Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 9, No. 2/2017

ON SPLIT EQUILIBRIUM PROBLEM, VARIATIONAL INEQUALITY PROBLEM AND FIXED POINT PROBLEM FOR MULTI-VALUED MAPPINGS*

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Abstract

In this paper, we propose an algorithm involving a step-size selected in such a way that its implementation does not require the computation or an estimate of the spectral radius. Using our algorithm we proved strong convergence theorem for common solution of a split equilibrium problem, a variational inequality problem and fixed point problem for multi-valued quasi-nonexpansive mappings in real Hilbert spaces. Our result generalizes some important and recent results in the literature.

MSC: 47H09; 47H10; 49J20; 49J40

keywords: Split equilibrium problem; variational inequality problem; fixed point problem; maximal monotone mapping; inverse strongly monotone; Multi-valued Quasi-nonexpansive mappings; Hilbert space.

*Accepted for publication in revised form on October 3-rd, 2017

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Paper written with financial Support of Department of Science and Technology - National Research Foundation Center of Excellence in Mathematical and Statistical Sciences

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1 Introduction

In this paper, we assume that H_1 and H_2 are real Hilbert spaces with inner product $\langle . \rangle$ and norm $\|.\|$. Let D be a nonempty closed and bounded subset of H_1. Let $f_n : D \to H_1$ be uniformly convergence sequence of contraction mappings. Then there exists a real numbers $\rho_n \in (0, 1)$ such that

$$\|f_n(x) - f_n(y)\| \leq \rho_n \|x - y\|, \quad \forall x, y \in D.$$

A subset C of H_1 is called proximinal if, for each $x \in H_1$, there exists $c \in C$ such that

$$\|c - x\| = \inf\{\|x - y\| : y \in C\} = d(x, C).$$

Let C be a closed convex and nonempty subset of H_1. It is well known that in a Hilbert space, closed and convex sets are Proximinal (see for example, [11, 29]). In the sequel, we denote by $CB(H_1)$ the collection of all nonempty, closed and bounded subsets of H_1.
We say that a mapping $T : C \to C$ is nonexpansive if, for all $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

The mapping T is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \quad \forall x, y \in C.$$

The mapping $T : H_1 \to H_1$ is said to be

(i) monotone if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x.y \in H_1;$$

(ii) α - strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|x - y\|, \quad \forall x, y \in H_1;$$

(iii) β- inverse strongly monotone, if there exists a constant $\beta > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \beta \|Tx - Ty\|^2, \quad \forall x, y \in H_1.$$

The Hausdorff metric $\mathcal{H}$ on $CB(H_1)$ is defined by

$$\mathcal{H}(A, B) := \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}, \quad \forall A, B \in CB(H),$$

where $d(x, A) := \inf_{y \in A} d(x, y)$.

Definition 1. *Let $S : H \to CB(H)$ be a multi-valued mapping. An element $p \in H$ is said to be a fixed point of S, if $p \in Sp$. The mapping S is said to be: (i) nonexpansive, if $\mathcal{H}(Sx, Sy) \leq \|x - y\|,\ \forall x, y \in H$;*
(ii) quasi-nonexpansive, if $Fix(S) \neq \emptyset$ and $\mathcal{H}(Sx, Sy) \leq \|x - p\|,\ \forall x \in H$, $p \in Fix(S)$.

A Fixed Point Problem (FPP, for short) for multi-valued quasi-nonexpansive mapping S is to find $x \in C$ such that

$$x \in Sx. \tag{1}$$

The solution set of FPP (1) is denoted by Fix(S). Fixed point theory for multi-valued mappings has many useful application in various fields, in particular, game theory and mathematical economics. Thus, it is natural to extend the known fixed point results for single-valued mappings to the setting of multi-valued mappings. Several authors have investigated the approximations of fixed point of multi-valued nonexpansive mappings in the literature (see, for example,[11, 12, 24, 28, 29, 32, 33, 34]).

A mapping P_C is said to be the metric projection of H_1 onto C if for every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \ \ \forall x, y, \in H_1. \tag{2}$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{3}$$

and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \quad \forall x \in H_1,\ y \in C. \tag{4}$$

More information on metric projection can also be found in ([10], section 3).

The Variational Inequality Problem (in short, VIP) is to find $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \quad \forall y \in C, \tag{5}$$

where $B : C \to H_1$ is a nonlinear mapping. The solution set of VIP (5) is denoted by Γ.

For solving the VIP in a finite-dimensional Euclidean space $\mathbb{R}^n$, Korpelevich [13] introduced an iterative method so-called extragradient method.

Motivated by the idea of Korpelevich extragradient method, Nadezhkina and Takahashi [21] introduced an iterative method for finding the common element of the set $\mathrm{Fix}(T) \cap \Gamma$ and proved the strong convergence theorem. For related works, see [15, 16, 22, 23, 25, 30, 31].

A set valued mapping $\mathcal{T} : H_1 \to 2^{H_1}$ is called monotone if for all $x, y \in H_1$, $u \in \mathcal{T}x$ and $v \in \mathcal{T}y$ imply $\langle x - y, u - v\rangle \geq 0$. A monotone mapping $\mathcal{T} : H_1 \to 2^{H_1}$ is maximal if the graph $G(\mathcal{T})$ of $\mathcal{T}$ is not properly contained in the graph of any other monotone mapping.

It is well known that a monotone mapping $\mathcal{T}$ is maximal if and only if for $(x, u) \times H_1 \times H_1$, $\langle x-y, u-v\rangle \geq 0$, for every $(y, v) \in G(\mathcal{T})$ implies $u \in \mathcal{T}x$. Let $B : C \to H_1$ be an inverse strongly monotone mapping and let $N_C x$ be the normal cone to C at $x \in C$, i.e., $N_C x := \{z \in H_1 : \langle y - x, z\rangle \geq 0,\ \forall\ y \in C\}$. Define

$$\mathcal{T}x = \begin{cases} Bx + N_C x, & \forall x \in C, \\ \emptyset, & \forall x \notin C. \end{cases}$$

Then $\mathcal{T}$ is maximal monotone and $0 \in \mathcal{T}x$ if and only if $x \in \Gamma$, see [21].

Further, it is easy to see that

$$x \in \Gamma \Longleftrightarrow x = P_C(x - \lambda Bx), \quad \lambda > 0.$$

For a bifunction $F : C \times C \to \mathbb{R}$, the Equilibrium Problem (in short, EP) is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{6}$$

This was introduced by Blum and Oettli [2]. The solution set of EP (6) is denoted by EP(F).

Combettes and Hirstoaga [9] introduced and studied an iterative method for finding the best approximation to the solution of the EP (6) and proved a strong convergence theorem. Subsequently, Takahashi and Takahashi [35] introduced another iterative scheme for finding the common element of the set $EP(F) \cap \mathrm{Fix}(T)$. Using the idea of Takahashi and Takahashi [35], Plubtieng and Punpaeng [27] introduced the general iterative method for finding the common element of the set $EP(F) \cap \mathrm{Fix}(T) \cap \Gamma$. Also Liu et al. [16] introduced and studied an iterative method, an extention of the viscosity approximation method, for finding the common element of the set $\cap_{i=1}^{\infty}\mathrm{Fix}(T_i) \cap EP(F) \cap \Gamma$, where $\{T_i\}_{i=1}^{\infty}$ is an infinite family of nonexpansive mappings.

