Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 9, No. 2/2017

RANDOM FUNCTIONAL EVOLUTION EQUATIONS WITH STATE-DEPENDENT DELAY*

Amel Benaissa† Mouffak Benchohra‡ Juan J. Nieto§

Abstract

Our aim in this work is to study the existence of mild solutions of a functional differential equation with delay and random effects. We use a random fixed point theorem with stochastic domain to show the existence of mild random solutions.

MSC: 34G20, 34K20, 34K30

Key words : Random fixed point, Functional differential equation, mild random solution, finite delay, semigroup theory.

1 Introduction

Functional evolution equations with state-dependent delay appear frequently in mathematical modeling of several real world problems and for this reason the study of this type of equations has received great attention in the last few years, see for instance [1, 9, 19, 20]. Functional differential

*Accepted for publication in revised form on July 26-th, 2017

†amel2222@yahoo.com Laboratory of Mathematics, University of Sidi Bel-Abbès PO Box 89, 22000 Sidi Bel-Abbès, Algeria

‡benchohra@univ-sba.dz Laboratory of Mathematics, University of Sidi Bel-Abbès PO Box 89, 22000 Sidi Bel-Abbès, Algeria

§juanjose.nieto.roig@usc.es Departamento de Análisis Matemático, Facultad de Matemáticas Universidad de Santiago de Compostela, Santiago de Compostela, Spain

equations and inclusions involving state-dependent delay are considered in [8, 15, 23, 31, 34, 35, 36]. Some models are studied in [24, 25].

Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. On the other hand, the nature of a dynamic system in engineering or natural sciences depends on the accuracy of the information we have concerning the parameters that describe that system. If the knowledge about a dynamic system is precise then a deterministic dynamical system arises. Unfortunately in most cases the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of a dynamic system are of statistical nature, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random differential equations, as natural extensions of deterministic ones, arise in many applications Random operator theory is needed for the study of various classes of random equations [30]. The theory of random fixed point theorems was initiated by the Prague school of probabilistic in the 1950s. The interest in this subject enhanced after publication of the survey paper by Bharucha Reid [6]. Random fixed point theory has received much attention in recent years (see [5, 26, 27, 32]).

On the other hand, the stochastic differential equation with delay is a special type of stochastic functional differential equations. Delay differential equations arise in many biological and physical applications, and it often forces us to consider variable or state-dependent delays. The stochastic functional differential equations with state-dependent delay have many important applications in mathematical models of real phenomena, and the study of this type of equations has received much attention in recent years. Guendouzi and Benzatout [14] studied the existence of mild solutions for a class of impulsive stochastic differential inclusions with state-dependent delay. Sakthivel and Ren [29] studied the approximate controllability of fractional differential equations with state-dependent delay. Benaissa *et al.* [3, 4] obtained local existence results of mild solutions for two classes of functional random evolution equations with delay.

In this work we prove the existence of mild solutions of the following functional evolution differential equation with delay and random effects (random parameters) of the form:

$$y'(t,w) = A(t)y(t,w) + f(t, y_{\rho(t,y_t)}(\cdot,w), w), \quad \text{a.e. } t \in J := [0,\infty) \qquad (1)$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \qquad (2)$$

where (Ω, F, P) is a complete probability space, $w \in \Omega$ where $f : J \times \mathcal{B} \times \Omega \to E$, $\phi : (-\infty, 0] \times \Omega \to E$ are given random functions, $\{A(t)\}_{0 \le t < +\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t,s)\}_{(t,s) \in J \times J}$ for $0 \le s \le t < +\infty$, $\mathcal{B}$ is the phase space to be specified later, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$, and $(E, |\cdot|)$ is a real Banach space. For any function y defined on $(-\infty, +\infty) \times \Omega$ and any $t \in J$ we denote by $y_t(\cdot, w)$ the element of $\mathcal{B}$ defined by $y_t(\theta, w) = y(t+\theta, w), \theta \in (-\infty, 0]$. Here $y_t(\cdot, w)$ represents the history of the state from time $-\infty$, up to the present time t. We assume that the histories $y_t(\cdot, w)$ to some abstract phases $\mathcal{B}$, to be specified later. To our knowledge, the literature on the global existence of random evolution equations with delay is very limited, so the present paper can be considered as a contribution to this issue.

