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SYMMETRY PROPERTIES OF THE COULOMB POTENTIAL WITH A LINEAR DEPENDENCE ON ENERGY*

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Abstract

The D-dimensional Schrödinger equation for a Coulomb potential with a coupling constant depending linearly on energy is analytically solved. The energy spectrum in the asymptotic regime of the slope parameter is found to be fully determined up to a scale only by its quantum numbers. The raising and lowering operators for this limiting model are determined from the recurrence properties of the associated solutions. It is shown that they satisfy the commutation relations of an $SU(1,1)$ algebra and act on wave-functions which are normalized differently from the case of the usual bound state problem for an energy independent Coulomb potential.

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1 Introduction

Exactly solvable problems associated to quantum systems have always attracted much attention due to their simple but elegant algebraic structure.

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This relation between exact solvability and the algebraization of the problem was observed in the early stages of the quantum physics foundations by Dirac [1] and Schrödinger [2]. Their findings lead to the introduction of the factorization method [3, 4], which is based on the simple premise that if the Schrödinger equation is factorizable, then its solutions can be obtained algebraically in terms of creation and annihilation operators. These operators usually close into a dynamical algebra [5] aided by an additional operator which do not change the wave function. Depending on the structure of the Hamiltonian operator in terms of these operators, the associated algebra can be a spectrum generating one or a universal covering one, also called dynamical symmetry [6]. Therefore, exact solvability of a problem is said to be directly related to the symmetry properties of the modeled system. As a matter of fact, the factorization method is closely related to the super-symmetric quantum mechanics [7] which offers a very useful concept of shape invariance [8] as a measure to characterize and search for solvable potentials [9, 10] and which was shown to have an algebraic structure [11].

The number of potentials which lead to an exactly solvable Schrödinger equation is unfortunately scarce. The Coulomb, Kratzer, Harmonic Oscillator, Davidson, Morse, Pöschl-Teller, Scarf, Rosen-Morse, Eckart, Infinite Square Well and Nathanzon are just a few textbook examples of one-dimensional and multi-dimensional exactly solvable potentials. In some cases, such as Coulomb, Harmonic Oscillator or the Infinite Square Well, the associated eigenvalues depend up to a scale just on the quantum numbers which are symmetry-defined by the associated dynamical group. In consequence, this provides a unique insight into the structure and dynamics of the modeled physical system through its underlying dynamical symmetry. This special analytical characteristic of the exactly solvable models can be extended to non-solvable problems by following few venues. One way to do this is by relaxing the condition of having the whole spectrum in a closed form. Basically, such extensions lead to spectral problems that are analytically solved only for a part of the entire energy spectrum. Such partially algebraized models are called quasi-exactly solvable [12, 13, 14, 15]. The energy dependent variations of the exactly solvable potentials represent an alternative direction [16, 17].

As the differential Schrödinger equation with an energy dependent (non-local) potential is solved similarly as in the case of the original local potential, it retains at least a part of the original analytical structure. The eigenvalues, however, are not determined from a linear correspondence, as in the usual case, but by solving a higher order polynomial equation [18]. This aspect, coupled with the modification of the quantum phase space of the

problem, stemming from its iterative behaviour, hinders the identification of the dynamical symmetries associated to such problems. Nevertheless, there are indications that some of symmetry aspects are preserved from the unperturbed problem, such as for example the super-symmetric factorization of the Hamiltonian [19]. However, this is possible to verify only in case of the linear dependence, which allows the transformation of the non-local Schrödinger equation into a usual quantum problem [16, 19].