Censor et al. [6] introduced and studied some iterative method for the following Split Variational Problem (in short, SVIP): Find $x^* \in C$ such that

$$\langle f(x^*), x - x^*\rangle \geq 0, \quad \forall \ \ x \in C, \tag{7}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } \langle g(y^*), y - y^* \rangle \geq 0, \quad \forall y \in Q, \tag{8}$$

where $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are nonlinear mapping and $A : H_1 \to H_2$ is a bounded linear operator. The special case of SVIP (7) and (8) is split zero problem and split feasibility problem which has already been studied and used in practice as a model in intensity-modulated radiation therapy treatment planning, see [7, 8]. Recently, Moudafi [20] introduced an iterative method, an extension of a method by Censor et al. [6] for the following split monotone variational inclusion:

$$\text{Find } \; x^* \in H_1 \; \text{ such that } \; 0 \in f(x^*) + B_1(x^*),$$

and such that

$$y^* = Ax^* \in H_2 \; \text{ solves } \; 0 \in g(y^*) + B_2(y^*),$$

where $B_i : H_i \to 2^{H_i}$ is a set-valued mapping for $i = 1, 2$. Later on Byrne et al. [4] generalize and extend the work of Censor et al. [6] and Moudafi [20].

Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonnilear bifunctions and $A : H_1 \to H_2$ be bounded linear operator, then the Split Equilibrium Problem (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{9}$$

and such that

$$y^* = Ax^* \in Q \; \text{ solves } \; F_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{10}$$

When looked upon separately, we observed that (9) is the classical Equilibrium Problem (EP) and we denote it's solution set by $EP(F_1)$. The SEP (9) and (10) constitute a pair of equilibrium problems which we have to solve so that the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution the EP (9) in H_1 is the solution set of EP (10) in another space H_2. The solution set of SEP (9) and (10) is denoted by

$$\Omega = \{p \in EP(F_1) : Ap \in EP(F_2)\}.$$

Motivated by the work of Censor et al.[5, 6], Moudafi [20], Byrne et al [4], Liu et al. [16], Kazmi and Rizvi [14] obtained the following converges result.

Theorem 1. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D : C \to H_1$ be a τ-inverse strongly monotone mapping. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are two bifunctions satisfying Assumption 2.1 and F_2 is upper semicontinuous in first argument. Let $S : C \to C$ be a nonexpansive mapping such that $\Theta := F(S) \cap \Omega \cap \Gamma \neq \emptyset$. For a given $x_0 = v \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} u_n = J_{r_n}^{F_1}(x_n + \gamma A^*(J_{r_n}^{F_2} - I)Ax_n); \\ y_n = P_C(u_n - \lambda_n D u_n); \\ x_{n+1} = \alpha_n v + \beta_n x_n + \gamma_n S y_n, \end{cases} \tag{11}$$

where $r_n \subset (0, \infty)$, $\lambda_n \in (0, 2\tau)$ and $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the following

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty}, \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\tau$, and $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(v) $\lim_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

(vi) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right) = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Theta$, where $z = P_\Theta v$.

Motivated by the work of Kazmi and Rizvi [14], we introduce an iterative scheme for approximating a common solution of SEP(9) and (10), VIP (5) and FPP (1) for multi-valued quasi-nonexpansive in real Hilbert space. Using our proposed algorithm we prove strong convergence theorem for approximating a common solution of SEP(9) and (10), VIP (5) and FPP (1) for multi-valued quasi-nonexpansive in real Hilbert. In all our results the variable step-size is selected in such a way that its implementation does not involve the computation or an estimate of the spectral radius.

2 Preliminaries

In this section, we give some definitions, lemmas and results that are needed in the main results. Also, we make the following assumptions on the bifunctions F in order to solve the equilibrium problem. [2] Let $F : C \times C \to \mathbb{R}$ be a bifunction.We assume that F satisfy the following:

(i) $F(x, x) = 0, \ \ \forall\ x \in C$;

(ii) F is monotone, i.e $F(x, y) + F(y, x) \leq 0, \forall\ x \in C$;

(iii) for each $x, y, z \in C$, $\limsup_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(iv) for each $x \in C$, $y \longmapsto F(x, y)$ is convex and lower semicontinuous.

Example of a bifunction that satisfies the above assumptions can be found in [36].

Definition 2. *A multi-valued mapping $S : H \to CB(H)$ is said to be demiclosed at the origin if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $d(x_n, Sx_n) \to 0,\ n \to \infty$, we have $x \in Sx$.*

Definition 3. *A single-valued mapping $S : H \to H$ is said to be demiclosed at the origin if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $Sx_n \to 0,\ n \to \infty$, we have $Sx = 0$.*

Lemma 1. *[9] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption* 2.1. *For $r > 0$ and $x \in C$, we have the following:*

(B1) there exist $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0\ \forall y \in C, \tag{12}$$

(B2) if we define a resolvent mapping $J_r^F : H \to C$ by

$$J_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $x \in H$. Then the following conclusions hold:

(i) for each $x \in H,\ J_r^F(x) \neq \emptyset$,

(ii) J_r^F is nonempty and single-valued;

(iii) J_r^F *is firmly nonexpansive, i.e.,*

$$\|J_r^F x - J_r^F y\|^2 \leq \langle J_r^F x - J_r^F y, x - y\rangle, \quad \forall x, y \in H;$$

(iv) $F(J_r^F) = EP(F)$;

(v) $EP(F)$ *is closed and convex.*

Lemma 2. *[26] Let* $(X, \langle ., . \rangle)$ *be an inner product space, then for all* $x, y \in X$ *and* $\alpha, \beta, \gamma \in [0, 1]$ *with* $\alpha + \beta + \gamma = 1$, *we have*

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 &= \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 \\ &- \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.to \end{aligned}$$

Lemma 3. *Let* H *be a Hilbert space, then*

$$2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2, \quad \forall\ x, y \in H.$$

and

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 4. *[37] Let* $\{a_n\}$ *be a sequence of non-negative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \delta_n, \quad n \geq 0,$$

where

(i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\limsup \sigma_n \leq 0$,

(iii) $\delta_n \geq 0$, $\sum_{n=0}^{\infty} < \infty$.

Then $a_n \to 0$ *as* $n \to \infty$.

