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COMBINATORICS OF HANKEL RELATIONS*

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Abstract

We investigate the problem to determine the defining equations of the algebraic variety of Hankel two-planes in the projective space. We compute the first and the second partial lifting of the Machado's binomial relations, by applying tecniques of Sagbi bases theory.

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1 Introduction

In the study of toric ideals and of canonical bases of subalgebras many authors are interested on the problem to degenerate an arbitrary parametrically presented variety X into a toric variety Y. The basic idea is to degenerate the algebra generators into monomials and therefore the algebra polynomial relations to binomials relations. We can see how this can be accomplished if X is a Grassmann variety, since there is a beautiful link between the toric ideal I of the toric degeneration of X and the Grassmann-Plücker ideal of X, whose initial ideal, with respect to a fixed weighted term order ([13]) on the monomials of the polynomial ring of the presentation of

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X, coincides with I. In general for subvarieties of X, the previous result is not true. For the Hankel variety $H(r,n)$, subvariety of $\mathbb{G}(r,n)$ (introduced in [10] by Giuffrida and Maggioni and later studied in [9], [4] and [6]), the result is true for $r = 1$, since we have the basic result of Conca, Herzog, Valla ([2]). Some combinatorial results are given in [5] and [7]. Nevertheless, the toric deformation of $H(r,n)$ is known, in the sense that Machado ([11]) established all binomial relations of the degenerate variety of $H(r,n)$. Then, an open problem is to find the relations of $H(r,n)$, starting from Machado relations and most proofs depend currently from Sagbi bases theory. By employing this theory, in [8] we obtain an algorithm that permits us to write all relations of $H(2,5)$. In the same paper we give the list of partial liftings of the binomial relations of Machado, consisting of polynomials $p_i = \sum_t m_{it}$, m_{it} a term. In some cases they are effective relations for $H(2,n)$. It is not clear if it will be possible to obtain all effective relations. For this proposal, we must work on the 4th term of the partial liftings and to extend the procedure adopted in [8] by the algorithm, for obtaining terms m_{it}, t ≥ 5. In this paper we present in detail all developments of the results that are built on the partial liftings for $H(2,n)$(only two cases are proved in detail in [8]). Our results are the starting point for a systematic study of the effective relations of $H(2,n)$, for any n. More precisely, in the main theorem of Section 2 we describe the basic techniques for all the steps occurring to obtain the 5th monomial in the virtual relations and the next monomials. Section 1 contains notations, some known results on Machado relations and the partial liftings ([8],[11]). Some results of this note have been conjectured by using the software CoCoA [1].

2 Preliminaries

A matrix of the form

$$H_{r,n} = \begin{pmatrix} x_1 & x_2 & \cdots & \cdots & x_n \\ x_2 & x_3 & \cdots & x_n & x_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{r-1} & x_r & \cdots & \cdots & x_{n+r-1} \\ x_r & x_{r+1} & \cdots & x_{n+r-1} & x_{n+r} \end{pmatrix}$$

is called *Hankel matrix*, whose entries belong to a commutative ring R. We consider generic Hankel matrices $H_{r,n}$, then the entries are indeterminates. Let K be a field and $S = K[x_1, x_2, \ldots, x_{n+r}]$ the polynomial ring over K in $n+r$ indeterminates. We denote by $[i_1 i_2 \ldots i_r]$ the r-minor with columns

$i_1 < i_2 < \ldots i_r$. Let $<$ be the lexicographical order induced by $x_1 > x_2 > \ldots > x_{n+r}$. Then

$$in_<[i_1 i_2 \ldots i_r] = x_{i_1} x_{i_2+1} \ldots x_{i_r+r-1},$$

where the monomial $x_{i_1} x_{i_2+1} \ldots x_{i_r+r-1}$ is the product of monomials corresponding to the main diagonal of the minor $[i_1 i_2 \ldots i_r]$.

Denote by $A_{2,n}$ the K-algebra over K generated by the initial monomials $x_{i_1} x_{i_2+1} x_{i_3+2}$ with $1 \leq i_1 < i_2 < i_3 \leq n$ of the $3-$minors of $H_{3,n}$. Moreover, let $T = K[y_{i_1 i_2 i_3} : 1 \leq i_1 < i_2 < i_3 \leq n]$ be the polynomial ring in the variables $y_{i_1 i_2 i_3}$ and let $\psi : T \to A_{2,n}$ be the K-algebra homomorphism with $y_{i_1 i_2 i_3} \mapsto x_{i_1} x_{i_2+1} x_{i_3+2}$. Each monomial of degree d in T can be identified with a $d \times 3$ matrix

$$\begin{pmatrix} i_{11} & i_{12} & i_{13} \\ i_{21} & i_{22} & i_{23} \\ \vdots & \vdots & \vdots \\ i_{d1} & i_{d2} & i_{d3} \end{pmatrix}$$

such that $(i_{11} i_{12} i_{13}) \geq (i_{21} i_{22} i_{23}) \geq \cdots \geq (i_{d1} i_{d2} i_{d3})$. In particular a monomial of degree two in T corresponds to a matrix of the form

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

with $a < b < c$, $d < e < f$ and $(a, b, c) \geq (d, e, f)$.

Theorem 1. (*Machado*[*11*]) *With the assumptions and notations introduced, one has:*

The kernel $J = \ker \psi$ is generated by the following type of relations

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} a & e & c \\ d & b & f \end{pmatrix} \quad \textit{with} \quad e < b,\ c \leq f,$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} a & e & f \\ d & b & c \end{pmatrix} \quad \textit{with} \quad e < b,\ f < c,$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} a & b & f \\ d & e & c \end{pmatrix} \quad \textit{with} \quad b \leq e,\ f < c,$$

and assuming that $a \leq d$, $b \leq e$, $c \leq f$ one has

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} a & d-1 & c \\ b+1 & e & f \end{pmatrix} \quad \textit{with} \quad b \ll d,\, e-c \le 1,\, d-1 < c$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} a & b & e-1 \\ d & c+1 & f \end{pmatrix} \quad \textit{with} \quad d-b \le 1,\, c \ll e,\, c+1 < f$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} a & d-1 & e-1 \\ b+1 & c+1 & f \end{pmatrix} \quad \textit{with} \quad b \ll d,\, c \ll e.$$

Here we set $i \ll j$ if $j - i \geq 2$.

