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A REPRESENTATION OF DE MOIVRE'S FORMULA OVER PAULI-QUATERNIONS*

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Abstract

We investigate algebraic and analytic properties of Pauli-quaternions. Also, we compose a polar form and De Moivre's formula over Pauli-quaternions and research their characteristics by using the isomorphism of the Pauli matrices.

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1 Introduction

In mathematical physics, the Pauli matrices are Hermitian and unitary which are elements of a set of three 2×2 complex matrices as follows:

Definition 1. [*5*] [*p*.213] *The Pauli matrices are real* (2×2)*-matrices which are linear combinations of the basis matrices*

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

whose multiplication rules are $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbf{1}$, $\sigma_1\sigma_2 = i\sigma_3 = -\sigma_2\sigma_1$, $\sigma_2\sigma_3 = i\sigma_1 = -\sigma_3\sigma_2$ *and* $\sigma_3\sigma_1 = i\sigma_2 = -\sigma_1\sigma_3$.

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These matrices have been introduced by the physicist Wolfgang Pauli. In quantum mechanics, the Pauli matrices are complex matrices which has the interaction of the spin with an external electromagnetic field. The real linear span of $\{I, i\sigma_1, i\sigma_2, i\sigma_3\}$ is isomorphic to the real algebra of quaternions $\mathbb{H}$. The isomorphism from $\mathbb{H}$ to this set is given by the following map which is reversed signs for the Pauli matrices: $1 \mapsto I,\ i \mapsto -i\sigma_1,\ j \mapsto -i\sigma_2,\ k \mapsto -i\sigma_3$ (see [1]). Alternatively, the isomorphism can be acted by a map using the Pauli matrices in reversed order such that $1 \mapsto I,\ i \mapsto i\sigma_3,\ j \mapsto i\sigma_2,\ k \mapsto i\sigma_1$. Specially, many studies have been derived in physics and mathematics. Altmann [2] presented a consistent description of the geometric and quaternionic treatment of rotation operators by the fundamentals of symmetries and matrices. Arvo [3] approached in graphics programming by using techniques for representing polar forms of matrices and variables. Ata and Yayli [4] gave one-to-one correspondence between the elements of the unit split three-sphere with the complex hyperbolic special unitary matrices. Cho [6] studied Euler's formula and De Moivre's formula for complex numbers are generalized for quaternions. Farebrother et. al [7] and Jafari and Yayli [8] established that 48 distinct ordered sets of three 4×4 skew-symmetric serve as the basis of an algebra of quaternions and studied their properties over quaternions. Kim and Shon [9] gave and refined a polar coordinate expression over split quaternions related to hyperholomorphic functions.

Based on these studies, we give attention to apply to De Moivre's formula over quaternions consisting of Pauli matrices. We investigate algebraic and analytic properties of Pauli-quaternions. Also, we compose a polar form and De Moivre's formula over Pauli-quaternions by using the isomorphism of the Pauli matrices.

2 Preliminaries

We introduce definitions and notations of the Pauli matrix and quaternions represented by Pauli matrices. For detailed definitions and properties of Pauli matrix, we refer to [10]. A Pauli-quaternion is defined as $p = x_0\mathbf{1} + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ and a set of Pauli-quaternions is denoted by $\mathbb{H}_P$. The conjugate of a Pauli-quaternion, denoted by p^*, is defined as $p^* = x_0\mathbf{1} - x_1\sigma_1 - x_2\sigma_2 - x_3\sigma_3$. We give the product for any Pauli-quaternions

$p, q \in \mathbb{H}_P$ as follows:

$$\begin{aligned} pq &= (x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)\mathbf{1} + \{(x_0y_1 + x_1y_0) + i(x_2y_3 - x_3y_2)\}\sigma_1 \\ &\quad + \{(x_0y_2 + x_2y_0) + i(x_3y_1 - x_1y_3)\}\sigma_2 \\ &\quad + \{(x_0y_3 + x_3y_0) + i(x_1y_2 - x_2y_1)\}\sigma_3. \end{aligned}$$

Moreover, we have $pp^* = p^*p = (x_0^2 - x_1^2 - x_2^2 - x_3^2)\mathbf{1}$. Also, the Pauli-quaternion product may be written as