3 Main Result

We now state and prove the following theorem

Theorem 2. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_n : H_1 \to H_1$ be a sequence of ρ_n-contractive mappings with $0 < \underline{\rho} \leq \rho_n \leq \bar{\rho} < 1$ and $\{f_n(x)\}$ is uniformly convergent for any $x \in D$, where D is any bounded subset of H_1. Let $B : C \to H_1$ be τ-inverse strongly monotone mapping. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are two bifunctions satisfying Assumption 2.1 and F_2 is upper semicontinuous in the first argument. Let $S : H_1 \to CB(H_1)$ be a multi-valued quasi-nonexpansive mapping such that S is demiclosed at the origin, $Sp = \{p\}$ $\forall p \in F(S)$ and $\Upsilon := Fix(S) \cap \Omega \cap \Gamma \neq \emptyset$. For arbitrary $x_1 \in H_1$, define the iterative sequence $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ by*

$$\begin{cases} u_n = J_{r_n}^{F_1}(x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n), \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n x_n + \delta_n(\sigma w_n + (1-\sigma)y_n), \quad w_n \in Sx_n, \quad n \geq 1, \end{cases} \tag{15}$$

where $\gamma_n := \mu_n \frac{\|(J_{r_n}^{F_2} - I)Ax_n\|^2}{\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2}$ with $0 < a \leq \mu_n \leq b < 1$,
$r_n \in (0, \infty)$, $\lambda_n \in (0, 2\tau)$, $\sigma, \bar{\rho}, \underline{\rho} \in (0,1)$ and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions

(i) $\alpha_n + \beta_n + \delta_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty}, \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\tau$;

(iv) $\beta_n \geq \epsilon_1 > 0$ $\delta_n \geq \epsilon_2 > 0$.

Then the sequence $\{x_n\}$ converges strongly to $p \in \Upsilon$ where $p = P_\Upsilon f(p)$.

proof: *Let $p \in \Upsilon := Fix(S) \cap \Omega \cap \Gamma$, i.e $p \in \Omega$, we have $p = J_{r_n}^{F_1} p$ and $Ap = J_{r_n}^{F_2} Ap$. We then obtain*

$$\begin{aligned} \|u_n - p\|^2 &= \|J_{r_n}^{F_1}(x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n - p\|^2 \\ &= \|J_{r_n}^{F_1}(x_n + \gamma_n A_i^*(J_{r_n}^{F_2} - I)Ax_n - J_{r_n}^{F_1}(p)\|^2 \\ &\leq \|x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n - p\|^2 \\ &= \|x_n - p + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 \\ &\leq \|x_n - p\|^2 + 2\gamma_n \langle x_n - p, A^*(J_{r_n}^{F_2} - I)A_i x_n\rangle + \\ &\quad \gamma_n^2 \|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2. \end{aligned}$$

Thus,

$$\begin{aligned}\|u_n - p\|^2 &= \|x_n - p\|^2 + 2\gamma_n\langle x_n - p, A^*(J_{r_n}^{F_2} - I)Ax_n\rangle \\ &\quad +\gamma_n^2\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2. \end{aligned} \tag{16}$$

Since $Ap \in F(J_{r_n}^{F_2})$, *we have that*

$$\begin{aligned}\langle x_n - p \quad , \quad & A^*(J_{r_n}^{F_2} - I)Ax_n\rangle = \langle A(x_n - p), (J_{r_n}^{F_2} - I)Ax_n\rangle \\ = \ & \langle A(x_n) - A(p) + (J_{r_n}^{F_2} - I)Ax_n \\ & -(J_{r_n}^{F_2} - I)Ax_n, (J_{r_n}^{F_2} - I)Ax_n\rangle \\ = \ & \langle J_{r_n}^{F_2}Ax_n - Ap - (J_{r_n}^{F_2} - I)Ax_n, (J_{r_n}^{F_2} - I)Ax_n\rangle \\ \leq \ & \langle J_{r_n}^{F_2}Ax_n - Ap, (J_{r_n}^{F_2} - I)Ax_n\rangle - \|(J_{r_n}^{F_2} - I)Ax_n\|^2 \\ = \ & \frac{1}{2}\left[\|J_{r_n}^{F_2}Ax_n - Ap\|^2 + \|(J_{r_n}^{F_2} - I)Ax_n\|^2 - \|Ax_n - Ap\|^2\right] \\ & -\|(J_{r_n}^{F_2} - I)Ax_n\|^2 \\ = \ & \frac{1}{2}\left[\|J_{r_n}^{F_2}Ax_n - Ap\|^2 - \|Ax_n - Ap\|^2\right] - \frac{1}{2}(\|J_{r_n}^{F_2} - I)Ax_n\|^2 \\ = \ & \frac{1}{2}\left[\|J_{r_n}^{F_2}Ax_n - Ap\|^2 - \|Ax_n - Ap\|^2 - \|(J_{r_n}^{F_2} - I)Ax_n\|^2\right] \\ \leq \ & \frac{1}{2}\left[\|Ax_n - Ap\|^2 - \|Ax_n - Ap\|^2 - \|(J_{r_n}^{F_2} - I)Ax_n\|^2\right] \\ = \ & -\frac{1}{2}\|(J_{r_n}^{F_2} - I)Ax_n\|^2. \end{aligned}$$

Therefore,

$$\langle x_n - p, A^*(J_{r_n}^{F_2} - I)Ax_n\rangle \leq -\frac{1}{2}\|(J_{r_n}^{F_2} - I)Ax_n\|^2. \tag{17}$$

Substituting (17) *into* (16), *we get*

$$\begin{aligned}\|u_n - p\|^2 &\leq \|x_n - p\|^2 - \gamma_n\|(J_{r_n}^{F_2} - I)Ax_n\|^2 + \gamma_n^2\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 \\ &\leq \|x_n - p\|^2 \\ &\quad -\gamma_n^2\left[\|(J_{r_n}^{F_2} - I)Ax_n\|^2 - \gamma_n\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2\right]. \end{aligned} \tag{18}$$

Using the definition of γ_n, *we have*

$$\|u_n - p\| \leq \|x_n - p\|. \tag{19}$$

Now, we estimate

$$\begin{aligned}\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n Bu_n) - P_C(p - \lambda_n Bp)\|^2 \\ &\leq \|(u_n - \lambda_n Bu_n) - (p - \lambda_n Bp)\|^2 \\ &\leq \|u_n - p\|^2 - \lambda_n(2\tau - \lambda_n)\|Bu_n - Bp\|^2 \qquad (20) \\ &\leq \|u_n - p\|^2 \\ &\leq \|x_n - p\|^2.\end{aligned}$$

Let $z_n := \sigma w_n + (1-\sigma)y_n$. *Then*

$$\begin{aligned}\|z_n - p\| &= \|\sigma(w_n - p) + (1-\sigma)(y_n - p)\| \\ &\leq \sigma\|w_n - p\| + (1-\sigma)\|y_n - p\| \\ &\leq \sigma\mathcal{H}(P_S(x_n), P_S(p)) + (1-\sigma)\|x_n - p\| \\ &\leq \sigma\|x_n - p\| + (1-\sigma)\|x_n - p\| \\ &= \|x_n - p\|.\end{aligned}$$

By using (15), *we have that*

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n f_n(x_n) + \beta_n x_n + \delta_n z_n - p\| \\ &= \|\alpha_n(f_n(x_n) - f_n(p)) + \alpha_n(f_n(p) - p) + \beta_n(x_n - p) + \delta_n(z_n - p)\| \\ &\leq \alpha_n(\|f_n(x_n) - f_n(p)\| + \|f_n(p) - p\|) + \beta_n\|x_n - p\| + \delta_n\|z_n - p\| \\ &\leq \alpha_n(\|f_n(x_n) - f_n(p)\| + \|f_n(p) - p\|) + (1 - \alpha_n)\|x_n - p\|.\end{aligned}$$