The following theorem gives a criterion for the existence of a Sagbi basis which is a variation of the known criterion by Robbiano and Sweedler given in [12]. The proof is contained in [8].

Theorem 2. *Let $T = K[y_1, \ldots, y_m]$ be the polynomial ring over K in the variables $y_1, \ldots, y_m$, and let $\varphi : T \to A$ the K-algebra homomorphism with $y_i \mapsto a_i$ and $\psi : T \to in_<(A)$ the K-algebra homomorphism with $y_i \mapsto in_<(a_i)$ for $i = 1, \ldots, m$. Let $I = Ker\varphi$ and $f_1, \ldots, f_r$ be a set of binomial generators of $J = Ker\psi$. Then the following conditions are equivalent:*

(a) *$a_1, \ldots, a_m$ is a Sagbi basis of A.*

(b) *For each j, there exist monomials $m_1, \ldots, m_s \in T$ and $c_1, \ldots, c_s \in K$ such that*

 (i) $f_j + \sum_i^s c_i m_i \in I$.

 (ii) $in_<(\varphi(m_{i+1})) = in_<(\varphi(f_j + \sum_{k=1}^i c_k m_k)) < in_<(\varphi(m_i))$, $in_<(\varphi(f_j + c_1 m_1)) < in_<(\varphi(f_j))$.

If the equivalent conditions are satisfied, we call $f_j + \sum_i^s c_i m_i$ a lifting of f_j.

3 Computing the relations of $H(2, n)$

By Theorem 2, one deduced easily the following algorithm ([8]) that permits to obtain polynomial relations from the binomial relations:

1. Choose one of the binomial relations of the initial terms of the minors in the initial algebra listed in Theorem 1, replace in the relation the initial

terms by the corresponding minors to obtain the element $f_1 \in A_{2,n}$, and determine its initial term.

2. If $in_< f_1$ is not a product of the initial terms of two minors of $H_{3,n}$, then the relation f_1 is not liftable. In this case the minors of Hankel matrix do not form a Sagbi basis (in our case this never happened).

3. If $in_< f_1$ is a product of the initial terms of two minors m_1, m_2, then we add a suitable multiple of $m_1 m_2$ to f_1 to obtain f_2 with the property that $in_< f_2 < in_< f_1$.

4. Proceed recursively by step 3.

The algorithm is applied to determine the expression of the first and second lifting of the binomial relations of Machado.

By repeated applications of the algorithm, there is the researched list of liftings.

For the following the employed monomial order is the lexicographic order and the usual order of the variables is $x_1 > x_2 > \ldots > x_{n+2}$. Moreover, we will identify the symbol $\begin{bmatrix} a & d & c \\ e & b & f \end{bmatrix}$ with the corresponding product of minors

$$\begin{vmatrix} x_a & x_b & x_c \\ x_{a+1} & x_{b+1} & x_{c+1} \\ x_{a+2} & x_{b+2} & x_{c+2} \end{vmatrix} \begin{vmatrix} x_d & x_e & x_c \\ x_{d+1} & x_{e+1} & x_{c+1} \\ x_{d+2} & x_{e+2} & x_{c+2} \end{vmatrix}$$

and the difference between two symbols by $(i) - (j)$.

Theorem 3. *The binomial relations of the $K-$algebra $A_{2,n}$ have the following liftings and partial liftings:*

(I) $e < b,\ c \le f$

(IA) $c = f, \quad a < d < e < b < c$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & e & c \\ d & b & c \end{bmatrix} + \begin{bmatrix} a & d & c \\ e & b & c \end{bmatrix}$$

(IB) $c < f, \quad a < d < e < b < c < f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & e & c \\ d & b & f \end{bmatrix} + \begin{bmatrix} a & e & b \\ d & c & f \end{bmatrix} + \begin{bmatrix} a & d & c \\ e & b & f \end{bmatrix} + \cdots$$

(II) $e < b,\ f < c, \quad a < d < e < b \leq f < c$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & e & f \\ d & b & c \end{bmatrix} + \begin{bmatrix} a & d & f \\ e & b & c \end{bmatrix} + \begin{bmatrix} a & d & e \\ b & f & c \end{bmatrix} + \cdots$$

(III) $b \leq e,\ f < c$

(IIIA) $a = d, \quad a < b < e < f < c$

$$\begin{bmatrix} a & b & c \\ a & e & f \end{bmatrix} - \begin{bmatrix} a & b & f \\ a & e & c \end{bmatrix} + \begin{bmatrix} a & b & e \\ a & f & c \end{bmatrix}$$

(IIIB) $b = e, \quad a < d < b < f < c$

$$\begin{bmatrix} a & b & c \\ d & b & f \end{bmatrix} - \begin{bmatrix} a & b & f \\ d & b & c \end{bmatrix} + \begin{bmatrix} a & d & b \\ b & f & c \end{bmatrix}$$

(IIIC) $b \leq d, \quad a < b \leq d < e < f < c$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & b & f \\ d & e & c \end{bmatrix} + \begin{bmatrix} a & b & e \\ d & f & c \end{bmatrix}$$
$$+ \begin{bmatrix} a & b & f-2 \\ d+1 & e+1 & c \end{bmatrix} + \cdots$$

(IIID) $b > d, \quad a < d < b < e < f < c$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & b & f \\ d & e & c \end{bmatrix} + \begin{bmatrix} a & b & e \\ d & f & c \end{bmatrix} + \begin{bmatrix} a & d & b \\ e & f & c \end{bmatrix} + \cdots$$

(IV) $2 \leq d - b,\ e - c \leq 1,\ d - 1 < c$

(IVA) $a < b \ll d < e - 1 \leq c < e < f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & c \\ b+1 & e & f \end{bmatrix} + \begin{bmatrix} a & d & e-1 \\ b+1 & c & f \end{bmatrix} +$$
$$+ \begin{bmatrix} a & d & e \\ b+1 & c & f-1 \end{bmatrix} + \cdots$$