In this paper, one will study the bound states of the energy dependent Coulomb potential in D dimensions. The peculiarities of the wave equation with energy-dependent potentials are discussed in the next section. The usual problem of the energy independent Coulomb potential with bound states succinctly presented in Sec.3, is known to satisfy the $SU(1,1)$ algebra [20, 21]. The scope of this study is to identify if and in what conditions the non-local problem can be described in terms of the same algebra. To achieve this, one presents in Sec.4 the analytical structure of the eigensystem corresponding to a Coulomb potential whose coupling constant depends linearly on the energy of the system. An interesting saturation of the resulted excitation energy spectrum is observed when the slope of the energy dependence is increased [22]. This result is analytically recovered in Sec.5 by means of the asymptotic expansion [23] in terms of the slope describing the energy dependence. It is shown that in this regime, the energy spectrum as well as the wave functions become fully scaled in terms of the so called slope parameter. As a consequence, the energy spectrum in this large slope regime is once again completely determined up to a scale only by the associated quantum numbers. This result suggests the presence of a dynamical symmetry, which is sought here by means of the method of Ref.[24]. Basically, one deduces the ladder operators from the analytical expression of the wave functions and then checks the commutation relations they obey in order to establish their underlying algebra. However, this procedure must be performed in the Hilbert space of the ordinary quantum mechanics. The general procedure of reformulation of a problem for an energy dependent potential as an ordinary Schrödinger equation is explained in Sec.2. Its application to the large slope regime of the energy dependent Coulomb potential and the consequent identification of the associated ladder operators is dully explained in Sec.6, where one also discuses their group theoretical consequences. Finally, one closes with a few comments in the last Section.

2 Basic aspects of energy dependent potentials

General considerations regarding the modification of the usual rules of quantum mechanics when considering energy dependent potentials were discussed by several authors [16, 25]. By far, the most important implication is the modification of the scalar product as

$$\langle\Psi|\Phi\rangle = \int \Psi^*(\vec{r}) \left[1 - \frac{\partial V(E,\vec{r})}{\partial E}\right] \Phi(\vec{r}) d\vec{r}. \tag{2.1}$$

This amendment is essential for the conservation of the norm and assures that the continuity equation is satisfied [16, 25, 26, 27, 28]. The correction term emerges as an additional contribution to the continuity equation for the standard definition of the scalar product. One can immediately see that for the linear case, the integration metric is state independent. This special feature is very important, because only in this situation the wave-functions (4.5) can form a complete set of eigenstates [16]. Moreover, only for a linear energy dependence, the problem can be reformulated in the usual quantum mechanics with a standard scalar product [16, 19]. In what follows one will sketch the general procedure which is similar to the Darboux transform [29, 30, 31].

Considering the linear dependence on energy of a general one-dimensional potential to be of the form $V(E,x) = V_0(x) + EV_1(x)$, one can write down the associated Hamiltonian operator as

$$H_d = H_0 + V_1(x) H_d = \frac{1}{1 - V_1(x)} H_0, \quad H_d \Phi(x) = E\Phi(x), \tag{2.2}$$

where H_0 is the Hamiltonian of the unperturbed problem for the local potential $V_0(x)$ and $\Phi(x)$ is the eigenfunction of the deformed Hamiltonian H_d. The time dependent Schrödinger equation for H_d can be rewritten in the following form

$$i\frac{\partial}{\partial t}\left[1 - V_1(x)\right]\Phi(x,t) = H_0 \Phi(x,t), \tag{2.3}$$

where $\Phi(x,t) = e^{-iEt}\Phi(x)$ is the time dependent wave function. Making the change of function $\tilde{\Phi} = \sqrt{1 - V_1(x)}\Phi$, the above equation acquires the usual Schrödinger form

$$i\frac{\partial}{\partial t}\tilde{\Phi}(x,t) = \frac{1}{\sqrt{1 - V_1(x)}} H_0 \frac{1}{\sqrt{1 - V_1(x)}} \tilde{\Phi}(x,t) \tag{2.4}$$

for a new Hamiltonian

$$\tilde{H}_d = \sqrt{1-V_1(x)}H_d\frac{1}{\sqrt{1-V_1(x)}} = \frac{1}{\sqrt{1-V_1(x)}}H_0\frac{1}{\sqrt{1-V_1(x)}}, \qquad (2.5)$$

whose eigenvalue problem reads

$$\tilde{H}_d\tilde{\Phi}(x,t) = E\tilde{\Phi}(x,t). \qquad (2.6)$$

Note that $\tilde{H}_d$ has the same eigenvalue as H_d, acting however in a distinct Hilbert space, more specifically one with an ordinary scalar product associated to it [16]. Using one of the above definitions, one can write the differential form of $\tilde{H}_d$:

$$\begin{aligned}\tilde{H}_d &= \frac{1}{1-V_1(x)}\left\{-\frac{d^2}{dx^2}+V_0(x)-\frac{V_1'(x)}{1-V_1(x)}\frac{d}{dx}\right.\\ &\left.-\frac{2V_1''(x)\left[1-V_1(x)\right]+3\left[V_1'(x)\right]^2}{4\left[1-V_1(x)\right]^2}\right\}. \qquad (2.7)\end{aligned}$$