2 Preliminaries

Let E be a Banach space with the norm $|\cdot|$ and $BC(J,E)$ the Banach space of all bounded and continuous functions y mapping J into E with the usual supremum norm

$$\|y\| = \sup_{t \in J} |y(t)|.$$

Let $B(E)$ denote the Banach space of bounded linear operators from E into E. A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see the classical monograph of Yosida [37]). Let $L^1(J,E)$ denote the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^\infty |y(t)|\, dt.$$

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0)$ to E.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [17] and follow the terminology used in [21]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, +\infty) \to E$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold :
(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
(iii) There exist two functions $L(\cdot), M(\cdot) : [0, \infty) \to [0, \infty)$ independent of y with L continuous and bounded, and M bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq L(t) \sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function for each $t \in J$.

(A_3) The space $\mathcal{B}$ is complete.

Denote

$$K_\infty = \sup\{L(t) : t \in J\},$$

and

$$M_\infty = \sup\{M(t) : t \in J\}.$$

Definition 1. *A map $f : J \times \mathcal{B} \times \Omega \to E$ is said to be Carathéodory if*

(i) $t \to f(t, y, w)$ is measurable for all $y \in \mathcal{B}$ and for all $w \in \Omega$.

(ii) $y \to f(t, y, w)$ is continuous for almost each $t \in J$ and for all $w \in \Omega$.

(iii) $w \to f(t, y, w)$ is measurable for all $y \in \mathcal{B}$, and almost each $t \in J$.

In what follows, we assume that $\{A(t),\ t \geq 0\}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t.

Definition 2. *A family of bounded linear operators*

$$\{U(t,s)\}_{(t,s)\in\Delta} : U(t,s) : E \to E\ (t,s) \in \Delta := \{(t,s) \in J \times J : 0 \leq s \leq t < +\infty\}$$

is called en evolution system if the following properties are satisfied:

1. *$U(t,t) = I$ where I is the identity operator in E,*
2. *$U(t,s)\ U(s,\tau) = U(t,\tau)$ for $0 \leq \tau \leq s \leq t < +\infty$,*
3. *$U(t,s) \in B(E)$ the space of bounded linear operators on E, where for every $(s,t) \in \Delta$ and for each $y \in E$, the mapping $(t,s) \to U(t,s)\ y$ is continuous.*

More details on evolution systems and their properties are reported in the books by Ahmed [2], Engel and Nagel [11], Pazy [28] and Vrabie [33].

Lemma 1 (Corduneanu)**.** *[7]*
Let $C \subset BC(J,E)$ be a set satisfying the following conditions:

(i) *C is bounded in $BC(J,E)$;*

(ii) *the functions belonging to C are equicontinuous on any compact interval of J;*

(iii) *for any compact subinterval $\tilde{J}$ of J, and each $t \in \tilde{J}$, the set $C(t) := \{y(t) : y \in C\}$ is relatively compact;*

(iv) *the functions from C are equiconvergent, i.e., given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|y(t) - y(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $y \in C$.*

Then C is relatively compact in $BC(J,E)$.

Theorem 1. ***(Schauder fixed point**)[13]*
Let B be a closed, convex and nonempty subset of a Banach space E. Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of E. Then N has at least one fixed point in B.

Let Y be a separable Banach space with the Borel σ-algebra B_Y. A mapping $y : \Omega \longrightarrow Y$ is said to be a random variable with values in Y if for each $B \in B_Y, y^{-1}(B) \in F$. A mapping $T : \Omega \times Y \longrightarrow Y$ is called a random operator if $T(\cdot, y)$ is measurable for each $y \in Y$ and is generally expressed as $T(w,y) = T(w)y$; we will use these two expressions alternatively.
Next, we will give a very useful random fixed point theorem with stochastic domain.

Definition 3. *[10] Let C be a mapping from Ω into 2^Y. A mapping $T : \{(w,y) : w \in \Omega \wedge y \in C(w)\} \longrightarrow Y$ is called 'random operator with stochastic domain C' if and only if C is measurable (i.e., for all closed $A \subseteq Y, \{w \in \Omega : C(w) \cap A \neq \emptyset\} \in F$) and for all open $D \subseteq Y$ and all $y \in Y, \{w \in \Omega : y \in C(w) \wedge T(w,y) \in D\} \in F$. T we be called 'continuous' if every $t(w)$ is continuous. For a random operator T, a mapping $y : \Omega \longrightarrow Y$ is called 'random (stochastic) fixed point of T' if and only if for p-almost all $w \in \Omega, y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subseteq Y, \{w \in \Omega : y(w) \in D\} \in F$('$y$ is measurable').*