(IVB) $d = c = e-1, \quad a < b \ll e-1 < e < f-1 < f$

$$\begin{bmatrix} a & b & e-1 \\ e-1 & e & f \end{bmatrix} - \begin{bmatrix} a & e-2 & e-1 \\ b+1 & e & f \end{bmatrix} + \begin{bmatrix} a & e-1 & e \\ b+1 & e-1 & f-1 \end{bmatrix} +$$
$$+ \begin{bmatrix} a & e-2 & e \\ b+2 & e & f-1 \end{bmatrix} + \cdots$$

(IVB1) $d = c = e-1, e = f-1, \quad a < b \ll e-1 < e < e+1$

$$\begin{bmatrix} a & b & e-1 \\ e-1 & e & e+1 \end{bmatrix} - \begin{bmatrix} a & e-2 & e-1 \\ b+1 & e & e+1 \end{bmatrix} + \begin{bmatrix} a & e-1 & e \\ b+1 & e-1 & e \end{bmatrix} +$$
$$+ \begin{bmatrix} a+1 & e-2 & e-1 \\ b & e & e+1 \end{bmatrix} + \cdots$$

(IVC) $d < e-1, c = e, \quad a < b \ll d < e-1 < e < f$

$$\begin{bmatrix} a & b & e \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & e \\ b+1 & e & f \end{bmatrix} + \begin{bmatrix} a & d & e \\ b+1 & e-1 & f \end{bmatrix}$$
$$- \begin{bmatrix} a & d & e-1 \\ b+1 & e & f \end{bmatrix} + \cdots$$

(IVD) $d = e-1, c = e, \quad a < b \ll e-1 < e < f-1 < f$

$$\begin{bmatrix} a & b & e \\ e-1 & e & f \end{bmatrix} - \begin{bmatrix} a & e-2 & e \\ b+1 & e & f \end{bmatrix} + \begin{bmatrix} a & e-1 & e \\ b+1 & e-1 & f \end{bmatrix} +$$
$$+ \begin{bmatrix} a & e-1 & e \\ b+1 & e & f-1 \end{bmatrix} + \cdots$$

(IVD1) $d = e-1, c = e, f = e+1, \quad a < b \ll e-1 < e < e+1$

$$\begin{bmatrix} a & b & e \\ e-1 & e & e+1 \end{bmatrix} - \begin{bmatrix} a & e-2 & e \\ b+1 & e & e+1 \end{bmatrix} + \begin{bmatrix} a & e-1 & e \\ b+1 & e-1 & e+1 \end{bmatrix} +$$
$$+ \begin{bmatrix} a+1 & e-2 & e \\ b & e & e+1 \end{bmatrix} + \cdots$$

(IVE) $\quad a < b \ll d < e < c < f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & c \\ b+1 & e & f \end{bmatrix} + \begin{bmatrix} a & d & c \\ b+1 & e-1 & f \end{bmatrix}$$
$$+ \begin{bmatrix} a & d & c-1 \\ b+1 & e & f \end{bmatrix} + \cdots$$

(V) $d-b\le 1,\ 2\le e-c,\ c+1<f$

(VA) $b=d-1,\ d<c,\ c+3<f,\quad a<b<d<c\ll e<f$

$$\begin{bmatrix} a & d-1 & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & e-1 \\ d & c+1 & f \end{bmatrix} + \begin{bmatrix} a & d-1 & e \\ d & c+1 & f-1 \end{bmatrix} + \\ - \begin{bmatrix} a & d-1 & e-1 \\ d & c+2 & f-1 \end{bmatrix} + \cdots$$

(VA1) $b=d-1,\ d=c,\ d=f-3,\quad a<d-1<d\ll d+2<d+3$

$$\begin{bmatrix} a & d-1 & d \\ d & d+2 & d+3 \end{bmatrix} - \begin{bmatrix} a & d-1 & d+1 \\ d & d+1 & d+3 \end{bmatrix} + \begin{bmatrix} a & d-1 & d+2 \\ d & d+1 & d+2 \end{bmatrix} + \\ - \begin{bmatrix} a+1 & d & d+1 \\ d-1 & d & d+3 \end{bmatrix} + \cdots$$

(VB) $b=d-1,\ d<c,\ c+3=f\quad a<b<d<c\ll c+2<c+3$

$$\begin{bmatrix} a & d-1 & c \\ d & c+2 & c+3 \end{bmatrix} - \begin{bmatrix} a & d-1 & c+1 \\ d & c+1 & c+3 \end{bmatrix} + \begin{bmatrix} a & d-1 & c+2 \\ d & c+1 & c+2 \end{bmatrix} + \\ - \begin{bmatrix} a & d & c+2 \\ d & c & c+2 \end{bmatrix} + \cdots$$

(VC) $d\le b,\ c+3<f\quad a<d\le b<c\ll e<f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & b & e-1 \\ d & c+1 & f \end{bmatrix} + \begin{bmatrix} a & b & e \\ d & c+1 & f-1 \end{bmatrix} + \\ + \begin{bmatrix} a & b & c+1 \\ d & e & f-1 \end{bmatrix} + \cdots$$

(VC1) $d\le b,\ c+3=f,\quad a<d\le b<c\ll e<f$

$$\begin{bmatrix} a & b & c \\ d & c+2 & c+3 \end{bmatrix} - \begin{bmatrix} a & b & c+1 \\ d & c+1 & c+3 \end{bmatrix} + \begin{bmatrix} a & b & c+2 \\ d & c+1 & c+2 \end{bmatrix} + \\ - \begin{bmatrix} a & b+1 & c+1 \\ d & c & c+3 \end{bmatrix} + \cdots$$

(VI) $2\le d-b,\ 2\le e-c$

(VIA) $c < d, \quad a < b < c < d < e < f, c+3 < f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & e-1 \\ b+1 & c+1 & f \end{bmatrix} + \begin{bmatrix} a & d-1 & e \\ b+1 & c+1 & f-1 \end{bmatrix}$$
$$+ \begin{bmatrix} a & d & e \\ b+1 & c+1 & f-2 \end{bmatrix} + \cdots$$