3 D-dimensional wave equation for a Coulomb potential with bound states

For a particle moving in a spherically symmetric potential in D dimensions, the radial equation has the form

$$\left[-\frac{d^2}{dr^2}-\frac{D-1}{r}\frac{d}{dr}+\frac{l(l+D-2)}{r^2}+V(r)-E_{nl}\right]\Psi_{nl}(r)=0, \qquad (3.1)$$

where l is the angular momentum. Making the change of function $\Phi_{nl}(r) = r^{(D-1)/2}\Psi_{nl}(r)$ and using the notation $\lambda = l+(D-3)/2$, one can write the above equation in a Schrödinger canonical form as

$$\left[-\frac{d^2}{dr^2}+\frac{\lambda(\lambda+1)}{r^2}+V(r)-E_{n\lambda}\right]\Phi_{n\lambda}(r)=0, \qquad (3.2)$$

which is also independent on the dimension of the problem. One will solve now this equation for the Coulomb potential $V(r) = -1/r$ as in Ref.[32], and consider only the bound states with $E_{n\lambda} = -s^2$. Defining the new variable $x = 2sr$, one can rewrite Eq.(3.2) as:

$$\left[\frac{d^2}{dx^2}-\frac{\lambda(\lambda+1)}{x^2}+\frac{1}{2sx}-\frac{1}{4}\right]\Phi_{n\lambda}(x)=0. \qquad (3.3)$$

This equation is similar to the Whittaker differential equation [33]:

$$\left[\frac{d^2}{dx^2}-\frac{\mu^2-\frac{1}{4}}{x^2}+\frac{k}{x}-\frac{1}{4}\right]M_{k\mu}(x)=0. \tag{3.4}$$

Therefore, bearing in mind the correspondences

$$k=\frac{1}{2s},\ \mu=\lambda+\frac{1}{2}, \tag{3.5}$$

the solutions of Eq.(3.2) are just the Whittaker functions $M_{k\mu}$ [34] which can be expressed in terms of hypergeometric functions of the first kind ${}_1F_1(a,b;x)$ [35] as:

$$M_{k\mu}(x)=x^{\mu+\frac{1}{2}}e^{-\frac{x}{2}}\,{}_1F_1\left(\mu+\frac{1}{2}-k,2\mu+1;x\right). \tag{3.6}$$

Unfortunately, this function diverges at $x\to\infty$. In order to assure regularity at infinity, the hypergeometric function must be replaced with an associated Laguerre polynomial by means of the relation:

$${}_1F_1(-n;2\mu+1;x)=\frac{n!\,(2\mu)!}{(2\mu+n)!}L_n^{2\mu}(x). \tag{3.7}$$

This can be achieved only if the first argument of the hypergeometric function is a negative integer:

$$\mu+\frac{1}{2}-k=\lambda+1-\frac{1}{2s}=-n. \tag{3.8}$$

This condition together with the fact that $s^2=-E_{n\lambda}$ provides an equation for determining the energy $E_{n\lambda}$, whose solution is

$$E_{n\lambda}=-\frac{1}{4(n+\lambda+1)^2}. \tag{3.9}$$

The corresponding dimensionally reduced wave function is then expressed in terms of $x=2s_{n\lambda}r$ as:

$$\Phi_{n\lambda}(x)=\mathcal{N}_{n\lambda}x^{\lambda+1}e^{-\frac{x}{2}}L_n^{2\lambda+1}(x), \tag{3.10}$$

where

$$s_{n\lambda}=\sqrt{-E_{n\lambda}}=\frac{1}{2(n+\lambda+1)}. \tag{3.11}$$

The normalization $\mathcal{N}_{n\lambda}$ constant is determined from the condition

$$\int_0^\infty |\Phi_{n\lambda}(r)|^2 \, dr = 1. \tag{3.12}$$

Using the properties of the associated Laguerre polynomials, one can easily obtain its analytical expression

$$\mathcal{N}_{n\lambda} = \sqrt{\frac{n!}{(n+2\lambda+1)!(n+\lambda+1)}}. \tag{3.13}$$

