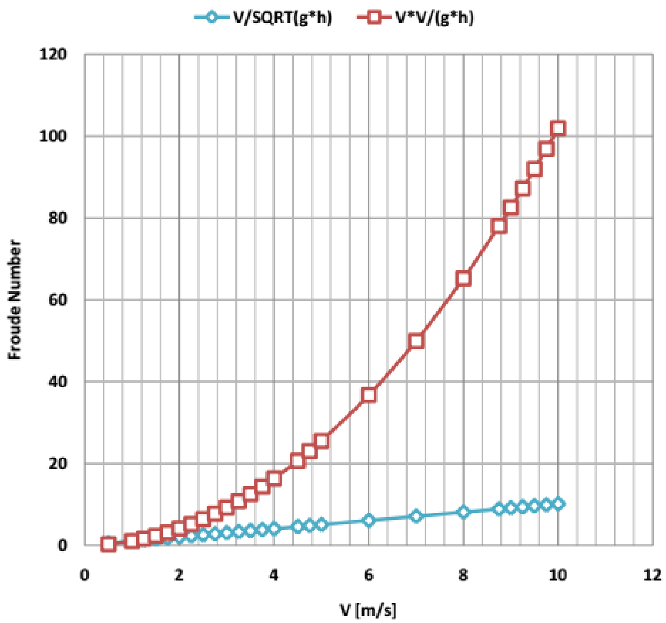


Fig.7. Froude number vs. velocity with constant characteristic length *L*, relation (3) and (4).

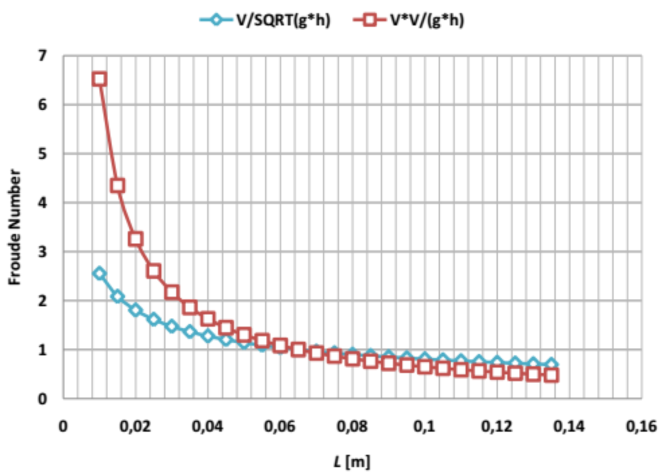


Fig. 8. Froude number vs. characteristic length *L* with constant velocity, relation (3) and (4).

We note that in the first case (Fig. 7), the Froude number has a parabolic evolution with respect to (4) and the hydraulic jump phenomena is preserved with the same significance as in Table 2, while in the second case (Fig. 8) the differences between the Froude calculated with relation (3) or (4) are insignificant, if the flow regime is subcritical.

The design of a fish ladder takes into account the flow conditions and the plunging flow regime. Consequently, the length of the hydraulic jump and the sequent depths will be calculated [24, 25].

From this point of view, it is very important how the Froude number is calculated because the sizing formulae used depend on the Froude.

The sequent depth at the exit of the jump is calculated as:

$$h_2 = 0.5 h_1 \cdot \left(\sqrt{1 + 8 \cdot \mathrm{Fr}_1} - 1\right), \tag{12}$$

if the Froude number is calculated with formula (3) and

$$h_2 = 0.5 h_1 \cdot \left(\sqrt{1 + 8 \cdot \mathrm{Fr}_1^2} - 1\right), \tag{13}$$

if the Froude is calculated with (4).