Hence,

$$\|x_{n+1} - p\| \leq \alpha_n(\|f_n(x_n) - f_n(p)\| + \|f_n(p) - p\|) + (1 - \alpha_n)\|x_n - p\|. \quad (21)$$

By the uniform convergence of $\{f_n(x)\}$ *on* D, *there exists* $M > 0$ *such that* $\|f_n(p) - p\| \leq M$, $\forall n \geq 1$. *Hence, we have*

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n\rho_n\|x_n - p\| + \alpha_n\|f_n(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\ &\leq \alpha_n\bar{\rho}\|x_n - p\| + \alpha_n\|f_n(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\ &= (\alpha_n\bar{\rho} + (1 - \alpha_n)\|x_n - p\| + \alpha_n\|f_n(p) - p\| \\ &= (1 - \alpha_n(1 - \bar{\rho}))\|x_n - p\| + \alpha_n\|f_n(p) - p\| \\ &= (1 - \alpha_n(1 - \bar{\rho}))\|x_n - p\| + \alpha_n(1 - \bar{\rho})\frac{\|f_n(p) - p\|}{1 - \bar{\rho}} \\ &\leq \max\left\{\|x_n - p\|, \frac{M}{1 - \bar{\rho}}\right\} \\ &\leq \vdots \\ &\leq \max\left\{\|x_1 - p\|, \frac{M}{1 - \bar{\rho}}\right\}.\end{aligned}$$

Therefore, $\{x_n\}$ is bounded.
By (15) *and* (19), *we have that*

$$\begin{aligned}
&\|x_{n+1}-p\|^2 = \|\alpha_n f_n(x_n)+\beta_n x_n+\delta_n z_n - p\|^2\\
&= \|\beta_n(x_n-p)+\delta_n(z_n-p)+\alpha_n(f_n(x_n)-f_n(p))+\alpha_n(f_n(p)-p)\|^2\\
&\le \|\beta_n(x_n-p)+\delta_n(z_n-p)+\alpha_n(f_n(x_n)-f_n(p))\|^2\\
&+2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle\\
&\le \|\beta_n(x_n-p)+\delta_n(z_n-p)\|^2+\alpha_n^2\|(f_n(x_n)-f_n(p))\|^2\\
&+2\alpha_n\langle \beta_n(x_n-p)+\delta_n(z_n-p), f_n(x_n)-f_n(p)\rangle + 2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle\\
&= \beta_n(\beta_n+\delta_n)\|x_n-p\|^2+\delta_n(\beta_n+\delta_n)\|z_n-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2\\
&+\alpha_n^2\|(f_n(x_n)-f_n(p))\|^2+2\alpha_n\langle\beta_n(x_n-p)+\delta_n(z_n-p), f_n(x_n)-f_n(p)\rangle\\
&+2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle\\
&\le (\beta_n+\delta_n)^2\|x_n-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2\\
&+\alpha_n^2\|f_n(x_n)-f_n(p)\|^2+2\alpha_n\langle\beta_n(x_n-p)+\delta_n(z_n-p), f_n(x_n)-f_n(p)\rangle\\
&+2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle\\
&= (1-\alpha_n)^2\|x_n-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2\\
&+\alpha_n^2\|f_n(x_n)-f_n(p)\|^2+2\alpha_n\langle\beta_n(x_n-p)+\delta_n(z_n-p), f_n(x_n)-f_n(p)\rangle\\
&\quad +2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle\\
&\le (1-\alpha_n)^2\|x_n-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2\\
&+\alpha_n^2\rho_n^2\|x_n-p\|^2+2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle+2\alpha_n\|\beta_n(x_n-p)\\
&+\delta_n(z_n-p)\|\|f_n(x_n)-f_n(p)\|\\
&\le (1-\alpha_n)^2\|x_n-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2\\
&\quad +\alpha_n^2\rho_n^2\|x_n-p\|^2+2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle\\
&+2\bar{\rho}\alpha_n(1-\alpha_n)\|x_n-p\|^2. \qquad (22)
\end{aligned}$$

We divide the rest of the proof into two cases.
Case 1. *Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\|x_n-p\|\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{\|x_n-p\|\}$ converges and $\|x_n-p\|-\|x_{n+1}-p\|\to 0$ as $n\to\infty$. By* (15), *we obtain*

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \le \alpha_n^2\|f_n(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\delta_n\|z_n-p\|^2\\
&\le \alpha_n\|f_n(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\delta_n(\sigma\|x_n-p\|^2+(1-\sigma)\|y_n-p\|^2)\\
&\le \alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\sigma\delta_n)\|x_n-p\|^2+(1-\sigma)\|u_n-p\|^2.
\end{aligned}$$

This implies that

$$-\|u_n-p\|^2 \le \frac{1}{(1-\sigma)\delta_n}\Big[\alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\sigma\delta_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2\Big].$$

From (17) *and* (18), *we have that*

$$\gamma_n\Big[\|(J_{r_n}^{F_2}-I)Ax_n\|^2-\gamma_n\|A^*(J_{r_n}^{F_2}-I)Ax_n\|^2\Big]\leq\|x_n-p\|^2-\|u_n-p\|^2$$
$$\leq\|x_n-p\|^2-\frac{1}{(1-\sigma)\delta_n}\|x_{n+1}-p\|^2+\frac{\alpha_n}{(1-\sigma)\delta_n}\|f_n(x_n)-p\|^2+$$
$$\frac{\beta_n+\sigma\delta_n}{(1-\sigma)\delta_n}\|x_n-p\|^2$$
$$=\frac{1-\alpha_n}{(1-\sigma)\delta_n}\|x_n-p\|^2-\frac{1-\alpha_n}{(1-\sigma)\delta_n}\|x_{n+1}-p\|^2+\frac{\alpha_n}{(1-\sigma)\delta_n}\|f_n(x_n)-p\|^2$$
$$=\frac{1}{(1-\sigma)\delta_n}\Big[\|x_n-p\|^2-\|x_{n+1}-p\|^2\Big]+$$
$$\frac{\alpha_n}{(1-\sigma)\delta_n}\Big[\|f_n(x_n)-p\|^2-\|x_n-p\|^2\Big].$$

Since $\alpha_n\to 0$ *as* $n\to\infty$ *by condition (i), we have that*

$$\gamma_n\Big[\|(J_{r_n}^{F_2}-I)Ax_n\|^2-\gamma_n\|A^*(J_{r_n}^{F_2}-I)Ax_n\|^2\Big]\to 0\quad as\quad n\to\infty,$$

which by definition of γ_n *implies that*

$$\frac{\mu_n(1-\mu_n)\|(J_{r_n}^{F_2}-I)Ax_n\|^4}{\|A^*(J_{r_n}^{F_2}-I)Ax_n\|^2}\to 0\quad as\quad n\to\infty.$$

Since $0<a\leq\mu_n\leq b<1$ *and* $\|A^*(J_{r_n}^{F_2}-I)Ax_n\|$ *is bounded, we have that*

$$\|(J_{r_n}^{F_2}-I)Ax_n\|\to 0\quad as\quad n\to\infty. \tag{23}$$

Now,

$$\|A^*(J_{r_n}^{F_2}-I)Ax_n\|=\|A\|\|(J_{r_n}^{F_2}-I)Ax_n\|\to 0\quad as\quad n\to\infty. \tag{24}$$