The ladder operators for this system are [21]:

$$\hat{K}_\pm = \pm x\frac{d}{dx} + K_0 + \frac{x}{2}, \tag{3.14}$$

where

$$\hat{K}_0 = -x\frac{d^2}{dx^2} + \frac{\lambda(\lambda+1)}{x} + \frac{x}{4}. \tag{3.15}$$

Together, these operators form an $SU(1,1)$ algebra. It is important to mention that the operators (3.14) are understood as ladder operators not for the eigenfunctions (3.10) but for their tilted (squeezed) variations [20, 36] which have the same functional form as (3.10) but are normalized as:

$$\int_0^\infty \left[\Phi^S_{n\lambda}(r)\right]^2 \frac{1}{r} dr = 1. \tag{3.16}$$

The associated norms are then given as:

$$\mathcal{N}^S_{n\lambda} = \sqrt{\frac{n!}{(n+2\lambda+1)!}}. \tag{3.17}$$

These tilted functions are called Coulomb Sturmians [37] due to their close relationship with Sturm-Liouville theory and are heavily employed now in atomic and molecular physics [38]. Another aspect worth mentioning is the fact that operators (3.14) depend implicitly on energy of the system through the coordinate $x = 2s_{n\lambda}r$. That is they transform a function of variable depending on n and corresponding to an energy $E_{n\lambda}$ into a function of variable with $n \pm 1$ having an energy $E_{n\pm1\,\lambda}$.

4 Coulomb potential with a coupling constant linearly dependent on energy

The solutions of the Eq.(3.2) for a Coulomb potential with a coupling constant linearly dependent on energy:

$$V(E'_{n\lambda};r)=-\frac{1+aE'_{n\lambda}}{r}, \tag{4.1}$$

are found by following closely the steps from the previous section. There are however some particularities which come from the redefinition of the parameter k, which in the present case is identified as:

$$k'=\frac{1+as'^2}{2s'}, \tag{4.2}$$

with s' defined through $s'^2=-E'_{n\lambda}$. Plugging this expression in the condition (3.8), one obtains a new quadratic equation for determining the energy:

$$E'_{n\lambda}=-\frac{(1+aE'_{n\lambda})^2}{4(n+\lambda+1)^2}. \tag{4.3}$$

The physically meaning solution of the above equation presents itself as

$$E'_{n\lambda}=\frac{1}{a^2}\left[-2\epsilon_{n\lambda}^2-a+2\epsilon_{n\lambda}\sqrt{\epsilon_{n\lambda}^2+a}\right], \tag{4.4}$$

where $\epsilon_{n\lambda}=n+\lambda+1$. In what concerns the associated dimensionally suppressed wave function, it is given similarly as (3.10)

$$\Phi^E_{n\lambda}(x)=\mathcal{N}^E_{n\lambda}x^{\lambda+1}e^{-\frac{x}{2}}L_n^{2\lambda+1}(x), \tag{4.5}$$

where $x=2rs'_{n\lambda}$ while superscript "E" indicates that this wave-function corresponds to an energy dependent potential. $s'_{n\lambda}$ is extracted from Eq.(4.3) and has the simple expression

$$s'_{n\lambda}=\sqrt{-E'_{n\lambda}}=\frac{1+aE'_{n\lambda}}{2\epsilon_{n\lambda}}. \tag{4.6}$$

The normalization constant $\mathcal{N}^E_{n\lambda}$ is determined this time from the following condition:

$$\int_0^\infty \left|\Phi^E_{n\lambda}(r)\right|^2\left(1+\frac{a}{r}\right)dr=1, \tag{4.7}$$

where one employed the modified scalar product (2.1) particularized for the potential (4.1). Relying once again on the properties of the associated Laguerre polynomials, one can easily obtain the new norm:

$$\begin{aligned}\mathcal{N}_{n\lambda}^{E} &= \sqrt{\frac{n!s'_{n\lambda}}{(n+2\lambda+1)!(n+\lambda+1+as'_{n\lambda})}} \\ &= \sqrt{\frac{n!}{(n+2\lambda+1)!}}\sqrt{\frac{\sqrt{\epsilon_{n\lambda}^2+a}-\epsilon_{n\lambda}}{a\sqrt{\epsilon_{n\lambda}^2+a}}}. \end{aligned} \tag{4.8}$$