(VIA1) $c < d, c+3 = f, \quad a < b < c < c+1 < c+2 < c+3$

$$\begin{bmatrix} a & b & c \\ c+1 & c+2 & c+3 \end{bmatrix} - \begin{bmatrix} a & c & c+1 \\ b+1 & c+1 & c+3 \end{bmatrix} + \begin{bmatrix} a & c & c+2 \\ b+1 & c+1 & c+2 \end{bmatrix} +$$
$$+ \begin{bmatrix} a+1 & c & c+1 \\ b & c+1 & c+3 \end{bmatrix} + \cdots$$

(VIB) $d \leq c, c+3 < f, \quad a < b \ll d \leq c \ll e < f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & e-1 \\ b+1 & c+1 & f \end{bmatrix} + \begin{bmatrix} a & d-1 & e \\ b+1 & c+1 & f-1 \end{bmatrix} +$$
$$+ \begin{bmatrix} a & d-1 & e-1 \\ b+1 & c+2 & f-1 \end{bmatrix} + \cdots$$

(VIB1) $d \leq c, \quad a < b \ll d \leq c \ll c+2 < c+3$

$$\begin{bmatrix} a & b & c \\ d & c+2 & c+3 \end{bmatrix} - \begin{bmatrix} a & d-1 & c+1 \\ b+1 & c+1 & c+3 \end{bmatrix} + \begin{bmatrix} a & d-1 & c+2 \\ b+1 & c+1 & c+2 \end{bmatrix} +$$
$$+ \begin{bmatrix} a & d & c+1 \\ b+1 & c+1 & c+2 \end{bmatrix} + \cdots$$

Proof. (IA) $c = f, \quad a < d < e < b < c$. See [8] for the proof.

(IB) $c < f \quad a < d < e < b < c < f$

Consider the binomial relation

$$\begin{pmatrix} a & b & c \\ d & e & c \end{pmatrix} - \begin{pmatrix} a & e & c \\ d & b & f \end{pmatrix}.$$

Replace it by the difference of the products of the corresponding minors:

$$\begin{bmatrix} a & b & c \\ d & e & c \end{bmatrix} - \begin{bmatrix} a & e & c \\ d & b & f \end{bmatrix} = \quad (1) - (2)$$

$$= \begin{vmatrix} x_a & x_b & x_c \\ x_{a+1} & x_{b+1} & x_{c+1} \\ x_{a+2} & x_{b+2} & x_{c+2} \end{vmatrix} \begin{vmatrix} x_d & x_e & x_c \\ x_{d+1} & x_{e+1} & x_{c+1} \\ x_{d+2} & x_{e+2} & x_{c+2} \end{vmatrix} +$$

$$- \begin{vmatrix} x_a & x_e & x_c \\ x_{a+1} & x_{e+1} & x_{c+1} \\ x_{a+2} & x_{e+2} & x_{c+2} \end{vmatrix} \begin{vmatrix} x_d & x_b & x_f \\ x_{d+1} & x_{b+1} & x_{f+1} \\ x_{d+2} & x_{b+2} & x_{f+2} \end{vmatrix}.$$

The monomials of (1), with $\{a, d, e+1\}$ in their support, consist of the unique monomial $x_a x_d x_{e+1} x_{b+1} x_{c+2} x_{f+2}$ that vanishes with a product of (2), while $f_1 = x_a x_d x_{e+1} x_{b+2} x_{c+1} x_{f+2}$ of (1) and $f_2 = x_a x_d x_{e+1} x_{b+2} x_{c+2} x_{f+1}$ of (2) do not vanish. Then the $in_<((1)-(2)) = f_1$, that gives

$$\begin{bmatrix} a & e & b \\ d & c & f \end{bmatrix} = (3).$$

Now in (1) and (2) the products with $\{a, d, e+1\}$ in the support are f_1 and f_2 that vanish with products of (3). Then we consider the products with $\{a, d, e+2\}$ in their support, $x_a x_d x_{e+2} x_{b+2} x_{c+1} x_{f+1}$, $x_a x_d x_{e+2} x_{b+1} x_{c+2} x_{f+1}$ that vanish with products of (2) and (3), in (2) $x_a x_d x_{e+2} x_{b+1} x_{c+1} x_{f+2}$ vanishes with a monomial of (3). Finally we consider the monomials with $\{a, d+1\}$ in their support starting by x_a, x_{d+1}, x_e that are only in (1): $f_3 = x_a x_{d+1} x_e x_{b+1} x_{c+2} x_{f+2}$ and $x_a x_{d+1} x_e x_{b+2} x_{c+1} x_{f+2}$. Then $in_<((1) - (2) + (3)) = f_3$ that gives:

$$\begin{bmatrix} a & d & c \\ e & b & f \end{bmatrix}.$$

But the procedure can continue and we can obtain other pieces in the lifting.

(II) $e < b$ and $f < c$, $\quad a < d < e < b < f < c$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & e & f \\ d & b & c \end{bmatrix} = (4) - (5).$$

Consider first the monomials of (4) with $\{a, d\}$ in the support:
$x_a x_d x_{e+1} x_{b+1} x_{f+2} x_{c+2}$, $x_a x_d x_{e+2} x_{b+1} x_{f+1} x_{c+2}$, $x_a x_d x_{e+1} x_{b+2} x_{f+2} x_{c+1}$,

$x_a x_d x_{e+2} x_{b+2} x_{f+1} x_{c+1}$ vanish with products of (5). Then we consider the monomials with $\{a, d+1\}$ in the support: $f_1 = x_a x_{d+1} x_e x_{b+1} x_{f+2}$ x_{c+2} do not vanish. Then $in_<((4)-(5)) = f_1$ gives:

$$\begin{bmatrix} a & d & f \\ e & b & c \end{bmatrix} = (6).$$

Now in (6) there are not monomials whose support contains $\{a, d\}$ and we consider the monomials containing the variables indexed by $a, d+1$. The monomials f_1 and $x_a x_{d+1} x_e x_{b+2} x_{f+2} x_{c+1}$ of (4) vanish with monomials of (6). The monomials $x_a x_{d+1} x_{e+1} x_{b+2} x_{f+2} x_c$, $x_a x_{d+1} x_{e+1} x_b x_{f+2} x_{c+2}$ of (5) vanish with monomials of (6). The monomial $f_2 = x_a x_{d+1} x_{e+2} x_b x_{f+1} x_{c+2}$ of (8) does not vanish. Then $in_<((7)-(8)+(9)) = f_2$ gives

$$\begin{bmatrix} a & d & e \\ b & f & c \end{bmatrix}.$$

(IIIA) $a = d, \quad a < b < e < f < c$

$$\begin{bmatrix} a & b & c \\ a & e & f \end{bmatrix} - \begin{bmatrix} a & b & f \\ a & e & c \end{bmatrix} = (7) - (8).$$