Remark 1. *If $C(w) \equiv Y$, then the definition of random operator with stochastic domain coincides with the definition of random operator.*

Lemma 2. *[10] Let $C : \Omega \longrightarrow 2^Y$ be measurable with $C(w)$ closed, convex and solid (i.e., int $C(w) \neq \emptyset$) for all $w \in \Omega$. We assume that there exists measurable $y_0 : \Omega \longrightarrow Y$ with $y_0 \in$ int $C(w)$ for all $w \in \Omega$. Let T be a continuous random operator with stochastic domain C such that for every $w \in \Omega, \{y \in C(w) : T(w)y = y\} \neq \emptyset$. Then T has a stochastic fixed point.*

Let y be a mapping of $J \times \Omega$ into X. y is said to be a stochastic process if for each $t \in J, y(t, \cdot)$ is measurable.

3 Existence of mild solutions

Now we give our main existence result for problem (1)-(2). Before starting and proving this result, we give the definition of the mild random solution.

Definition 4. *A stochastic process $y : J \times \Omega \to E$ is said to be random mild solution of problem (1)-(2) if $y(t, w) = \phi(t, w),\ t \in (-\infty, 0]$ and the restriction of $y(\cdot, w)$ to the interval J is continuous and satisfies the following integral equation:*

$$y(t, w) = U(t, 0)\phi(0, w) + \int_0^t U(t, s) f(s, y_{\rho(s,y_s)}(\cdot, w), w)ds,\ t \in J. \tag{3}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s, \varphi) : (s, \varphi) \in J \times \mathcal{B}, \rho(s, \varphi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to (-\infty, \infty)$ is continuous. Additionally, we introduce following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 2. *The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [21].*

Lemma 3. *([18], Lemma 2.4) If $y : (-\infty, T] \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T \sup\{|y(\theta)|; \theta \in [0, max\{0, s\}]\}, \ s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^{\phi} = \sup\limits_{t \in \mathcal{R}(\rho^-)} L^{\phi}(t)$.

We will need to introduce the following hypothesis which are be assumed there after:

(H_1) There exists a constant $M \geq 1$ and $\alpha > 0$ such that

$$\|U(t,s)\|_{B(E)} \leq Me^{-\alpha(t-s)} \quad \text{for every } (s,t) \in \Delta.$$

(H_2) The function $f : J \times \mathcal{B} \times \Omega \to E$ is Carathéodory.

(H_3) There exist functions $\psi : J \times \Omega \to [0, \infty)$ and $p : J \times \Omega \to [0, \infty)$ such that for each $w \in \Omega$, $\psi(\cdot, w)$ is a continuous nondecreasing function and $p(\cdot, w)$ integrable with:

$$|f(t,u,w)| \leq p(t,w)\ \psi(\|u\|_{\mathcal{B}}, w) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B},$$

(H_4) For each $(t,s) \in \Delta$ we have

$$\lim_{t \to +\infty} \int_0^t e^{-\alpha(t-s)} p(s,w) ds = 0.$$

(H_5) There exists a random function $R : \Omega \longrightarrow (0, \infty)$ such that:

$$M\|\phi\|_{\mathcal{B}} + M\ \psi\Big((M + L^{\phi})\|\phi\|_{\mathcal{B}} + KR(w), w\Big)\|p\|_{L^1} \leq R(w),$$

(H_6) For each $w \in \Omega, \phi(\cdot, w)$ is continuous and for each $t, \phi(t, \cdot)$ is measurable.

Theorem 2. *Suppose that hypotheses (H_ϕ) and $(H_1) - (H_6)$ are valid, then the random with delay problem (1)-(2) has at least one mild random solution on $(-\infty, \infty)$.*

Proof. Let Y be the space defined by

$$Y = \{y : (-\infty, +\infty) \to E \text{ such that } y|_J \in BC(J, E) \text{ and } y_0 \in \mathcal{B}\},$$

we denote by $y|_J$ the restriction of y to J, endowed with the uniform convergence topology and $N : \Omega \times Y \to Y$ be the random operator defined by:

$$(N(w)y)(t) = \begin{cases} \phi(t, w), & \text{if } t \in (-\infty, 0] \\ U(t, 0)\phi(0, w) + \displaystyle\int_0^t U(t, s) & \\ f(s, y_{\rho(s,y_s)}(\cdot, w), w)ds, & \text{if } t \in J. \end{cases} \tag{4}$$

Then we show that the mapping defined by (4) is a random operator. To do this, we need to prove that for any $y \in Y$, $N(\cdot)(y) : \Omega \longrightarrow Y$ is a random variable. Then we prove that $N(\cdot)(y) : \Omega \longrightarrow Y$ is measurable as a mapping $f(t, y, \cdot), t \in J, y \in Y$ is measurable by assumption (H_2) and (H_6).