From the last expression one can deduce that the norm is real for $a > 0$, but also for negative values of a provided $|a| > \epsilon_{n\lambda}^2$. Nevertheless, a coherent quantum theory demands a positive definite density of probability distribution, which in our case is defined as:

$$\rho_{n\lambda}^{E} = \left|\Psi_{n\lambda}^{E}(r)\right|^2\left(1+\frac{a}{r}\right)r^{D-1} = \left|\Phi_{n\lambda}^{E}(r)\right|^2\left(1+\frac{a}{r}\right). \tag{4.9}$$

This is condition is fulfilled for all states only for $a > 0$.

5 Large slope regime

From the dependence of the energy function (4.4) on the slope parameter a for different combinations of the quantum numbers $\epsilon_{n\lambda} = n+\lambda+1$, one can observe that the whole spectrum normalized to the ground state energy presents a saturation at very high values of a [22]. This can be easily verified by considering the asymptotic expansion of the energy (4.4) as function of a,

$$E_{n\lambda}^{as} = -\frac{1}{a} + 2\frac{\epsilon_{n\lambda}}{a^{3/2}}. \tag{5.1}$$

This amounts to the neglecting of terms a^k with $k \geq 2$. From this result one can see that the excitation energy in respect to the ground state energy becomes a quantity which depends solely on the quantum numbers n and λ by means of $\epsilon_{n\lambda}$ when expressed in units of $a^{-3/2}$. This scaling property is also reflected in the wave-functions which become

$$\Phi_{n\lambda}^{as}(y) = \mathcal{N}_{n\lambda}^{as} y^{\lambda+1}e^{-\frac{y}{2}}L_n^{2\lambda+1}(y), \tag{5.2}$$

where the new variable is $y = 2r/\sqrt{a}$ on account of

$$s'_{as} = \frac{1}{\sqrt{a}}. \tag{5.3}$$

The above expression is obtained by using Eq.(5.1) in the last expression of Eq.(4.6).

The norm $N_{n\lambda}^{as}$ can be calculated either by making an asymptotic expansion of the expression (4.8) relative to a or from the condition:

$$\int_0^\infty |\Phi_{n\lambda}^{as}(r)|^2 \left(\frac{a}{r}\right) dr = 1. \tag{5.4}$$

Here, consistent with the large a regime, one also made the approximation

$$1 + \frac{a}{r} \approx \frac{a}{r} \tag{5.5}$$

for the correction term of the modified scalar product. Both procedures offer the same result for the norm in the asymptotic regime:

$$\mathcal{N}_{n\lambda}^{as} = \sqrt{\frac{n!}{a(n+\lambda+1)!}}. \tag{5.6}$$

The normalization of the wave-functions for this large a limit is very similar to that of the Coulomb Sturmians in Eq.(3.16). Note however that the scaled variable of the asymptotic wave-function no longer depends on the energy of the system as was happening in the cases of simple and energy dependent Coulomb potentials.

6 Symmetry properties of the asymptotic limit

The fact that the excitation spectrum is dependent only on quantum numbers up to a scaling factor, suggests the presence of a dynamical symmetry associated to the modeled system. In order to find the symmetry group governing the Schrödinger equations in this particular limit, one must find the associated ladder operators. To do this one must first rewrite the problem in the usual quantum mechanical space with non-distorted integration metric. Fortunately this is possible just for Schrödinger problems with potentials exhibiting a linear energy dependence. For the central potential considered in this study, one must extend the procedure presented in Sec.2 to the multidimensional case. Alternatively, one can use directly the one-dimensional example by making the correspondences:

$$V_0(r) = \frac{\lambda(\lambda+1)}{r^2} - \frac{1}{r}, \;\; V_1(r) = -\frac{a}{r}, \tag{6.1}$$

and considering the radial variable r instead of x. The equivalent Hamiltonian is then defined as

$$\tilde{H}_d = \frac{r}{r+a}\left[-\frac{d^2}{dr^2} - \frac{a}{r(r+a)}\frac{d}{dr} + \frac{\lambda(\lambda+1)}{r^2} + \frac{a(4r+a)}{4r^2(r+a)^2} - \frac{1}{r}\right]. \quad (6.2)$$