Consider first the monomials with $\{a, b+1\}$ in the support(since there are not monomials with a and b): in (7) $x_a^2 x_{b+1} x_{e+1} x_{f+2} x_{c+2}$ vanishes with a monomial of (8), while $f_1 = x_a^2 x_{b+1} x_{e+2} x_{f+1} x_{c+2}$ does not vanish. Then $in_<((7)-(8)) = f_1$ that gives

$$+\begin{bmatrix} a & b & e \\ a & f & c \end{bmatrix} = (9).$$

Now the remaining monomials in the sum $(7)-(8)+(9)$ vanish at all. Then (IIIA) is a relation.

(IIIB) $b = e, \quad a < d < b < f < c$

$$\begin{bmatrix} a & b & c \\ d & b & f \end{bmatrix} - \begin{bmatrix} a & b & f \\ d & b & c \end{bmatrix} = (10) - (11).$$

Consider first the monomials with the variables indexed by a, d: the monomials of (10) $x_a x_d x_{b+1} x_{b+1} x_{f+2} x_{c+2}$, $x_a x_d x_{b+1} x_{b+2} x_{f+1} x_{c+2}$,

$x_a x_d x_{b+1} x_{b+2} x_{f+2} x_{c+2}$, $x_a x_d x_{b+2} x_{b+2} x_{f+1} x_{c+1}$ vanish with monomials of (11). The monomials with $\{a, d+1\}$ in the support are $x_a x_{d+1} x_b x_{b+1} x_{f+2} x_{c+2}$ of (10) that vanishes with a monomial of (11) and $f_1 = x_a x_{d+1} x_b x_{b+2} x_{f+1} x_{c+2}$ of (11) that does not vanish. Then $in_<((10) - (11)) = f_1$ that gives:

$$+\begin{bmatrix} a & d & b \\ b & f & c \end{bmatrix} \qquad (12).$$

Now the remaining monomials in $(10)-(11)+(12)$ vanish at all. Then (IIIB) is a relation.

(IIIC) $b \le d, \quad a < b < e < f < c$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & b & f \\ d & e & c \end{bmatrix} = (13) - (14).$$

Consider first the monomials with the variables indexed by $a, b+1$: $x_a x_{b+1} x_d x_{e+1} x_{f+2} x_{c+2}$ of (13) vanishes with a monomial of (14), while the monomial $f_1 = x_a x_{b+1} x_d x_{e+2} x_{f+1} x_{c+2}$ of (13) does not vanish. Then $in_<((13) - (14)) = f_1$ that we write as

$$+\begin{bmatrix} a & b & e \\ d & f & c \end{bmatrix} = (15)$$

Now the monomials f_1 of (13) and $x_a x_{b+1} x_d x_{e+2} x_{f+2} x_{c+1}$ of (14) vanish with monomials of (15). Consider the monomials with $\{a, d+1\}$ in the support. The monomials $x_a x_{b+1} x_{d+1} x_e x_{f+2} x_{c+2}$, $x_a x_{b+1} x_{d+1} x_{e+2} x_f x_{c+2}$ of (13) vanish with monomials of (14) and (15). The monomial $x_a x_{b+1} x_{d+1} x_{e+2} x_{f+2} x_c$ of (14) vanishes with a monomial of (15). The monomial $f_2 = x_a x_{b+1} x_{d+1} x_f x_{e+2} x_{c+2}$ of (14) does not vanish. Then the $in_<((13) - (14) + (15)) = f_2$ that gives:

$$+\begin{bmatrix} a & d & e \\ b+1 & f-1 & c \end{bmatrix}.$$

(IIID) $b > d, \quad a < d < b < e < f < c$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & b & f \\ d & e & c \end{bmatrix} = (16) - (17).$$

Consider the monomials with the variables indexed by a, d:
$x_a x_d x_{b+1} x_{e+1} x_{f+2} x_{c+2}$ of (16) vanishes with a monomial of (17) while

$f_1 = x_a x_d x_{b+1} x_{e+2} x_{f+1} x_{c+2}$ of (16) does not vanish. Then the $in_<((16) - (17)) = f_1$ that gives:

$$+\begin{bmatrix} a & b & e \\ d & f & c \end{bmatrix} = (18).$$

Now f_1 of (16) and $x_a x_d x_{b+1} x_{e+2} x_{f+2} x_{c+1}$ of (17) vanish with monomials of (18). Consider the monomials containing the variables indexed by $a, d, b+2$: $x_a x_d x_{b+2} x_{e+1}\ x_{f+2} x_{c+1}$, $x_a x_d x_{b+2} x_{e+2} x_{c+1} x_{f+1}$ of (16) vanish with monomials of (17) and (18); $x_a x_d x_{b+2} x_{e+1} x_{f+1} x_{c+2}$ of (17) vanishes with monomials of (18). Consider the monomials with $\{a, d+1, b+1\}$ in the support: $x_a x_{d+1} x_{b+1} x_{e+2} x_f x_{c+2}$, $x_a x_{d+1} x_{b+1} x_e x_{f+2}$ x_{c+2} of (16) vanish with monomials of (17) and (18). $x_a x_{d+1} x_{b+1} x_{e+2}$ $x_{f+2} x_c$ of (17) vanishes with a monomial of (18). Consider the monomials with $\{a, d+1, b+2\}$ in the support: $f_2 = x_a x_{d+1} x_{b+2} x_e x_{f+1} x_{c+2}$ of (17) does not vanish. Then $in_<((16) - (17) + (18)) = f_2$ that gives

$$+\begin{bmatrix} a & d & b \\ e & f & c \end{bmatrix}.$$

The expression is not yet a relation since all the monomials do not vanish. It is easy to check that the monomial $x_{a+1} x_{d+2} x_b x_{e+1} x_f x_{c+2}$ is the next lifting.