Let $D : \Omega \longrightarrow 2^Y$ be defined by:

$$D(w) = \{y \in Y : \|y\| \le R(w)\}.$$

$D(w)$ is bounded, closed, convex and solid for all $w \in \Omega$. Then D is measurable by lemma 17 in [16].

Let $w \in \Omega$ be fixed, If $y \in D(w)$, from Lemma 3 follows that

$$\|y_{\rho(t,y_t)}\|_{\mathcal{B}} \le (M + L^{\phi})\|\phi\|_{\mathcal{B}} + KR(w),$$

and for each $y \in D(w)$, by (H3) and (H5), we have for each $t \in J$

$$\begin{aligned} |(N(w)y)(t)| &\le M\|U(t, 0)\|_{B(E)}\|\phi\|_{\mathcal{B}} + M \int_0^t \|U(t, s)\|_{B(E)}|f(s, y_{\rho(s,y_s)}, w)| \, ds \\ &\le Me^{-\alpha t}\|\phi\|_{\mathcal{B}} + M \int_0^t e^{-\alpha(t-s)}p(s, w) \ \psi\left(\|y_{\rho(s,y_s)}\|_{\mathcal{B}}, w\right) \, ds \\ &\le M\|\phi\|_{\mathcal{B}} + M \int_0^t p(s, w) \ \psi\Big((M_\infty + L^{\phi})\|\phi\|_{\mathcal{B}} + K_\infty R(w), w\Big) \, ds \\ &\le M\|\phi\|_{\mathcal{B}} + M \ \psi\Big((M_\infty + L^{\phi})\|\phi\|_{\mathcal{B}} + K_\infty R(w), w\Big) \|p\|_{L^1} \\ &\le R(w). \end{aligned}$$

This implies that N is a random operator with stochastic domain D and $N(w) : D(w) \longrightarrow D(w)$ for each $w \in \Omega$.

Step 1: N is continuous.
Let y^n be a sequence such that $y^n \longrightarrow y$ in Y. Then

$$\begin{aligned}|(N(w)y^n)(t)-(N(w)y)(t)| &\le \int_0^t \|U(t,s)\|_{B(E)}\Big|f(s,y^n_{\rho(s,y^n_s)},w)- \\ &\qquad f(s,y_{\rho(s,y_s)},w)\Big|\,ds\\ &\le M\int_0^t e^{-\alpha(t-s)}\Big|f(s,y^n_{\rho(s,y^n_s)},w)-f(s,y_{\rho(s,y_s)},w)\Big|\,ds.\end{aligned}$$

Since $f(s,\cdot,w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$|(N(w)y^n)(t)-(N(w)y)(t)| \to 0 \text{ as } n\to+\infty.$$

Thus N is continuous.

Step 2: We prove that for every $w\in\Omega, \{y\in D(w) : N(w)y=y\}\neq\emptyset$. For prove this we apply Schauder's theorem. $N(D(w))$ is relatively compact. To prove the compactness, we will use Corduneanu's lemma.

(a) Firstly, it is clear that the assumption (i) is holds. Then we will show that $N(D(w))$ is equicontinuous set for each closed bounded interval $[0,T]$ in J. Let $\tau_1,\tau_2\in[0,T]$ with $\tau_2>\tau_1$, $D(w)$ be a bounded set, and $y\in D(w)$. Then