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & -ix_3 & ix_2 \\ x_2 & ix_3 & x_0 & -ix_1 \\ x_3 & -ix_2 & ix_1 & x_0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Let the inner product over Pauli-quaternions $\mathcal{I}(p) = x_0^2 - x_1^2 - x_2^2 - x_3^2$ and then, the norm of Pauli-quaternions is

$$\mathcal{N}(p) = \sqrt{|pp^*|} = \sqrt{|x_0^2 - x_1^2 - x_2^2 - x_3^2|}.$$

For $p = x_0\mathbf{1} + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$, if $\mathcal{N}(p) = 1$, then p is called unit Pauli-quaternions. Also, spacelike and timelike Pauli-quaternions have multiplicative inverse, denoted by p^{-1}, of $p \in \mathbb{H}_P^*$, where $\mathbb{H}_P^* = \mathbb{H}_P \setminus E$ with

$$E = \{x_0\mathbf{1} + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \mid x_0^2 = x_1^2 + x_2^2 + x_3^2\},$$

and their property $pp^{-1} = p^{-1}p = 1$. On the other hands, lightlike Pauli-quaternions have no inverses. Referring [8] and [11], we give the following definition:

Definition 2. *p is spacelike if $\mathcal{I}(p) < 0$, p is lightlike if $\mathcal{I}(p) = 0$, p is timelike if $\mathcal{I}(p) > 0$.*

Proposition 1. *The set of spacelike Pauli-quaternions is not closed under multiplication for Pauli-quaternions. On the other hand, the set of timelike Pauli-quaternions forms a group under multiplication for Pauli-quaternions.*

Proof. For any two elements p and q of the set of spacelike Pauli-quaternions, the equations $\mathcal{I}(p) = x_0^2 - x_1^2 - x_2^2 - x_3^2 < 0$ and $\mathcal{I}(q) = y_0^2 - y_1^2 - y_2^2 - y_3^2 < 0$ are satisfied. However, the product pq satisfies the following equation:

$$\begin{aligned} \mathcal{I}(pq) &= (x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)^2 + \{(x_0y_1 + x_1y_0) + i(x_2y_3 - x_3y_2)\}^2 \\ &\quad + \{(x_0y_2 + x_2y_0) + i(x_3y_1 - x_1y_3)\}^2 \\ &\quad + \{(x_0y_3 + x_3y_0) + i(x_1y_2 - x_2y_1)\}^2 \\ &= (x_0^2 - x_1^2 - x_2^2 - x_3^2)(y_0^2 - y_1^2 - y_2^2 - y_3^2). \end{aligned}$$

Since $\mathcal{I}(p) < 0$ and $\mathcal{I}(q) < 0$, we have $\mathcal{I}(pq) > 0$. Thus, the set of spacelike Pauli-quaternions is not closed under multiplication. On the other hand, if p and q are elements of the set of timelike Pauli-quaternions, then $\mathcal{I}(p) > 0$ and $\mathcal{I}(q) > 0$ are satisfied. Also, since $\mathcal{I}(pq) > 0$ is satisfied, the set of timelike Pauli-quaternions is a group under multiplication for Pauli-quaternions. $\square$

Similarly, the vector part of any spacelike quaternion is spacelike, but vector part of any timelike quaternion can be spacelike, timelike and null.

3 De Moivre's Formula for Pauli-quaternions

Now, we express any Pauli-quaternions in a polar form similar to Pauli-quaternions and quaternions. There are two types of polar forms of the Pauli-quaternions which are referred by [8] and [10] as follows:
1) If p is spacelike, that is, $\mathcal{I}(p) = x_0^2 - x_1^2 - x_2^2 - x_3^2 < 0$, then

$$p = \mathcal{N}(p)(\sinh\theta + \overrightarrow{v}\cosh\theta),$$

where

$$\cosh\theta = \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{\mathcal{N}(p)} \quad \text{and} \quad \sinh\theta = \frac{x_0}{\mathcal{N}(p)}$$

with

$$\mathcal{N}(p) = \sqrt{-x_0^2 + x_1^2 + x_2^2 + x_3^2}, \quad \overrightarrow{v} = \frac{x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

and $\overrightarrow{v}^2 = 1$.