(IVA) $a < b \ll d < e-1 \leq c < e < f$. See [8] for the proof.

(IVB) $d = c = e-1 \quad a < b \ll e-1 < e < f-1 < f$

$$\begin{bmatrix} a & b & e-1 \\ e-1 & e & f \end{bmatrix} - \begin{bmatrix} a & e-2 & e-1 \\ b+1 & e & f \end{bmatrix} = (19) - (20).$$

Consider the monomials with $\{a, b+1, e-1\}$ in the support (since there are not monomials with x_a and x_b): in (19) $x_a x_{b+1} x_{e-1} x_{e+1} x_{e+1} x_{f+2}$, $-x_a x_{b+1} x_{e-1} x_{e+1} x_{e+2} x_{f+1}$ vanish with monomials of (20). Considering the monomials with $\{a, b+1, e\}$, in (19) $x_a x_{b+1} x_e x_e x_{e+1} x_{f+2}$ vanishes with a monomial of (20) while $f_1 = x_a x_{b+1} x_e x_e x_{e+2} x_{f+1}$ of (20) does not vanish. Then $in_<((19) - (20)) = f_1$ that we write as

$$\begin{bmatrix} a & e-1 & e \\ b+1 & e-1 & f-1 \end{bmatrix} = (21).$$

Now in (21) there are not monomials with $\{a, b+1, e-1\}$ in the support and consider again monomials containing x_a, x_{b+1}, x_e: $x_a x_{b+1} x_e x_{e+1}^2$

x_{f+1}, $x_a x_{b+1} x_e x_{e+1} x_{e+2} x_f$ of (19) vanish with monomials of (21). Consider the monomials with $\{a, b+1, e+1\}$ in the support: $x_a x_{b+1} x_{e+1}^3$ x_f of (19) vanishes with a product of (21). Then consider the monomials containing the variables indexed by $a, b+2, e-1$: $x_a x_{b+2} x_{e-1} x_e x_{e+1} x_{f+2}$, $x_a x_{b+2} x_{e-1} x_e x_{e+2} x_{f+1}$ of (19) vanish with products of (20) and (21), while $f_2 = x_a x_{b+2} x_{e-1} x_{e+1}^2 x_{f+1}$ of (21) does not vanish. Then $in_<((19) - (20) + (21)) = f_2$ that we write as

$$\begin{bmatrix} a & e-2 & e-1 \\ b+2 & e & f-1 \end{bmatrix}.$$

(IVB1) $d = c = e-1$, $e+1 = f \quad a < b \ll e-1 < e < e+1$

$$\begin{bmatrix} a & b & e-1 \\ e-1 & e & e+1 \end{bmatrix} - \begin{bmatrix} a & e-2 & e-1 \\ b+1 & e & e+1 \end{bmatrix} = (22) - (23).$$

Consider first the monomials with $\{a, b+1, e-1\}$ in the support (since there are not monomials with x_a and x_b): in (22) $x_a x_{b+1} x_{e-1} x_{e+1}^2 x_{e+3}$, $x_a x_{b+1} x_{e-1} x_{e+1} x_{e+2}^2$ vanish with monomials of (23). Considering the monomials with x_a, x_{b+1}, x_e, in (22) we have $x_a x_{b+1} x_e^2 x_{e+1} x_{e+3}$, that vanishes with a monomial of (23); in (23) we have only $f_1 = x_a x_{b+1} x_e^2$ x_{e+2}^2. Then $in_<((22) - (23)) = f_1$ that we write as

$$\begin{bmatrix} a & e-1 & e \\ b+1 & e-1 & e \end{bmatrix} = (24).$$

Now in (24) there are not monomials with x_a, x_{b+1}, x_{e-1} in the support and we consider the monomials with x_a, x_{b+1}, x_e: $2x_a x_{b+1} x_e x_{e+1}^2 x_{e+2}$ of (22) vanishes with a monomial of (24). Consider the monomials with $\{a, b+1, e+1\}$ in the support: $x_a x_{b+1} x_{e+1}^4$ of (22) vanishes with a monomial of (24). Then we consider the monomials containing x_a, x_{b+2}, x_{e-1}. It easy to check that in the sum $(22) - (23) + (24)$ all the monomials with the variable x_a vanish. Then we consider monomials containing x_{a+1}, x_b. They are only in (22) where $f_2 = x_{a+1} x_b x_{e-1} x_{e+1}^2 x_{e+3}$ does not vanish. Then $in_<((22)-(23)+(24)) = f_2$ that we write as

$$\begin{bmatrix} a+1 & e-2 & e-1 \\ b & e & e+1 \end{bmatrix}.$$

(IVC) $d < e-1$, $c = e, \quad a < b \ll d < e-1 < e < f$

$$\begin{bmatrix} a & b & e \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & e \\ b+1 & e & f \end{bmatrix} = (25) - (26).$$

Consider first the monomials with $\{a, b+1, d\}$ in the support: in (25) $x_a x_{b+1} x_d x_{e+1} x_{e+2} x_{f+2}$, $x_a x_{b+1} x_d x_{e+2} x_{e+2} x_{f+1}$ vanish with monomials of (26). Then we consider the the monomials with $\{a, b+1, d+1\}$ in the support: $f_1 = x_a x_{b+1} x_{d+1} x_e x_{e+2} x_{f+2}$ of (25) does not vanish and then $in_<((25) - (26)) = f_1$ that we write as

$$\begin{bmatrix} a & d & e \\ b+1 & e-1 & f \end{bmatrix} = (27).$$

Now we consider the products in $(25), (26), (27)$ containing the variables indexed by $a, b+1, d+1, e+1$: $f_2 = x_a x_{b+1} x_{d+1} x_{e+1} x_{e+1} x_{f+1}$ does not vanish. Then $in_<((25) - (26) + (27)) = f_2$ that we write as

$$-\begin{bmatrix} a & d & e-1 \\ b+1 & e & f \end{bmatrix}.$$

(IVD) $d = e-1$ $c = e$ $\quad a < b \ll e-1 < e < f-1 < f$

$$\begin{bmatrix} a & b & e \\ e-1 & e & f \end{bmatrix} - \begin{bmatrix} a & e-2 & e \\ b+1 & e & f \end{bmatrix} = (28) - (29).$$