$$\begin{aligned} & |(N(w)y)(\tau_2)-(N(w)y)(\tau_1)|\\ \le\ & \|U(\tau_2,0)-U(\tau_1,0)\|_{B(E)}\|\phi\|_{\mathcal{B}}\\ +\ & \Big|\int_0^{\tau_1}[U(\tau_2,s)-U(\tau_1,s)]f(s,y_{\rho(s,y_s)},w)ds\Big|\\ +\ & \Big|\int_{\tau_1}^{\tau_2}U(\tau_2,s)f(s,y_{\rho(s,y_s)},w)\Big|ds\\ \le\ & \|U(\tau_2,0)-U(\tau_1,0)\|_{B(E)}\|\phi\|_{\mathcal{B}}\\ +\ & \int_0^{\tau_1}|U(\tau_2,s)-U(\tau_1,s)||f(s,y_{\rho(s,y_s)},w)|\,ds\\ +\ & \int_{\tau_1}^{\tau_2}|U(\tau_2,s)f(s,y_{\rho(s,y_s)},w)|\,ds\end{aligned}$$

$$\begin{aligned}
&\leq \; |U(\tau_2,0)-U(\tau_1,0)|\|\phi\|_{\mathcal{B}}+\psi\Big((M_\infty+L^\phi)\|\phi\|_{\mathcal{B}}+K_\infty R(w),w\Big) \\
&\qquad \int_0^{\tau_1}|U(\tau_2,s)-U(\tau_1,s)|p(s,w)ds \\
&+ \; M\psi\Big((M_\infty+L^\phi)\|\phi\|_{\mathcal{B}}+K_\infty R(w),w\Big)\int_{\tau_1}^{\tau_2}p(s,w)ds.
\end{aligned}$$

The right-hand of the above inequality tends to zero as $\tau_2-\tau_1\to 0$, thus N is bounded and equicontinuous.

Next, let $w\in\Omega$ be fixed (therefore we do not write 'w' in the sequel) but arbitrary.

(b) Now we will prove that $Z(t,w)=\{(N(w)y)(t):y\in D(w)\}$ is precompact in E. Let $t\in[0,T]$ be fixed and let ϵ be a real number satisfying $0<\epsilon<t$. For $y\in D(w)$ we define

$$(N_\epsilon(w)y)(t)=U(t,0)\phi(0,w)+U(t,t-\epsilon)\int_0^{t-\epsilon}U(t-\epsilon,s)f(s,y_{\rho(s,y_s)},w)\,ds.$$

Since $U(t,s)$ is a compact operator and the set $Z_\epsilon(t,w)=\{(N_\epsilon(w)y)(t): y\in D(w)\}$ is the image of bounded set of E then $Z_\epsilon(t,w)$ is precompact in E for every ϵ, $0<\epsilon<t$. Moreover

$$\begin{aligned}
&|(N(w)y)(t)-(N_\epsilon(w)y)(t)| \\
\leq\; &\int_{t-\epsilon}^t\|U(t,s)\|_{B(E)}|f(s,y_{\rho(s,y_s)},w)|ds \\
\leq\; &M\psi\Big((M+L^\phi)\|\phi\|_{\mathcal{B}}+KR(w),w\Big)\,e^{-\alpha(t-s)}\int_{t-\epsilon}^t p(s,w)ds.
\end{aligned}$$

The right-hand side tends to zero as $\epsilon\to 0$, then $N(w)y$ converges uniformly to $N_\epsilon(w)y$ which implies that $Z(t,w)=\{(N(w)y)(t):y\in D(w)\}$ is precompact in E.

(c) Finally, it remains to show that N is equiconvergent.

Let $y\in D(w)$, then from (H_1), (H_3) we have

$$|(N(w)y)(t)|\leq Me^{-\alpha t}\|\phi\|_{\mathcal{B}}+M\;\psi\Big((M_\infty+L^\phi)\|\phi\|_{\mathcal{B}}+K_\infty R(w),w\Big)$$

$$\int_0^t e^{-\alpha(t-s)} p(s,w)\, ds,$$

it follows immediately by (H_4) that $|(N(w)y)(t)| \longrightarrow 0$ as $t \to +\infty$. Then

$$\lim_{t\to+\infty} |(N(w)y)(t) - (N(w)y)(+\infty)| = 0,$$

which implies that N is equiconvergent.

A consequence of Steps 1-2 and (a), (b), (c), we can conclude that $N(w) : D(w) \to D(w)$ is continuous and compact. From Schauder's theorem, we deduce that $N(w)$ has a fixed point $y(w)$ in $D(w)$. Since $\bigcap_{w\in\Omega} D(w) \neq \emptyset$, and the hypothesis that a measurable selector of int$D$ exists holds, Lemma 2 implies that the random operator N has a stochastic fixed point $y^*(w)$, which is a mild solution of the random problem (1)-(2).