By (22), *we have that*

$$\beta_n\delta_n\|x_n-z_n\|^2\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_nM_1,$$

for some $M_1>0$ *and this implies that*

$$\|x_n-z_n\|\to 0,\ n\to\infty.$$

Hence,

$$\|x_{n+1}-x_n\|=\|\alpha_nf_n(x_n)+\beta_nx_n+\delta_nz_n-(\alpha_nx_n+\beta_nx_n+\delta_nx_n)\|$$
$$\leq\alpha_n\|f_n(x_n)-x_n\|+\alpha_n\|x_n-z_n\|\to 0, n\to\infty.$$

Similarly,

$$\|x_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + \|x_n - z_n\| \to 0, n \to \infty.$$

From (15) *and the fact that* $J_{r_n}^{F_1}$ *is firmly nonexpansive, we have that*

$$\begin{aligned}
&\|u_n - p\|^2 = \|J_{r_n}^{F_1}(x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n) - p\|^2 \\
&= \|J_{r_n}^{F_1}(x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n) - J_{r_n}^{F_1}(p)\|^2 \\
&= \langle u_n - p, x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n - p\rangle \\
&\leq \frac{1}{2}[\|u_n - p\|^2 + \|x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n - p\|^2 \\
&-\|u_n - p - (x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n - p)\|^2] \\
&\leq \frac{1}{2}[\|u_n - p\|^2 + \|x_n - p + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 \\
&-\|u_n - x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\|^2] \\
&\leq \frac{1}{2}[\|u_n - p\|^2 + \|x_n - p + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 \\
&-[\|u_n - x_n\|^2 + \gamma_n\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 - 2\langle u_n - x_n, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle]] \\
&= \frac{1}{2}[\|u_n - x_n\|^2 + \|x_n - p + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 - \|u_n - x_n\|^2 \\
&-\gamma_n\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 + 2\langle u_n - x_n, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle] \\
&= \frac{1}{2}[\|u_n - x_n\|^2 + \|x_n - p\|^2 + \gamma_n\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 \\
&+2\langle x_n - p, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle - \|u_n - x_n\|^2 - \gamma_n\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 \\
&+2\langle u_n - x_n, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle] \\
&\leq \frac{1}{2}[\|x_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\langle x_n - p, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle \\
&+2\langle u_n - x_n, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle] \\
&\leq \|x_n - p\|^2 - \frac{1}{2}\|u_n - x_n\|^2 + \langle u_n - p, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle \\
&\leq \|x_n - p\|^2 - \frac{1}{2}\|u_n - x_n\|^2 + \gamma_n\|u_n - p\|\|A^*(J_{r_n}^{F_2} - I)Ax_n\|.
\end{aligned}$$

That is

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \frac{1}{2}\|u_n - x_n\|^2 + \gamma_n\|u_n - p\|\|A^*(J_{r_n}^{F_2} - I)Ax_n\|. \quad (25)$$

Hence,

$$\begin{aligned}
&\frac{1}{2}\|u_n - x_n\|^2 \leq \|x_n - p\|^2 - \|u_n - p\|^2 + \gamma_n\|u_n - p\|\|A^*(J_{r_n}^{F_2} - I)Ax_n\| \\
&\leq \|x_n - p\|^2 - \frac{1}{(1-\sigma)\delta_n}\|x_{n+1} - p\|^2 + \frac{\alpha_n}{(1-\sigma)\delta_n}\|f_n(x_n) - p\|^2 + \\
&\frac{\beta_n + \sigma\delta_n}{(1-\sigma)\delta_n}\|x_n - p\|^2 + \gamma_n\|u_n - p\|\|A^*(J_{r_n}^{F_2} - I)Ax_n\| \\
&= \frac{1-\alpha_n}{(1-\sigma)\delta_n}\|x_n - p\|^2 - \frac{1-\alpha_n}{(1-\sigma)\delta_n}\|x_{n+1} - p\|^2 + \frac{\alpha_n}{(1-\sigma)\delta_n}\|f_n(x_n) - p\|^2 \\
&+\gamma_n\|u_n - p\|\|A^*(J_{r_n}^{F_2} - I)Ax_n\| \\
&= \frac{1}{(1-\sigma)\delta_n}\Big[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big] + \\
&+\frac{\alpha_n}{(1-\sigma)\delta_n}\Big[\|f_n(x_n) - p\|^2 - \|x_n - p\|^2\Big] + \gamma_n\|u_n - p\|\|A^*(J_{r_n}^{F_2} - I)Ax_n\|,
\end{aligned}$$

and this implies that

$$\|u_n - x_n\| \to 0, n \to \infty,$$

and

$$\|z_n - u_n\| \leq \|u_n - x_n\| + \|x_n - z_n\| \to 0, n \to \infty.$$

Next, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \leq \alpha_n\|f_n(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \delta_n\|z_n - p\|^2 \\
&\leq \alpha_n\|f_n(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \delta_n(\sigma\|x_n - p\|^2 + (1-\sigma)\|y_n - p\|^2) \\
&= \alpha_n\|f_n(x_n) - p\|^2 + (\beta_n + \sigma\delta_n)\|x_n - p\|^2 + (1-\sigma)\delta_n\|y_n - p\|^2 \\
&\leq \alpha_n\|f_n(x_n) - p\|^2 + (\beta_n + \sigma\delta_n)\|x_n - p\|^2 + \\
&(1-\sigma)\delta_n[\|u_n - p - \lambda_n(Bu_n - Bp)\|^2] \\
&\leq \alpha_n\|f_n(x_n) - p\|^2 + (\beta_n + \sigma\delta_n)\|x_n - p\|^2 \\
&+(1-\sigma)\delta_n\left[\|u_n - p\|^2 - 2\lambda_n\langle u_n - p, Bu_n - Bp\rangle + \lambda_n^2\|Bu_n - Bp\|^2\right] \\
&\leq \alpha_n\|f_n(x_n) - p\|^2 + (\beta_n + \sigma\delta_n)\|x_n - p\|^2 + (1-\sigma)\delta_n\|x_n - p\|^2 \\
&-(1-\sigma)2\delta_n\lambda_n\tau\|Bu_n - Bp\|^2 + (1-\sigma)\delta_n\lambda_n^2\|Bu_n - Bp\|^2 \\
&\leq \alpha_n\|f_n(x_n) - p\|^2 + (\beta_n + \sigma\delta_n)\|x_n - p\|^2 + (1-\sigma)\delta_n\|x_n - p\|^2 \\
&+(1-\sigma)\delta_n\lambda_n(\lambda_n - 2\tau)\|Bu_n - Bp\|^2,
\end{aligned}$$

and so

$$\begin{aligned}
&\lambda_n(2\tau-\lambda_n)\|Bu_n-Bp\|^2 \le \|x_n-p\|^2 - \frac{1}{(1-\sigma)\delta_n}\|x_{n+1}-p\|^2\\
&+\frac{\alpha_n}{(1-\sigma)\delta_n}\|f_n(x_n)-p\|^2 + \frac{\beta_n+\sigma\delta_n}{(1-\sigma)\delta_n}\|x_n-p\|^2\\
&= \frac{1-\alpha_n}{(1-\sigma)\delta_n}\|x_n-p\|^2 - \frac{1-\alpha_n}{(1-\sigma)\delta_n}\|x_{n+1}-p\|^2\\
&+\frac{\alpha_n}{(1-\sigma)\delta_n}\|f_n(x_n)-p\|^2\\
&= \frac{1}{(1-\sigma)\delta_n}\Big[\|x_n-p\|^2 - \|x_{n+1}-p\|^2\Big]\\
&+\frac{\alpha_n}{(1-\sigma)\delta_n}\Big[\|f_n(x_n)-p\|^2 - \|x_n-p\|^2\Big].
\end{aligned}$$