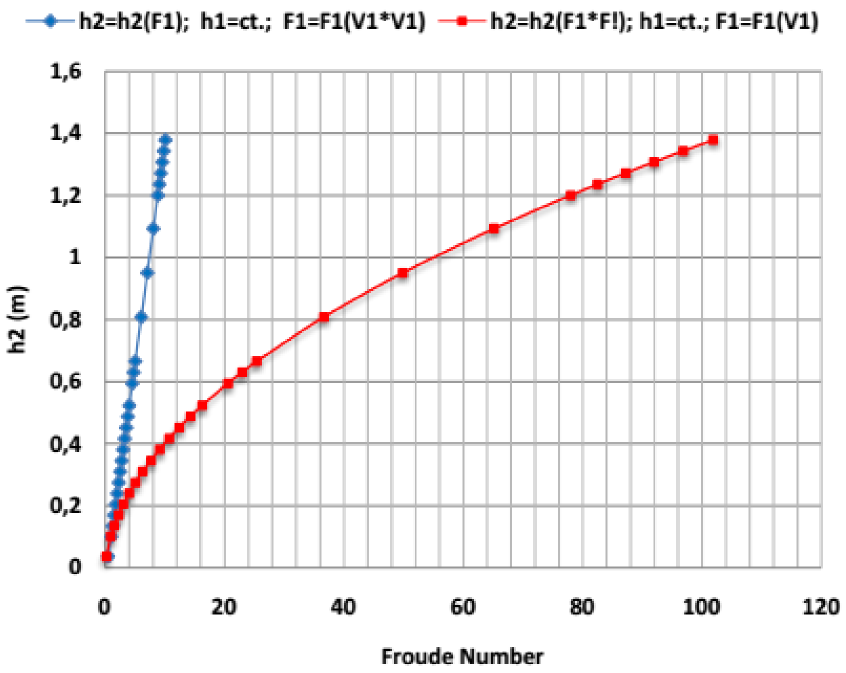


Fig. 9. Sequent depths in hydraulic jump with h_1 = ct. and h_2 calculated with (12) associated with (3) and with (13) associated with (4) for Froude number.

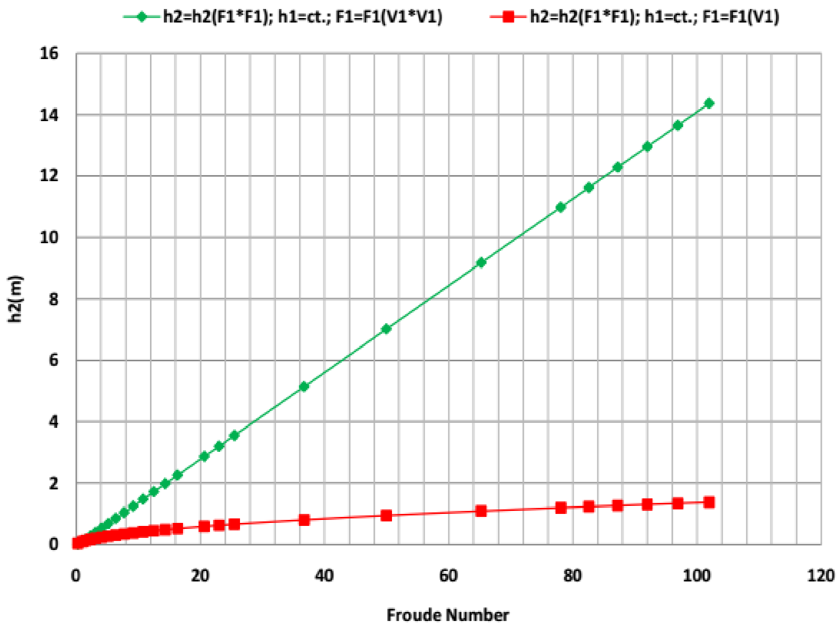


Fig. 10. Sequent depths in hydraulic jump with h_1 = ct. and h_2 calculated with (13) associated with (3) and with (13) associated with (4) for Froude number.

In Figure 9, the sequent depth is plotted as the function of the Froude number using the formulae written above and considering the depth at the entrance of the hydraulic jump h_1 to be constant. The h_2 values obtained are comparable.

In Figure 10, we give an example regarding the size of h_2 when it is calculated with (13) and the Froude number is given by (3), instead of (4) as it is right. The results are compared but not comparable because, as it is natural, the depth of h_2 is bigger than in the case of formula (13) associated with (4), for the Froude number.

Final Remarks

The Froude number continues to amaze us even today, 150 years after its definition by William Froude. The Froude number can be used in the most unexpected research fields such as anthropology or robots industry. The paper highlights how the Froude was defined and used in the similarity analysis and other research topics, too. It also shows the need to use the Froude number expression in correspondence with the studied phenomenon. The difference between the linear variation of the Froude number according to the speed and compared with the parabolic Froude number variation is plotted even if the Froude number expression is obvious, in relations (3) and (4).

We emphasized the need to specify the calculation formula for the Froude number in any problem of sizing or graphical representation, whenever we wish to refer to the results from the literature.

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