4 An example

Consider the following functional partial differential equation:

$$\frac{\partial}{\partial t} z(t,x,w) = a(t,x)\frac{\partial^2}{\partial x^2} z(t,x,w) + C_0(w)b(t)\int_{-\infty}^0 F(z(t+\sigma(t,z(t+s,$$

$$\text{x,w)),x,w))ds, x} \in [0,\pi],\ t\geq 0,\ w\in\Omega \tag{5}$$

$$z(t,0,w) = z(t,\pi,w) = 0,\ t\geq 0, w\in\Omega \tag{6}$$

$$z(s,x,w) = z_0(s,x,w),\ s\in(-\infty,0],\ x\in[0,\pi], w\in\Omega, \tag{7}$$

where $a(t,\xi)$ is a continuous function which is uniformly Hölder continuous in t, C_0 is a real-valued random variable, $b \in L^1(J;(0,+\infty))$, $F:\mathbb{R}\to\mathbb{R}$ is continuous, $z_0 : (-\infty,0]\times[0,\pi]\times\Omega \to \mathbb{R}$ and $\sigma : J\times\mathbb{R}\to\mathbb{R}$ are given functions.

Suppose that $E = L^2[0,\pi]$, (Ω, F, P) is a complete probability space. Take and define $A(t) : D(A(t)) \subset E \to E$ by $A(t)v(\cdot) = a(t,\cdot)v''(\cdot)$ with domain

$$D(A) = \{v\in E, v, v'\text{are absolutely continuous},\ v''\in E,\ v(0)=v(\pi)=0\}.$$

Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumption (H_1) (see [12, 22]).

Let $\mathcal{B} = BCU((-\infty, 0]; E)$: the space of uniformly bounded continuous functions endowed with the following norm:

$$\|\phi\| = \sup_{s \leq 0} |\phi(s)| \qquad \text{for } \phi \in \mathcal{B}$$

For $\phi \in BCU((-\infty, 0]; E), x \in [0, \pi]$ and $w \in \Omega$, Set

$$y(t, x, w) = z(t, x, w), t \in [0, T]$$

$$\phi(s, x, w) = z_0(s, x, w), \; s \in (-\infty, 0],$$

$$f(t, \phi(x), w) = C_0(w)b(t) \int_{-\infty}^{0} F(z(t + \sigma(t, z(t + s, x, w)), x, w))ds,$$

and

$$\rho(t, \phi)(x) = \sigma(t, z(t, x, w)).$$

Let $\phi \in \mathcal{B}$ be such that (H_ϕ) holds, and let $t \to \phi_t$ be continuous on $\mathcal{R}(\rho^-)$.

Then the problem (1)-(2) in an abstract formulation of the problem (5)-(7), and conditions $(H_1) - (H_6)$ are satisfied. Theorem 2 implies that the random problem (5)-(7) has at least one random mild solutions.

Acknowledgement. The authors are grateful to the referee for the helpful remarks.

References

[1] S. Abbas and M. Benchohra, *Advanced Functional Evolution Equations and Inclusions*, Developments in Mathematics, **39**, Springer, Cham, 2015.

[2] N.U. Ahmed, *Semigroup Theory with Applications to Systems and Control*, Harlow John Wiley & Sons, Inc., New York, 1991.

[3] A. Benaissa and M. Benchohra, Functional differential equations with state-dependent delay and random effects. *Rom. J. Math. Comput. Sci.* **5** (2015), 84-94.

[4] A. Benaissa, M. Benchohra, and J. Graef, Functional differential equations with delay and random effects. *Stoch. Anal. Appl.* **33** (2015), 1083-1091.

[5] I. Beg, and N. Shahzad, Random fixed points of random multivalued operators on polish spaces, *Nonlinear Anal.* **20** (1993), 835-847.

[6] A. T. Bharucha-Reid, Fixed point theorem in probabilistic analysis, *Bull. Amer. Math. Soc* **82** (1976), 641-645.

[7] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Acadimic Press, New York, 1973.

[8] O. Diekmann and K. Korvasov, Linearization of solution operators for state-dependent delay equations: A simple example, *Disc. Cont. Dynam. Systems- Series A* **36** (2016), 137-149.

[9] J. P. C. dos Santos, On state-dependent delay partial neutral functional integrodifferential equations, *Appl. Math. Comput* **100** (2010), 1637-1644.