Since $\alpha_n \to 0$ *and* $0 < \lim_{n\to\infty}\lambda_n = \lambda < 2\tau$, *we have that*

$$\lim_{n\to\infty}\|Bu_n - Bp\| = 0. \tag{26}$$

Since P_C *is firmly nonexpansive and* $(I-\lambda_n B)$ *is nonexpansive by* (15), *we have*

$$\begin{aligned}
&\|y_n-p\|^2 = \|P_C(u_n-\lambda_n Bu_n) - P_C(p-\lambda_n Bp)\|^2\\
&\le \langle y_n-p, u_n-\lambda_n Bu_n-(p-\lambda_n Bp)\rangle\\
&= \frac{1}{2}(\|y_n-p\|^2 + \|(I-\lambda_n B)u_n-(I-\lambda_n B)p\|^2 -\\
&\|y_n-u_n+\lambda_n(Bu_n-Bp)\|^2)\\
&\le \frac{1}{2}(\|y_n-p\|^2 + \|u_n-p\|^2 - \|y_n-u_n+\lambda_n(Bu_n-Bp)\|^2)\\
&= \frac{1}{2}(\|y_n-p\|^2 + \|u_n-p\|^2 - \|y_n-u_n\|^2 - \lambda_n^2\|Bu_n-Bp\|^2\\
&-2\lambda_n\langle y_n-u_n, Bu_n-Bp\rangle)\\
&\le \frac{1}{2}(\|y_n-p\|^2 + \|u_n-p\|^2 - \|y_n-u_n\|^2 - \lambda_n^2\|Bu_n-Bp\|^2\\
&+2\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|),
\end{aligned}$$

and so

$$\begin{aligned}
\|y_n-p\|^2 &\le \|u_n-p\|^2 - \|y_n-u_n\|^2 - \lambda_n^2\|Bu_n-Bp\|^2\\
&\quad + 2\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|\\
&\le \|x_n-p\|^2 - \|y_n-u_n\|^2 + 2\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|.
\end{aligned} \tag{27}$$

From (15) *and* (27), *we have*

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\sigma\delta_n)\|x_n-p\|^2+(1-\sigma)\delta_n\|y_n-p\|^2\\ &\leq \alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\sigma\delta_n)\|x_n-p\|^2\\ &+(1-\sigma)\delta_n[\|x_n-p\|^2-\|y_n-u_n\|^2+2\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|]\\ &\leq \alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\sigma\delta_n)\|x_n-p\|^2\\ &+(1-\sigma)\delta_n\|x_n-p\|^2-(1-\sigma)\delta_n\|y_n-u_n\|^2+\\ &2(1-\sigma)\delta_n\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|y_n-u_n\|^2 \leq \frac{1}{(1-\sigma)\delta_n}\Big[\|x_n-p\|^2-\|x_{n+1}-p\|^2\Big]+\frac{\alpha_n}{(1-\sigma)\delta_n}\Big[\|f_n(x_n)-p\|^2\\ -\|x_n-p\|^2\Big]+2\lambda_n(\|y_n\|+\|u_n\|)\|Bu_n-Bp\|.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$ *and both* $\{y_n\}$ *and* $\{u_n\}$ *are bounded by* (26), *we have*

$$\lim_{n\to\infty}\|y_n-u_n\|=0. \tag{28}$$

From $z_n=\sigma w_n+(1-\sigma)y_n$, *we get*

$$\|w_n-y_n\|=\frac{1}{\sigma}[\|z_n-u_n\|+\|y_n-u_n\|]\to 0,\quad n\to\infty.$$

So,

$$\|w_n-x_n\|\leq\|w_n-y_n\|+\|y_n-u_n\|+\|u_n-x_n\|\to 0,\quad n\to\infty.$$

Therefore,

$$d(x_n,Sx_n)\leq\|x_n-w_n\|\to 0,\quad n\to\infty. \tag{29}$$

Since $\{x_n\}$ *is bounded, there exists a subsequence* $\{x_{n_j}\}$ *of* $\{x_n\}$ *such that* $x_{n_j}\rightharpoonup x^*\in H_1$ *and*

$$\limsup_{n\to\infty}\langle f(p)-p,x_n-p\rangle=\limsup_{j\to\infty}\langle f(p)-p,x_{n_j}-p\rangle.$$

By demiclosedness principle for multi-valued map S *at zero and* (29), *we have that* $x^*\in F(S)$.

Next, we show that $x^*\in EP(F_1)$. *Since* $u_n=J_{r_n}^{F_1}x_n$, *we have*
$F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in C.$
It follows from monotonicity of F_1 *that* $\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq F_1(y,u_n)$ *and*

hence $\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_n} \right\rangle \geq F_1(y, u_{n_i})$. *Since* $\|u_n - x_n\| \to 0$, $d(x_n, Sx_n) \leq \|x_n - w_n\| \to 0$, $n \to \infty$, *we get* $u_{n_i} \rightharpoonup x^*$ *and* $\frac{u_{n_i} - x_{n_i}}{r_n} \to 0$. *It follows by Assumption* 2.1(iv) *that* $0 \geq F_1(y, x^*)$, $\forall x^* \in C$. *For* $0 < t \leq 1$ *and* $y \in C$, *let* $y_t = ty + (1-t)x^*$. *Since* $y \in C$, $x^* \in C$, *we get* $y_t \in C$ *and* $F_1(y_1, x^*) \leq 0$. *So from Assumption* 2.1(i) *and* (iv), *we have*

$$0 = F_1(y_t, y_t) \leq tF(y_t, y) + (1-t)F_1(y_t, x^*) \leq tF_1(y_t, y).$$

Therefore $) \leq F_1(y_t, y)$. *From Assumption* 2.1(iii), *we have* $0 \leq F_1(x^*, y)$. *Hence* $x^* \in EP(F_1)$.

Next, we show that $Ax^* \in EP(F_2)$. *Since* $\|u_n - x_n\| \to 0$, $u_n \rightharpoonup x^*$ *as* $n \to \infty$ *and* $\{x_n\}$ *is bounded, there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_k\}$ *such that* $x_{n_k} \rightharpoonup x^*$ *and since* A *is a bounded linear operator so that* $Ax_{n_k} \rightharpoonup Ax^*$.

Now setting $v_{n_k} = Ax_{n_k} - J^{F_2}_{r_n} Ax_{n_k}$. *It follows from* (23) *that* $\lim_{k\to\infty} v_{n_k} = 0$ *and* $Ax_{n_k} - v_{n_k} = J^{F_2}_{r_{n_k}} Ax_{n_k}$.