Before performing an asymptotic expansion of this differential operator in terms of a, it is better to rewrite its corresponding stationary Schrödinger equation in the following form:

$$\left[-\frac{d^2}{dr^2} - \frac{a}{r(r+a)}\frac{d}{dr} + \frac{\lambda(\lambda+1)}{r^2} + \frac{a(4r+a)}{4r^2(r+a)^2} - \frac{1+aE'_{n\lambda}}{r} - E'_{n\lambda}\right]\tilde{\Phi}^E_{n\lambda}(r) = 0 \quad (6.3)$$

and then use the Eq.(4.6) to achieve the final differential equation ready for the asymptotic approximation:

$$\left[-\frac{d^2}{dr^2} - \frac{a}{r(r+a)}\frac{d}{dr} + \frac{\lambda(\lambda+1)}{r^2} + \frac{a(4r+a)}{4r^2(r+a)^2} - \frac{2s'_{n\lambda}\epsilon_{n\lambda}}{r} + \left(s'_{n\lambda}\right)^2\right]\tilde{\Phi}^E_{n\lambda}(r) = 0. \quad (6.4)$$

Using the results from the previous section, the equation for the high a regime is readily found

$$\left[-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{\lambda(\lambda+1)+\frac{1}{4}}{r^2} - \frac{2s_{as}\epsilon_{n\lambda}}{r} + s^2_{as}\right]\tilde{\Phi}^{as}_{n\lambda}(r) = 0, \quad (6.5)$$

where $s^{as}_{n\lambda}$ is given by Eq.(5.3) while

$$\tilde{\Phi}^{as}_{n\lambda}(r) = \sqrt{\frac{a}{r}}\Phi^{as}_{n\lambda}(r), \quad (6.6)$$

with $\Phi^{as}_{n\lambda}$ defined by Eq.(5.2). Making the change of variable $y = 2s_{as}r$ one obtains the eigenvalue equation for $\epsilon_{n\lambda}$:

$$\left[-y\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{\lambda(\lambda+1)+\frac{1}{4}}{y} + \frac{y}{4}\right]\tilde{\Phi}^{as}_{n\lambda}(y) = \epsilon_{n\lambda}\tilde{\Phi}^{as}_{n\lambda}(y), \quad (6.7)$$

with the same eigenfunction (6.6) rewritten in terms of y as

$$\tilde{\Phi}^{as}_{n\lambda}(y) = \tilde{\mathcal{N}}^{as}_{n\lambda}y^{\lambda+\frac{1}{2}}e^{-\frac{y}{2}}L^{2\lambda+1}_n(y). \quad (6.8)$$

The norm of this function is calculated using the ordinarily defined scalar product and has the following expression:

$$\tilde{N}^{as}_{n\lambda} = \sqrt{\frac{2n!}{\sqrt{a}(n+2\lambda+1)!}}. \tag{6.9}$$

It can be easily checked that Eq.(6.7) can be brought to a form resembling the Laguerre differential equation, from where one can deduce from this treatment, which is independent from that of the last section that, its eigenvalue is indeed $\epsilon_{n\lambda} = n + \lambda + 1$.

The next step is to study the symmetry properties of this wave-function. For this, one needs to obtain the ladder operators which transforms the state with n into a state with $n \pm 1$ and the same λ. These can be found through different ways, such as the factorization method [21] or by using the properties of the associated Laguerre polynomials [24]. The second procedure is the option chosen for this task. Using the identity

$$\frac{d}{dy}L^{\alpha}_{n}(y) = \frac{1}{y}\left[nL^{\alpha}_{n}(y) - (n+\alpha)L^{\alpha}_{n-1}(y)\right], \tag{6.10}$$

one can write down the following transformation:

$$\left[-y\frac{d}{dy} - \frac{y}{2} + \epsilon_{n\lambda} - \frac{1}{2}\right]\tilde{\Phi}^{as}_{n\lambda}(y) = \frac{\tilde{N}^{as}_{n\lambda}}{\tilde{N}^{as}_{n-1\,\lambda}}(n+2\lambda+1)\tilde{\Phi}^{as}_{n-1\,\lambda}(y). \tag{6.11}$$