Proposition 2. *Let $p = \sinh\theta + \overrightarrow{v}\cosh\theta$ be a unit spacelike quaternion. Then we have $p^n = \sinh n\theta + \overrightarrow{v}\cosh n\theta$ if $|n|$ is odd, and $p^n = \cosh n\theta + \overrightarrow{v}\sinh n\theta$ if $|n|$ is even for $n \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers.*

Proof. For $n = 1$, it is trivial. By the properties of hyperbolic functions, we have $p^2 = \sinh^2\theta + \cosh^2\theta + 2\overrightarrow{v}\sinh\theta\cosh\theta = \cosh 2\theta + \overrightarrow{v}\sinh 2\theta$ and $p^3 = \sinh 3\theta + \overrightarrow{v}\cosh 3\theta$. Furthermore, for $n = -1$, by the properties of the multiplicative inverse element of $\mathbb{H}_P$, we have $p^{-1} = -\sinh\theta + \overrightarrow{v}\cosh\theta$ and $p^{-2} = (p^2)^{-1} = \cosh 2\theta - \overrightarrow{v}\sinh 2\theta$. If the calculation process is repeated, we can obtain the result. $\square$

2) If p is timelike, $\mathcal{I}(p) = x_0^2 - x_1^2 - x_2^2 - x_3^2 > 0$, then

$$p = \mathcal{N}(p)(\cosh\theta + \overrightarrow{w}\sinh\theta),$$

where

$$\sinh\theta = \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{\mathcal{N}(p)} \quad \text{and} \quad \cosh\theta = \frac{x_0}{\mathcal{N}(p)},$$

with

$$\mathcal{N}(p) = \sqrt{x_0^2 - x_1^2 - x_2^2 - x_3^2}, \quad \overrightarrow{w} = \frac{x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

and $\overrightarrow{w}^2 = 1$.

Proposition 3. *Let* $p = \cosh\theta + \overrightarrow{w}\sinh\theta$ *be a unit spacelike quaternion. Then we have*

$$p^n = \cosh n\theta + \overrightarrow{w}\sinh n\theta \tag{1}$$

for $n \in \mathbb{Z}$*, where* $\mathbb{Z}$ *is the set of integers.*

Proof. Like the proof process of Proposition (3), by applying the induction, we can obtain the expression of Equation (1). □

We introduce the $\mathbb{R}$-linear transformations representing left and right multiplication in $\mathbb{H}_P$ by using the De Moivre's formula for a corresponding matrix representation. Let p be a Pauli-quaternion, then $\varphi_{L_p} : \mathbb{H}_P \to \mathbb{H}_P$ and $\varphi_{R_p} : \mathbb{H}_P \to \mathbb{H}_P$ defined as follows: for $\chi \in \mathbb{H}_P$,

$$\varphi_{L_p}(\chi) = p\chi \quad \text{and} \quad \varphi_{R_p}(\chi) = \chi p, \tag{2}$$

respectively. The Hamilton's operator φ_{L_p} and φ_{R_p} can be written by the matrices:

$$A_{\varphi_{L_p}} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & -ia_3 & ia_2 \\ a_2 & ia_3 & a_0 & -ia_1 \\ a_3 & -ia_2 & ia_1 & a_0 \end{pmatrix}$$

and

$$A_{\varphi_{R_p}} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & ia_3 & -ia_2 \\ a_2 & -ia_3 & a_0 & ia_1 \\ a_3 & ia_2 & -ia_1 & a_0 \end{pmatrix},$$

respectively. For unit Pauli-quaternions χ, the mapping $\varphi_p : \mathbb{H}_P \to \mathbb{H}_P$ is defined by $\varphi_p = \varphi_{L_p} \circ \varphi_{R_p} = \varphi_{R_p} \circ \varphi_{L_p}$. If p and q are Pauli-quaternions and λ is a real number and φ_{L_p} and φ_{R_p} are operators as defined in equations $A_{\varphi_{L_p}}$ and $A_{\varphi_{R_p}}$, respectively, then the following properties hold:

Proposition 4. *Let χ, χ_1 and χ_2 be Pauli-quaternions and let α and β be real constants. Then,*
1) $\chi_1 = \chi_2$ *if and only if* $\varphi_{L_p}(\chi_1) = \varphi_{L_p}(\chi_2)$ $(\varphi_{R_p}(\chi_1) = \varphi_{R_p}(\chi_2))$,
2) $\varphi_{L_p}(\alpha\chi_1 + \beta\chi_2) = \alpha\varphi_{L_p}(\chi_1) + \beta\varphi_{L_p}(\chi_2)$,
3) $\varphi_{R_p}(\alpha\chi_1 + \beta\chi_2) = \alpha\varphi_{R_p}(\chi_1) + \beta\varphi_{R_p}(\chi_2)$,
4) $\varphi_{L_p}(\chi)\varphi_{R_p}(\chi) = \varphi_{R_p}(\chi)\varphi_{L_p}(\chi)$.