Therefore from Lemma 2.4, *we have*

$$F_2(Ax_{n_k} - v_{n_k}, z) + \frac{1}{r_{n_k}} \langle z - (Ax_{n_k} - v_{n_k}), (Ax_{n_k} - v_{n_k}) - Ax_{n_k} \rangle \geq 0, \quad \forall z \in Q.$$

Since F_2 *is upper semicontinuous in the first argument, taking limsup of the above inequality as* $k \to \infty$ *and using condition* (iv), *we obtain*

$$F_2(Ax^*, z) \geq 0, \quad \forall z \in Q,$$

which means that $Ax^* \in EP(F_2)$ *and hence* $x^* \in \Omega$.

Finally, by using the argument as in the proof of Theorem 3.1 *of [21], we can show that* $x^* \in \Gamma$. *Meanwhile, since* $\{f_n(x_n)\}$ *is uniformly convergent on* D, *we have*

$$\begin{aligned} \limsup_{n\to\infty} \langle f_n(p) - p, x_n - p \rangle &= \limsup_{j\to\infty} \langle f_{n_j}(p) - p, x_{n_j} - p \rangle \\ &= \langle f(p) - p, x^* - p \rangle \geq 0. \end{aligned}$$

By (22), *we get*

$$\begin{aligned} &\|x_{n+1} - p\|^2 \leq (1 - 2\alpha_n(1 - \bar{\rho}(1 - \alpha_n)))\|x_n - p\|^2 - \beta_n\delta_n\|x_n - z_n\|^2 \\ &+ \alpha_n^2(1 + \bar{\rho}^2)\|x_n - p\|^2 + 2\alpha_n\langle f_n(p) - p, x_{n+1} - p \rangle \\ &\leq (1 - 2\alpha_n(1 - \bar{\rho}))\|x_n - p\|^2 - \beta_n\delta_n\|x_n - z_n\|^2 \\ &+ \alpha_n \left[\alpha_n(1 + \bar{\rho}^2)\|x_n - p\|^2 + 2\langle f_n(p) - p, x_{n+1} - p \rangle \right]. \end{aligned} \tag{30}$$

Using Lemma 2.7 *we have that* $x_n \to p$ *as* $n \to \infty$.

Case 2. *Assume that*$\{\|x_n - p\|\}$ *is not a monotonically decreasing sequence. Set* $\Gamma_n = \|x_n - p\|^2$ *and Let* $\tau : \mathbb{N} \to \mathbb{N}$ *be a mapping for all* $n \geq n_0$ *(for some* n_0 *large enough) by*

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ *is non decreasing sequence such that* $\tau(n) \to \infty$ *as* $n \to \infty$ *and*

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.$$

This implies that $\|x_{\tau(n)} - p\| \leq \|x_{\tau(n)+1} - p\|, \forall n \geq n_0$. *Thus* $\lim_{n\to\infty} \|x_{\tau(n)} - p\|$ *exists. In a similar way as in case 1, we can show that*

$$\|A^*(J^{F_2}_{r_{\tau(n)}} - I)Ax_{\tau(n)}\| \to 0, \quad n \to \infty. \tag{31}$$

Similarly,

$$\|x_{\tau(n)} - w_{\tau(n)}\| \to 0, \quad n \to \infty, \tag{32}$$

so that

$$d(x_{\tau(n)}, Sx_{\tau(n)}) \leq \|x_{\tau(n)} - w_{\tau(n)}\| \to 0, \quad n \to \infty,$$

and

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0, \quad n \to \infty. \tag{33}$$

We can also show that

$$\|u_{\tau(n)} - x_{\tau(n)}\| \to 0, \quad n \to \infty,$$

$$\|w_{\tau(n)} - y_{\tau(n)}\| \to 0, \quad n \to \infty,$$

and

$$\|(J^{F_2}_{r_{\tau(n)}} - I)A_{x_{\tau(n)}}\| \to 0, \quad n \to \infty.$$

From the fact that $\{x_{\tau(n)}\}$ *is bounded, we have that there exists a subsequence of* $\{x_{\tau(n)}\}$, *denoted as* $\{x_{\tau(n)}\}$, *that converges weakly to* $x^* \in H_1$. *Since* $\|u_{\tau(n)} - x_{\tau(n)}\| \to 0$, *it follows that* $u_{\tau(n)} \rightharpoonup x^* \in H_1$. *As in Case 1, we can show that* $x^* \in \Upsilon$ *and*

$$\limsup_{n\to\infty} \langle f_{\tau(n)}(p) - p, x_{\tau(n)+1} - p\rangle \geq 0.$$

By (22)*, we get*

$$\begin{aligned}\|x_{\tau(n)+1}-p\|^2 &\leq (1-2\alpha_{\tau(n)}(1-\bar{\rho}(1-\alpha_{\tau(n)})))\|x_{\tau(n)}-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2\\ &\quad+\alpha_{\tau(n)}^2(1+\bar{\rho}^2)\|x_{\tau(n)}-p\|^2+2\alpha_{\tau(n)}\langle f_{\tau(n)}(p)-p,x_{\tau(n)+1}-p\rangle\\ &\leq (1-2\alpha_{\tau(n)}(1-\bar{\rho}))\|x_{\tau(n)}-p\|^2+\alpha_{\tau(n)}^2(1+\bar{\rho}^2)\|x_{\tau(n)}-p\|^2\\ &\quad+2\alpha_{\tau(n)}\langle f_{\tau(n)}(p)-p,x_{\tau(n)+1}-p\rangle.\end{aligned} \tag{34}$$

Which implies that (noting that $\Gamma_{\tau(n)}\leq\Gamma_{\tau(n)+1}$ *and* $\alpha_{\tau(n)}>0$*)*

$$\begin{aligned}2(1-\bar{\rho})\|x_{\tau(n)}-p\|^2 &\leq \alpha_{\tau(n)}(1+\bar{\rho}^2)\|x_{\tau(n)}-p\|^2\\ &\quad+2\langle f_{\tau(n)}(p)-p,x_{\tau(n)+1}-p\rangle.\end{aligned} \tag{35}$$

This implies that

$$\limsup_{n\to\infty}\|x_{\tau(n)}-p\|\leq 0.$$

Thus

$$\lim_{n\to\infty}\|x_{\tau(n)}-p\|=0. \tag{36}$$

Therefore,

$$\|x_{\tau(n)+1}-p\|\leq\|x_{\tau(n)}-p\|+\|x_{\tau(n)+1}-x_{\tau(n)}\|\to 0,\quad n\to\infty.$$

Furthermore, for $n\geq n_0$*, it is easy to see that* $\Gamma_{\tau(n)}\leq\Gamma_{\tau(n)+1}$ *if* $n\neq\tau(n)$ *(that is,* $\tau(n)<n$*), because* $\Gamma_j\leq\Gamma_{j+1}$ *for* $\tau(n)+1\leq j\leq n$*. As a consequence, we obtain for all* $n\geq n_0$*,*

$$0\leq\Gamma_n\leq\max\{\Gamma_{\tau(n)},\Gamma_{\tau(n)+}\}=\Gamma_{\tau(n)+1}.$$