This transformation defines the lowering operator

$$\hat{J}_{-} = -y\frac{d}{dy} - \frac{y}{2} + \epsilon_{n\lambda} - \frac{1}{2} \tag{6.12}$$

satisfying $\hat{J}_{-}\tilde{\Phi}^{as}_{n\lambda}(y) = j_{-}\tilde{\Phi}^{as}_{n-1\,\lambda}(y)$ with $j_{-} = \sqrt{n(n+2\lambda+1)}$.

Alternatively, using the relation

$$\frac{d}{dy}L^{\alpha}_{n}(y) = \frac{1}{y}\left[(x-n-\alpha-1)L^{\alpha}_{n}(y) + (n+1)L^{\alpha}_{n+1}(y)\right], \tag{6.13}$$

one obtains in a similar way the operator

$$\hat{J}_{+} = y\frac{d}{dy} - \frac{y}{2} + \epsilon_{n\lambda} + \frac{1}{2}, \tag{6.14}$$

whose action on the wave-function $\tilde{\Phi}^{as}_{n\lambda}(y)$ is

$$\hat{J}_{+}\tilde{\Phi}^{as}_{n\lambda}(y) = j_{+}\tilde{\Phi}^{as}_{n+1\,\lambda}(y) = \sqrt{(n+1)(n+2\lambda+2)}\tilde{\Phi}^{as}_{n+1\,\lambda}(y). \tag{6.15}$$

Now, we can calculate the action of the commutator between these ladder operators:

$$\left[\hat{J}_-, \hat{J}_+\right] \tilde{\Phi}^{as}_{n\lambda}(y) = 2(n+\lambda+1)\tilde{\Phi}^{as}_{n\lambda}(y) = 2\epsilon_{n\lambda}\tilde{\Phi}^{as}_{n\lambda}(y). \tag{6.16}$$

Knowing from Eq.(6.7) the operator corresponding to the eigenvalue $\epsilon_{n\lambda}$, one can define the third operator as

$$\hat{J}_0 = -y\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{\lambda(\lambda+1)+\frac{1}{4}}{y} + \frac{y}{4}, \tag{6.17}$$

which satisfies $\hat{J}_0\tilde{\Phi}^{as}_{n\lambda}(y) = j_0\tilde{\Phi}^{as}_{n\lambda}(y)$ with $j_0 = \epsilon_{n\lambda} = (n+\lambda+1)$. With this, the operators $\hat{J}_\pm$ will acquire the full differential representation:

$$\begin{aligned}\hat{J}_\pm &= \pm y\frac{d}{dy} - \frac{y}{2} + \hat{J}_0 \pm \frac{1}{2} \\ &= -y\frac{d^2}{dy^2} - (1\mp y)\frac{d}{dy} + \frac{\lambda(\lambda+1)+\frac{1}{4}}{y} - \frac{y}{4} \pm \frac{1}{2}.\end{aligned} \tag{6.18}$$

It is easy now to verify that the $\hat{J}_\pm$ and $\hat{J}_0$ operators satisfy the commutation relations:

$$\left[\hat{J}_-, \hat{J}_+\right] = 2\hat{J}_0, \quad \left[\hat{J}_0, \hat{J}_\pm\right] = \pm\hat{J}_\pm \tag{6.19}$$

specific to the $SU(1,1)$ symmetry group. The associated Casimir operator of this group is

$$\begin{aligned}\hat{C}_{SU(1,1)} &= \hat{J}_0(\hat{J}_0 - 1) - \hat{J}_+\hat{J}_- \\ &= \hat{J}_0(\hat{J}_0 + 1) - \hat{J}_-\hat{J}_+.\end{aligned} \tag{6.20}$$