Proof. From the definition of φ_{L_p} and φ_{L_p} (see (2)), we can obtain the above results. □

Theorem 1. *Let ψ_L be a mapping defined as*

$$\psi_L : (\mathbb{H}_P, +, \cdot) \ \rightarrow \ (M_{(4,\mathbb{R})}, \oplus, \otimes);$$

$$\psi_L(x_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) \ \rightarrow \ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & -ix_3 & ix_2 \\ x_2 & ix_3 & x_0 & -ix_1 \\ x_3 & -ix_2 & ix_1 & x_0 \end{pmatrix}$$

is an isomorphism.

Proof. By the definition of the equality of a matrix, we have that $\psi_L(x) = \psi_L(y)$, where $x = x_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ and $y = y_0 + y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3$, implies $x = y$, that is, ψ_L is injective. For any elements of $M_{(4,\mathbb{R})}$, since there is an element of $\mathbb{H}_P$ such that

$$\psi_L(x_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) \ \rightarrow \ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & -ix_3 & ix_2 \\ x_2 & ix_3 & x_0 & -ix_1 \\ x_3 & -ix_2 & ix_1 & x_0 \end{pmatrix}, \qquad (3)$$

the mapping ψ_L is subjective. Also, by using Equation (3), we calculate $\psi_L(xy)$ and $\psi_L(x)\psi_L(y)$. From comparing with $\psi_L(xy)$ and $\psi_L(x)\psi_L(y)$, we obtain that $\psi_L(xy) = \psi_L(x)\psi_L(y)$. Therefore, the mapping ψ_L is an isomorphism. □

Theorem 2. *Let ψ_R be a mapping defined as*

$$\psi_R : (\mathbb{H}_P, +, \cdot) \ \rightarrow \ (M_{(4,\mathbb{R})}, \oplus, \otimes);$$

$$\psi_R(x_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) \ \rightarrow \ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & ix_3 & -ix_2 \\ x_2 & -ix_3 & x_0 & ix_1 \\ x_3 & ix_2 & -ix_1 & x_0 \end{pmatrix}$$

is also an isomorphism.

Proof. Using the similar process of the proof of Theorem 1, we can also obtain that ψ_R is an isomorphism. $\square$

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References

[1] G. Arfken. *Mathematical Methods for Physicists.* Academic Press, Orlando, 1985.

[2] S.L. Altmann. *Rotations, Quaternions, and Double Groups.* Clarendon Press, Oxford, England, 3rd ed., 1986.

[3] J. Arvo. *Graphics Gems II.* Academic Press, New York, 1994.

[4] E. Ata, Y. Yayli. Split quaternions and semi-Euclidean projective spaces. *Chaos, Solitions and Fractals* 41:1910–1915, 2009.

[5] E.U. Condon, P.M. Morse. *Quantum Mechanics.* McGraw-Hill, New York, 1929.

[6] E. Cho. De-Moivre Formula for Quaternions. *Appl. Math. Lett.* 11(6):33–35, 1998.

[7] R.W. Farebrother, J. GroB, S. Troschke. Matrix Representaion of Quaternions. *Lin. Alg. Appl.* 362:251–255, 2003.

[8] M. Jafari, Y. Yayli. Matrix Theory over the Split Quaternions. *Int. Jour. Geom.* 3(2):57–69, 2014.

[9] J.E. Kim, K.H. Shon. Polar coordinate expression of hyperholomorphic functions on split quaternions in Clifford analysis. *Adv. Appl. Clifford Al.* 25(4):915–924, 2015.

[10] P. Longe. The properties of the Pauli matrices A, B, C and the conjugation of charge. *Physica* 32(3):603-610, 1966.

[11] M. Ozdemir, A. Ergin. Rotation with Unit Timelike Quaternions in Minkowski 3-Space. *J. Geom. Phys.* 56:322–336, 2006.