Hence $\lim\Gamma_n=0$*, that is,* $\{x_n\}$ *converges to* $\bar{x}$*. This completes the proof.*

If S *is a single-valued quasi-nonexpansive mapping. We obtain the following result.*

Corollary 1. *Let* H_1 *and* H_2 *be two real Hilbert spaces and* $C\subseteq H_1$ *and* $Q\subseteq H_2$ *be nonempty, closed and convex. Let* $A:H_1\to H_2$ *be a bounded linear operator and* A^* *the adjoint of* A*. Let* $f_n:H_1\to H_1$ *be a sequence of* ρ_n*-contractive mappings with* $0<\underline{\rho}\leq\rho_n\leq\bar{\rho}<1$ *and* $\{f_n(x)\}$ *is uniformly convergent for any* $x\in D$*, where* D *is any bounded subset of* H_1*. Let* $B:C\to H_1$ *be* τ*-inverse strongly monotone mapping. Assume that* $F_1:C\times C\to\mathbb{R}$ *and* $F_2:Q\times Q\to\mathbb{R}$ *are two bifunctions satisfying Assumption* 2.1 *and* F_2

is upper semicontinuous in the first argument. Let $S : H_1 \to H_1$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at the origin, $Sp = p$ $\forall p \in Fix\ (S)$ and $\Upsilon := Fix\ (S) \cap \Omega \cap \Gamma \neq \emptyset$. For arbitrary $x_1 \in H_1$, define the iterative sequence $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ by

$$\begin{cases} u_n = J_{r_n}^{F_1}(x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n), \\ y_n = P_C(u_n - \lambda_n Bu_n), \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n x_n + \delta_n(\sigma Sx_n + (1-\sigma)y_n), \quad n \geq 1, \end{cases} \tag{37}$$

where $\gamma_n := \mu_n \frac{\|(J_{r_n}^{F_2} - I)Ax_n\|^2}{\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2}$ with $0 < a \leq \mu_n \leq b < 1$, $r_n \in (0, \infty)$, $\lambda_n \in (0, 2\tau)$, $\sigma, \bar{\rho}, \underline{\rho} \in (0, 1)$ and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ are real sequences in $(0, 1)$ satisfying the following conditions

(i) $\alpha_n + \beta_n + \delta_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty}, \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\tau$,

(iv) $\beta_n \geq \epsilon_1 > 0 \quad \delta_n \geq \epsilon_2 > 0$.

Then the sequence $\{x_n\}$ converges strongly to $p \in \Upsilon$ where $p = P_\Upsilon f(p)$.

4 Applications

4.1 Split Monotone Inclusion Problem.

Let $G : H_1 \to 2^{H_1}$ be a multivalued mapping. The multi-valued mapping G is said to be monotone if for each $x, y \in H_1$ any $u \in G(x), v \in G(y)$, we have that

$$\langle u - v, x - y \rangle \geq 0.$$

A monotone multi-valued mapping G is said to be a maximal monotone mapping if the Graph$(G) = \{(x, u) \in H_1 \times H_1, u \in Gx\}$ is not properly contained in the graph of any other monotone mapping on H_1, see [1]. To every maximal monotone multi-valued mapping G, there is an associated mapping $J_\lambda^G : H_1 \to H_1$, is called the resolvent of G, defined by

$$J_\lambda^G(x) := (I + \lambda G)^{-1}(x), \quad \forall x \in H_1,$$

for some $\lambda > 0$, where I is the identity mapping on H_1. The resolvent mapping J_λ^G is single valued and firmly nonexpansive (hence nonexpansive)

(see for example, [3, 17, 18] for more details).
Let H_1 and H_2 be real Hilbert spaces. Let $G_1 : H_1 \to 2^{H_1}$ and $G_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings. Let $A : H_1 \to H_2$ be a bounded linear mapping. The split monotone inclusion problem (see, for example, [19]) is to find $x^* \in H_1$ such that

$$0 \in G_1(x^*), \qquad x^* \in Sx^* \tag{38}$$

and

$$0 \in G_2(Ax^*), \tag{39}$$

where $S : H_1 \to CB(H_1)$ is a multi-valued quasi-nonexpansive mapping. We shall denote by $\mho$, the solution set of (38) - (39). *That is,*

$$\mho = \{x^* \in H_1 : 0 \in G_1(x^*) \ \text{ and } \ G_2(Ax^*)\}.$$

Putting $F_1 = G_1$ and $F_2 = G_2$ in Theorem 3.1, *we obtain the following result*

Corollary 2. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_n : H_1 \to H_1$ be a sequence of ρ_n-contractive mappings with $0 < \underline{\rho} \leq \rho_n \leq \bar{\rho} < 1$ and $\{f_n(x)\}$ is uniformly convergent for any $x \in D$, where D is any bounded subset of H_1. Let $B : C \to H_1$ be τ-inverse strongly monotone mapping. Assume that $G_1 : H_1 \to 2^{H_1}$ and $G_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings. Let $S : H_1 \to CB(H_1)$ be a multi-valued quasi-nonexpansive mapping such that S is demiclosed at the origin, $Sp = \{p\}$ $\forall p \in F(S)$ and $\Upsilon := Fix(S) \cap \mho \cap \Gamma \neq \emptyset$. For arbitrary $x_1 \in H_1$, define the iterative sequence $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ by*

$$\begin{cases} u_n = J_\lambda^{G_1}(x_n + \gamma_n A^*(J_\lambda^{G_2} - I)Ax_n), \\ y_n = P_C(u_n - \lambda_n Bu_n), \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n + \delta_n(\sigma w_n + (1-\sigma)y_n), \quad w_n \in Sx_n, \quad n \geq 1, \end{cases} \tag{40}$$

where $\gamma_n := \mu_n \frac{\|(J_\lambda^{G_2} - I)Ax_n\|^2}{\|A^*(J_\lambda^{G_2} - I)Ax_n\|^2}$ with $0 < a \leq \mu_n \leq b < 1$,
$\lambda_n \in (0, 2\tau)$, $\sigma, \bar{\rho}, \underline{\rho} \in (0,1)$ and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions

(i) $\alpha_n + \beta_n + \delta_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty}, \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\tau$,

(iv) $\beta_n \geq \epsilon_1 > 0 \quad \delta_n \geq \epsilon_2 > 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $p \in \Upsilon$ *where* $p = P_\Upsilon f(p)$.

Remark 1. *Our result extends and compliments some recent results in the following ways:*

1. *Our result improves and extend the result of Kazmi and Rizvi [14], from single-valued nonexpansive to multi-valued quasi-nonexpansive mappings.*

2. *In contrast with other related methods, our algorithm does not require any estimate of some spectral radius. In all our results in this paper, our iterative scheme is proposed with a way of selecting the step-size* γ_n *such that its implementation does not need any prior information about the spectral radius of the operator* A^*A. *The constant step-size* γ *in the result of Kazmi and Rizvi [14], for example, depends on the spectral radius of the operator* A^*A *and we know that computing the spectral radius of this operator* A^*A *can be difficult to find at times. Therefore, our result improve and extend the result of Kazmi and Rizvi [14].*

Acknowledgement: *The first author acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DST-NRF COE-MaSS) Doctoral Bursary. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.*

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