Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 9, No. 2/2017

ON A DECOMPOSITION OF AUGMENTED MONOMIAL SYMETRIC FUNCTIONS*

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Abstract

We consider a recent result for expanding augmented monomial symmetric functions in terms of the power sum symmetric functions to illustrate a technique for proving and generating inequalities involving specializations of monomial symmetric functions.

MSC: 05E05, 11P81

keywords: symmetric functions, integer partitions, inequalities.

1 Introduction

Any positive integer n can be written as a sum of one or more positive integers, i.e.,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_r \ . \tag{1}$$

When the order of integers λ_i does not matter, this representation is known as an integer partition [1] and can be rewritten as

$$n = t_1 + 2t_2 + \cdots + nt_n \ ,$$

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where each positive integer i appears t_i times. In order to indicate that

$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r] \qquad \text{or} \qquad \lambda = [1^{t_1} 2^{t_2} \ldots n^{t_n}]$$

is an integer partition of n, we use the notation $\lambda \vdash n$.

Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ be a partition with $k \leqslant n$. Being given a set of variables $\{x_1, x_2, \ldots, x_n\}$, the monomial symmetric function

$$m_\lambda = m_{[\lambda_1, \lambda_2, \ldots, \lambda_k]}(x_1, x_2, \ldots, x_n)$$

on these variables is the sum of monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ and all distinct monomials obtained from it by a permutation of variables. For instance, with $\lambda = [2, 1, 1]$ and $n = 4$, we have:

$$\begin{aligned} m_{[2,1,1]} = {} & x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 + x_1^2 x_2 x_4 + x_1 x_2^2 x_4 + x_1 x_2 x_4^2 \\ & + x_1^2 x_3 x_4 + x_1 x_3^2 x_4 + x_1 x_3 x_4^2 + x_2^2 x_3 x_4 + x_2 x_3^2 x_4 + x_2 x_3 x_4^2 \ . \end{aligned}$$

In particular, when $\lambda = [k]$, we have the kth power sum symmetric function $p_k = p_k(x_1, x_2, \ldots, x_n)$, i.e.,

$$m_{[k]} = p_k = \sum_{i=1}^{n} x_i^k \ .$$

In every case $p_0(x_1, x_2, \ldots, x_n) = n$. Proofs and details about monomial symmetric functions can be found in Macdonald's book [2].

For each partition

$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k] = [1^{t_1} 2^{t_2} \cdots r^{t_r}] \ ,$$

with $k \leqslant n$, the augmented monomial symmetric function

$$\tilde{m}_\lambda = \tilde{m}_{[\lambda_1, \lambda_2, \ldots, \lambda_k]}(x_1, x_2, \ldots, x_n)$$

is defined by

$$\tilde{m}_\lambda = t_1! t_2! \cdots t_r! \cdot m_\lambda \ .$$

Recently, we introduced in [4, Theorem 1] a simple recursive formula for the expansion of the augmented monomial symmetric functions into power sum symmetric functions, i.e.,

$$\tilde{m}_{[\lambda_1, \lambda_2, \ldots, \lambda_k]} = p_{\lambda_k} \cdot \tilde{m}_{[\lambda_1, \lambda_2, \ldots, \lambda_{k-1}]} - \sum_{i=1}^{k-1} \tilde{m}_{[\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + \lambda_k, \lambda_{i+1}, \ldots, \lambda_{k-1}]} \ , \quad (2)$$

where $\tilde{m}$ and p are functions of n variables, $n \geqslant k$.

Example 1. *Replacing k by 2 in (2), we get*

$$\tilde{m}_{[\lambda_1,\lambda_2]} = p_{\lambda_1} p_{\lambda_2} - p_{\lambda_1+\lambda_2} \ . \tag{3}$$

Then, for $k = 3$, we obtain

$$\tilde{m}_{[\lambda_1,\lambda_2,\lambda_3]} = p_{\lambda_3} \cdot \tilde{m}_{[\lambda_1,\lambda_2]} - \tilde{m}_{[\lambda_1+\lambda_3,\lambda_2]} - \tilde{m}_{[\lambda_1,\lambda_2+\lambda_3]} \ . \tag{4}$$