Its eigenvalue problem reads

$$\hat{C}_{SU(1,1)}\Psi_{n,\lambda} = J(J-1)\Psi_{n,\lambda}, \tag{6.21}$$

where $J > 0$ denotes the Bargmann index of the $SU(1,1)$ symmetry group. Making use of the expressions for $j_\pm$ and j_0, one can identify $J = \lambda + 1$. Coming back to the angular momentum quantum number, one will have $J = l + (D-1)/2$. This last bit sets the allowed values of the Bargmann index to integer and half integer numbers. Since the group $SU(1,1)$ is a non-compact one, all its unitary irreducible representations are infinite dimensional. Therefore, the basis states are indexed by an additional non-negative integer M defined such that the eigenvalue equations:

$$\hat{C}_{SU(1,1)}|J,M\rangle = J(J-1)|J,M\rangle, \quad \hat{J}_0|J,M\rangle = (J+M)|J,M\rangle, \tag{6.22}$$

are simultaneously satisfied. This condition obviously imply $M = n$.

Finally, based on the expression (5.1) of the eigenvalue in the asymptotic regime of a, one can write down an effective Hamiltonian for it as

$$\hat{\tilde{H}}_{eff} = \frac{1}{a}\left(\frac{2}{\sqrt{a}}\hat{J}_0 - 1\right). \tag{6.23}$$

In an expanded form, this expression has the following differential realization:

$$\begin{aligned}\tilde{H}_{eff} &= \frac{2}{a^{3/2}}\left[-y\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{\lambda(\lambda+1)+\frac{1}{4}}{y} + \frac{y}{4} - \frac{\sqrt{a}}{2}\right] \\ &= \frac{1}{a}\left[-r\frac{d^2}{dr^2} - \frac{d}{dr} + \frac{\lambda(\lambda+1)+\frac{1}{4}}{r} + \frac{r}{a} - 1\right].\end{aligned} \tag{6.24}$$

Note that this operator differs from an operator obtained by performing the $a \gg r$ approximation on Eq.(6.2). This discrepancy comes from the fact that the asymptotic expansion in a must be performed on several levels, taking also into account the a-dependence from the action of the differential operators on the wave-functions. As the effective Hamiltonian of the asymptotically steep energy dependent Coulomb potential is proportional to a non-central element (not to the Casimir) of the $SU(1,1)$ algebra, it is said that this is the spectrum generating algebra for the considered problem [6].

7 Conclusions

The analytical structure of the Schrödinger equation for a Coulomb potential with bound states and a coupling which depends linearly on the total energy of the system, was investigated in order to ascertain the remnants of the symmetry properties governing the unperturbed problem. For the sake of generality, the study is performed for D dimensions. The non-local problem preserves some of the analytical structure of the local one. There are however some very important changes such as the modification of the scalar product and the fact that the eigenvalue is obtained from a higher order polynomial equation. These aspects, as well as other implications define a new Hilbert space associated to a non-local quantum theory. The linear dependence on energy of the potential adopted in this study provides a favorable environment to study the algebraic composition of the problem. This is due

to the fact that the non-local Schrödinger equation in this case can be rewritten in the usual quantum theory, and one can then apply well established group theoretical methods. Nevertheless, although the nonlocal problem is exactly solvable, it cannot be effectively factorized and consequently algebraized. The reason is the highly nonlinear dependence on energy of the reformulated Hamiltonian and its solutions. This obstacle vanishes if one considers very steep energy dependence by means of an asymptotic expansion relative to the slope parameter. After rewriting this problem into the ordinary quantum mechanical picture, one identified the raising and lowering operators for the transformed wave functions corresponding to the large slope regime of the energy dependence. This was done based on the properties of the associated Laguerre polynomials defining the eigenfunctions. It is then shown that these ladder operators, together with another operator which does not change the transformed wave-functions, satisfy the $SU(1,1)$ symmetry commutation relations. Moreover, the effective Hamiltonian of the asymptotic case is proportional with the complementary operator, such that one can conclude that the $SU(1,1)$ is the spectrum generating algebra for the bound states of the Coulomb potential with an asymptotically steep linear dependence on energy. The distinguishing aspect of the present realization of the $SU(1,1)$ dynamical symmetry, is that contrary to the unperturbed situation, the same algebra closes on wave-functions which are correctly normalized and not Sturmian contractions.

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