Consider the monomials with the variables indexed by $a, b+1, e-1$ (since there are not monomials with a and b): $x_a x_{b+1} x_{e-1} x_{e+1} x_{e+2} x_{f+2}$, $x_a x_{b+1} x_{e-1} x_{e+2} x_{e+2} x_{f+1}$ of (34) vanish with products of (29). Then we consider the products containing the variables indexed by $a, b+1, e$: $f_1 = x_a x_{b+1} x_e x_e x_{e+2} x_{f+2}$ of (28) does not vanish. Then $in_<((28) - (29)) = f_1$ that we write as

$$+\begin{bmatrix} a & e-1 & e \\ b+1 & e-1 & f \end{bmatrix} = (30).$$

Now we consider the monomials in $(28), (29), (30)$ containing the variables indexed by $a, b+1, e, e+1$: $f_2 = x_a x_{b+1} x_e x_{e+1} x_{e+2} x_{f+1}$ of (28) vanishes with a product of (29), $x_a x_{b+1} x_e x_{e+1} x_{e+1} x_{f+2}$ of (29) vanishes with a product of (30). But there is another product f_3 in (30) equal to f_2 that does not vanish. Then $in_<((28) - (29) + (30)) = f_3$ that we write as

$$+\begin{bmatrix} a & e-1 & e \\ b+1 & e & f-1 \end{bmatrix}.$$

(IVD1) $d = e-1$ $c = e =$, $f = e+1$ $\quad a < b \ll e-1 < e < e+1$

This case is the same of (IVD) until the first lifting, just putting $f = e+1$. Computing the second lifting, the monomial $x_a x_{b+1} x_e x_{e+1} x_{e+2}$ x_{e+2}, corresponding to f_3 of (VD), vanishes with a monomial of (28). Then we consider monomials with $\{a, b+1, e+1\}$ in the support: $x_a x_{b+1} x_{e+1} x_{e+1} x_{e+1} x_{e+2}$ vanishes with a product of (30). The monomials with $\{a, b+2, e-1\}$ in the support: $x_a x_{b+2} x_{e-1} x_{e+1} x_{e+1} x_{e+3}$ of (28) and $x_a x_{b+2} x_{e-1} x_e x_{e+2} x_{e+3}$ of (29) vanish with monomials of (30). The monomials containing the variables indexed by $a, b+2, e$: $2x_a x_{b+2} x_e x_{e+1} x_{e+1} x_{e+2}$ and $x_a x_{b+2} x_e x_e x_{e+1} x_{e+3}$ of (28) vanish with monomials of (29) and (30). The monomials having $\{a, b+2, e+1\}$ in the support are $x_a x_{b+2} x_{e+1} x_{e+1} x_{e+1} x_{e+1}$ of (28) that vanishes with a monomial of (30). The monomials containing the variables indexed by $a, b+3$ are $x_a x_{b+3} x_{e-1} x_e x_{e+2} x_{e+2}$ and $x_a x_{b+3} x_{e-1} x_{e+1} x_{e+1} x_{e+2}$ of (29) that vanish with monomials of (30).Finally, consider the monomials containing the variables indexed by $a+1, b$ that there exist only in (28):$f_3 = x_a x_b x_{e-1} x_{e+1} x_{e+2} x_{e+3}$, $f_4 = x_{a+1} x_b x_{e-1} x_{e+2} x_{e+2} x_{e+3}$. Then $in_<((28) - (29) + (30)) = f_3$ that we write as

$$\begin{bmatrix} a+1 & e-2 & e \\ b & e & e+1 \end{bmatrix}.$$

(IVE) $\quad a < b \ll d < e < c < f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & c \\ b+1 & e & f \end{bmatrix} = (31) - (32).$$

Consider first the products containing the monomials with the variables $a, b+1, d$ (since there are not monomials with variables indexed by a, b): in (31) $x_a x_{b+1} x_d x_{e+1}$
$x_{c+2} x_{f+2}$, $x_a x_{b+1} x_d x_{e+2} x_{c+2} x_{f+1}$ vanish with products of (32). Then we consider the products containing variables indexed by $a, b+1, d+1$: $f_1 = x_a x_{b+1} x_{d+1} x_{e+2} x_{c+2} x_f$ and $f_2 = x_a x_{b+1} x_{d+1} x_e x_{c+2} x_{f+2}$ of (31), $f_3 = x_a x_{b+1} x_{d+1} x_{e+1} x_{c+1} x_{f+2}$ and $f_4 = x_a x_{b+1} x_{d+1} x_{e+2} x_{c+1} x_{f+1}$ of (32); f_1, f_2, f_3, f_4 do not vanish and then $in_<((31) - (32)) = f_2$, that we write

$$\begin{bmatrix} a & d & c \\ b+1 & e-1 & f-1 \end{bmatrix} = (33).$$

Now we consider $f_5 = x_a x_{b+1} x_{d+1} x_{e+1} x_{c+2} x_{f+1}$ of (33) with $\{a, b+1, d+1\}$. f_1, f_2, f_3, f_4, f_5 do not vanish. Then $in_<((31)-(32)+(33)) = f_3$ that we write as

$$\begin{bmatrix} a & d & c-1 \\ b+1 & e & f \end{bmatrix}.$$

(VA) $b = d-1$, $d < c$, $c+3 < f \quad a < b < d < c \ll e < f$

$$\begin{bmatrix} a & d-1 & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & e-1 \\ d & c+1 & f \end{bmatrix} = (34) - (35).$$