By (3) and (4), we deduce that

$$\tilde{m}_{[\lambda_1,\lambda_2,\lambda_3]} = p_{\lambda_1} p_{\lambda_2} p_{\lambda_3} - p_{\lambda_1} p_{\lambda_2+\lambda_3} - p_{\lambda_2} p_{\lambda_1+\lambda_3} - p_{\lambda_3} p_{\lambda_1+\lambda_2} + 2p_{\lambda_1+\lambda_2+\lambda_3} \ .$$

In this paper, we consider (2) in order to derive some inequalities involving specializations of monomial symmetric functions.

2 Main results

In this section, we illustrate a technique for proving and generating inequalities involving specializations of monomial symmetric functions based on (2).

Theorem 1. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers, $n > 1$. If k, p and q are positive integers such that $k < n$, then*

$$\frac{n}{n-k} m_{[p,q^k]} \leqslant m_{[p]} m_{[q^k]} \leqslant \frac{n}{k} m_{[p+q,q^{k-1}]}, \quad p \neq q,$$

and

$$\frac{n(k+1)}{n-k} m_{[p^{k+1}]} \leqslant m_{[p]} m_{[p^k]} \leqslant \frac{n}{k} m_{[2p,p^{k-1}]},$$

where m are monomials symmetric functions in $x_1, x_2, \ldots, x_n$.

Proof. We start with the following identity

$$\tilde{m}_{[p,q^k]} = \tilde{m}_{[p]} \tilde{m}_{[q^k]} - k \tilde{m}_{[p+q,q^{k-1}]},$$

that is a very special case of (2). This identity can be rewritten as

$$m_{[p,q^k]} + m_{[p+q,q^{k-1}]} = m_{[p]} m_{[q^k]}. \tag{5}$$

For $p \neq q$, according to Muirhead's inequality [3, p. 87], we have

$$\frac{m_{[p,q^k]}}{\binom{n}{1,k,n-k-1}} \leqslant \frac{m_{[p+q,q^{k-1}]}}{\binom{n}{1,k-1,n-k}}, \tag{6}$$

where the usual symbol for the multinomial coefficient has been used. On the other hand, the case $p = q$ is given by

$$\frac{m_{[p^{k+1}]}}{\binom{n}{k+1,n-k-1}} \leqslant \frac{m_{[2p,p^{k-1}]}}{\binom{n}{1,k-1,n-k}}. \tag{7}$$

By (5)-(7), we arrive at our inequalities. □

Corollary 1. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers, $n > 1$. Then*

$$\frac{n}{n-1} \sum_{1\leqslant i<j\leqslant n} \left(x_i^p x_j^q + x_i^q x_j^p\right) \leqslant \left(\sum_{i=1}^{n} x_i^p\right)\left(\sum_{i=1}^{n} x_i^q\right) \leqslant n \sum_{i=1}^{n} x_i^{p+q},$$

where p and q are positive integers.

This corollary is the case $k = 1$ in Theorem 1. In addition, for $p = q$ we obtain the following Maclaurin's inequality

$$\frac{m_{[1]}}{\binom{n}{1}} \geqslant \sqrt{\frac{m_{[1^2]}}{\binom{n}{2}}}$$

and the AM-QM inequality

$$\frac{m_{[1]}}{n} \leqslant \sqrt{\frac{m_{[2]}}{n}}.$$

Theorem 2. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers, $n > 2$. If k, p, q and r are positive integers such that $k < n-1$, $p > q$ and $r \notin \{p, q, p-q\}$, then*

1. $\frac{n-1}{k} m_{[r+q,p,q^{k-1}]} + m_{[r+p,q^k]} \geqslant m_{[r]} m_{[p,q^k]}$,
2. $m_{[r,p,q^k]} + (k+1) m_{[r+p,q^k]} \geqslant m_{[r]} m_{[p,q^k]}$,
3. $\frac{n-1}{n-k-1} m_{[r,p,q^k]} + m_{[r+p,q^k]} \leqslant m_{[r]} m_{[p,q^k]}$,
4. $m_{[r,p,q^k]} + \frac{k+1}{k} m_{[r+q,p,q^{k-1}]} \leqslant m_{[r]} m_{[p,q^k]}$,

where m are monomials symmetric functions in $x_1, x_2, \ldots, x_n$.