[10] HW. Engl, A general stochastic fixed-point theorem for continuos random operators on stochastic domains. *Anal. Appl.* **66** (1978), 220-231.

[11] K.J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.

[12] A. Friedman, *Partial Differential Equations*, Holt, Rinehat and Winston, New York, 1969.

[13] A. Granas, and J. Dugundji, *Fixed Point Theory.* Springer-Verlag New York, 2003.

[14] T. Guendouzi and O. Benzatout, Existence of mild solutions for impulsive fractional stochastic differential inclusions with state-dependent delay, *Chin. J. Math.*, Article ID 981714 **21** (2014).

[15] X. He and R. de la Llave, Construction of quasi-periodic solutions of state-dependent delay differential equations by the parameterization method II: Analytic case, *J. Differential Equations*, 261 (3) (2016), 2068-2108.

[16] S. Itoh, Random fixed point theorems with an application to random differential equations in Banach space, *Anal. Appl* **67** (1979), 261-273.

[17] J. Hale, and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21** (1978), 11-41.

[18] E. Hernández, R. Sakthivel, and A. Tanaka, Existence results for impulsive evolution differential equations with state-dependent delay, *Electron. J. Differential Equations* **2008** (2008), 1-11.

[19] E. Hernández M. and M. A. McKibben, On state-dependent delay partial neutral functional-differential equations, *Appl. Math. Comput.* **186** (2007), 294-301.

[20] E. Hernández, A. Prokopczyk and L. Ladeira, A note on partial functional differential equations with state-dependent delay, *Nonlin. Anal.* **7** (2006), 510-519.

[21] Y. Hino, S. Murakami, and T. Naito, *Functional Differential Equations with Unbounded Delay*, Springer-Verlag, Berlin, 1991.

[22] S.G. Krein, *Linear Differential Equation in Banach Spaces,* Amer. Math. Soc.,Providence, 1971.

[23] W.-S. Li, Y.-K. Chang, and J.J. Nieto, Solvability of impulsive neutral evolution differential inclusions with state-dependent delay, *Math. Comput. Modelling* **49**(2009), 1920-1927.

[24] S. Li and Z. Xiang, Stability and (Formula presented.) -Gain Analysis for Positive Switched Systems with Time-Varying Delay Under State-Dependent Switching, *Circuits Systems Signal Process.* **35** (2016), 1045-1062.

[25] L.-F. Nie, Z.-D. Teng, J.J. Nieto, and I. H. Jung, State impulsive control strategies for a two-languages competitive model with bilingualism and interlinguistic similarity, *Phys. A : Statistical Mechanics Appl.* **430** (2015), 136-147.

[26] N. S. Papageorgiou, Random fixed point theorems for measurable multifunctions in Banach spaces, *Proc. Amer. Math. Soc.* **97** (1986), 507-514.

[27] N. S. Papageorgiou, Random fixed points and random differential inclusions, *Internat. J. Math. Math. Sci* **11** (1988), 551-560.

[28] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[29] R. Sakthivel and Y. Ren, Approximate controllability of fractional differential equations with state-dependent delay. *Results Math.* **63**(2013), 949–963.

[30] T.T. Soong, *Random Differential Equations in Science and Engineering*, Academic Press, New York, 1973.

[31] E. Stumpf, Local stability analysis of differential equations with state-dependent delay, *Discrete Contin. Dyn. Syst.- Series A* **36** (2016), 3445-3461.

[32] K. K. Tan and X. Z. Yuan, Random fixed point theorems and approximation in cones, *J. Math. Anal. Appl.* **185** (1994), 378-390.

[33] I. I. Vrabie, C_0*-Semigroups and Applications*, North-Holland Mathematics Studies 191, Elsevier, 2003.

[34] H.-O. Walther, Merging homoclinic solutions due to state-dependent delay. *J. Differential Equations* **259** (2015), 473-509.

[35] H.-O. Walther, A homoclinic loop generated by variable delay. *J. Dynam. Differential Equations* **27** (2015), 1101-1139.

[36] Z. Yan, Approximate controllability of fractional neutral integro-differential inclusions with state-dependent delay in Hilbert spaces, *MA J. Math. Control Inform.* **30** (2013), 443-462.

[37] K. Yosida, *Functional Analysis,* 6^{th} ed. Springer-Verlag, Berlin, 1980.