Theorem 3. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers, $n > 2$. If k, p, q and r are positive integers such that $k < n-1$, $p < q$ and $r \notin \{p, q, q-p\}$, then*

1. $m_{[r+q,p,q^{k-1}]} + (n-k)m_{[r+p,q^k]} \geqslant m_{[r]}m_{[p,q^k]}$,

2. $m_{[r,p,q^k]} + \frac{k+1}{k}m_{[r+q,p,q^{k-1}]} \geqslant m_{[r]}m_{[p,q^k]}$,

3. $\frac{n-k}{n-k-1}m_{[r,p,q^k]} + m_{[r+q,p,q^{k-1}]} \leqslant m_{[r]}m_{[p,q^k]}$,

4. $m_{[r,p,q^k]} + (k+1)m_{[r+p,q^k]} \leqslant m_{[r]}m_{[p,q^k]}$,

where m are monomials symmetric functions in $x_1, x_2, \ldots, x_n$.

Proof of Theorems 2 and 3. For $p \neq q$ and $r \notin \{p, q, |p-q|\}$, we consider the following case of (2)

$$\tilde{m}_{[r,p,q^k]} = \tilde{m}_{[r]}\tilde{m}_{[p,q^k]} - \tilde{m}_{[r+q,p,q^{k-1}]} - \tilde{m}_{[r+p,q^k]},$$

that can be written as

$$m_{[r,p,q^k]} + m_{[r+q,p,q^{k-1}]} + m_{[r+p,q^k]} = m_{[r]}m_{[p,q^k]}. \tag{8}$$

For $p > q$, taking into account the Muirhead inequality, we have

$$\frac{m_{[r,p,q^k]}}{\binom{n}{1,1,k,n-k-2}} \leqslant \frac{m_{[r+q,p,q^{k-1}]}}{\binom{n}{1,1,k-1,n-k-1}} \leqslant \frac{m_{[r+p,q^k]}}{\binom{n}{1,k,n-k-1}}. \tag{9}$$

In the case $p < q$, we get

$$\frac{m_{[r,p,q^k]}}{\binom{n}{1,1,k,n-k-2}} \leqslant \frac{m_{[r+p,q^k]}}{\binom{n}{1,k,n-k-1}} \leqslant \frac{m_{[r+q,p,q^{k-1}]}}{\binom{n}{1,1,k-1,n-k-1}}. \tag{10}$$

By (8)-(10), we derive our inequalities. □

The case $n = 3$ and $k = 1$ of Theorems 2 and 3 can be written as

Corollary 2. *Let a, b, c be positive real numbers. If p, q and r are positive integers such that $p < q$ and $r \notin \{p, q, q-p\}$, then*

1. $a^{r+p}(b^q + c^q) + b^{r+p}(a^q + c^q) + c^{r+p}(a^q + b^q)$
   $\geqslant a^p(b^r c^q + b^q c^r) + b^p(a^r c^q + a^q c^r) + c^p(a^r c^q + a^q c^r)$;

2. $(a^{r+p} + b^{r+p} + c^{r+p})(a^q + b^q + c^q) \leqslant (a^{r+q} + b^{r+q} + c^{r+q})(a^p + b^p + c^p)$.

3 Concluding remarks

A new technique for proving and generating inequalities has been introduced in this paper. Similar inequalities can be obtained if we consider other special cases of (2), for instance

$$\tilde{m}_{[p^2,q^k]} = \tilde{m}_{[p]}\tilde{m}_{[p,q^k]} - k\tilde{m}_{[p+q,p,q^{k-1}]} - \tilde{m}_{[2p,q^k]}, \qquad p \neq q,$$

that can be written as

$$2m_{[p^2,q^k]} + m_{[p+q,p,q^{k-1}]} + m_{[2p,q^k]} = m_{[p]}m_{[p,q^k]}, \qquad p \neq q.$$

In addition, one can show that Corollary 2 holds for any value of r.

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