We consider the monomials with variables indexed by a, d (since there are not monomials with variables indexed by a and $d-1$): $x_a x_d x_d x_{c+2}$ $x_{e+1} x_{f+2}$ of (34) vanishes with a product of (35), while $f_1 = x_a x_d x_d x_{c+2}$ $x_{e+2} x_{f+1}$ of (34) and $f_2 = x_a x_d x_d x_{c+3} x_{e+1} x_{f+1}$ of (35) do not vanish. Then $in_<((34)-(35)) = f_1$ that gives:

$$\begin{bmatrix} a & d-1 & e \\ d & c+1 & f-1 \end{bmatrix} = (36).$$

Now f_2 does not vanish with any product of (35) and (36). Then $in_<((34)-(35)+(36)) = f_2$ that gives:

$$-\begin{bmatrix} a & d-1 & e-1 \\ d & c+2 & f-1 \end{bmatrix}.$$

(VA1) $b = d-1$, $d = c$, $d = f-3, \quad a < d-1 < d \ll d+2 < d+3$

$$\begin{bmatrix} a & d-1 & d \\ d & d+2 & d+3 \end{bmatrix} - \begin{bmatrix} a & d-1 & d+1 \\ d & d+1 & d+3 \end{bmatrix} = (37) - (38).$$

We consider the monomials with $\{a, d\}$ in the support (since there are not monomials with variables indexed by a and $d-1$): in (37) $x_a x_d x_d x_{d+2} x_{d+3} x_{d+5}$ vanishes with a product of (38) while $f_1 = x_a x_d x_d$ $x_{d+2} x_{d+4} x_{d+4}$ of (37) and $f_2 = x_a x_d x_d x_{d+3} x_{d+3} x_{d+4}$ of (38) do not vanish. Then $in_<((37)-(38)) = f_1$ that we can write:

$$\begin{bmatrix} a & d-1 & d+2 \\ d & d+1 & d+2 \end{bmatrix} = (39).$$

Now in (37) and (38) the monomials with $\{a, d\}$ in the support are f_1 and f_2 that vanish with products of (39). Then we consider the monomials containing variables x_a, x_d, x_{d+1}. In (37) $x_a x_d x_{d+1}^2 x_{d+3} x_{d+5}$, $x_a x_d x_{d+1} x_{d+2}^2 x_{d+5}$ vanish with monomials of (38), $x_a x_d x_{d+1}^2 x_{d+4} x_{d+4}$ vanishes with a monomial of (39). In (37) and (39) $2x_a x_d x_{d+1} x_{d+2} x_{d+3} x_{d+4}$ vanishes with monomials of (38) and (39). In (38) $x_a x_d x_{d+1} x_{d+2} x_{d+3} x_{d+4}$, $x_a x_d x_{d+1} x_{d+3}^3$ vanish with products of (39). The products containing the variables x_a, x_d, x_{d+2} are in (39) $x_a x_d x_{d+2}^2 x_{d+3}^2$ and $x_a x_d x_{d+2}^3 x_{d+4}$ that vanish with products of (38) and (39); the products containing the variables x_a, x_{d+1} are in (37) $x_a x_d x_{d+1}^2 x_{d+2} x_{d+5}$ and $x_a x_{d+1}^2 x_{d+2} x_{d+3} x_{d+5}$ that vanish with products of (38), $x_a x_d x_{d+1}^2 x_{d+3} x_{d+4}$ and $x_a x_{d+1}^2 x_{d+2} x_{d+3}^2$ that vanish with products of (39); in (39) $x_a x_{d+1}^2 x_{d+2} x_{d+3}^2$ and $x_a x_{d+1} x_{d+2}^3 x_{d+3}$ that vanish with products of (40).The products containing the variables x_a, x_{d-1} are:
$x_{a+1} x_{d-1} x_{d+1} x_{d+3}^2 x_{d+4}$ and $x_{a+1} x_{d-1} x_{d+1} x_{d+2}^2 x_{d+5}$ in (37), $x_a x_{d-1} x_{d+1} x_{d+3}^3$ and $f_3 = x_{a+1} x_{d-1} x_{d+1}^2 x_{d+3} x_{d+5}$ in (44), $x_{a+1} x_{d-1} x_{d+1} x_{d+2} x_{d+3} x_{d+4}$, $x_{a+1} x_{d-1} x_{d+1}^2 x_{d+4}^2$ in (39). All of the previous products do not vanish and $in_<((37) - (38)) = f_3$ that we can write as

$$\begin{bmatrix} a+1 & d & d+1 \\ d-1 & d & d+3 \end{bmatrix}.$$

(VB) $b = d-1$, $d < c$, $c+3 = f$ $\quad a < d-1 < d < c \ll c+2 < c+3$

$$\begin{bmatrix} a & d-1 & c \\ d & c+2 & c+3 \end{bmatrix} - \begin{bmatrix} a & d-1 & c+1 \\ d & c+1 & c+3 \end{bmatrix} = (40) - (41).$$

We consider the monomials with variables x_a, x_d (since there are not monomials with x_a, x_{d-1}): in (40) $x_a x_d^2 x_{c+2} x_{c+3} x_{c+5}$ vanishes with a product of (41), while $f_1 = x_a x_d^2 x_{c+2} x_{c+4}^2$ (in (40)) and $f_2 = x_a x_d^2 x_{c+3}^2 x_{c+4}$ (in (41)) do not vanish. Then $in_<((40) - (41)) = f_1$ that gives

$$\begin{bmatrix} a & d-1 & c+2 \\ d & c+1 & c+2 \end{bmatrix} = (42).$$

Now f_2 does not vanish with any product of (40) and (42). Then $in_<((40) - (41) + (42)) = f_2$ and we have:

$$-\begin{bmatrix} a & d & c+2 \\ d & c & c+2 \end{bmatrix}.$$

(VC) $d \leq b,\, c+3 < f, \quad a < d \leq b < c \ll e < f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & b & e-1 \\ d & c+1 & f \end{bmatrix} = (43) - (44).$$

We consider first the monomials with variables x_a, x_d: $x_a x_d x_{b+1} x_{c+2} x_{e+1} x_{f+2}$ (in (43)) vanishes with a product of (44), while $f_1 = x_a x_d x_{b+1} x_{c+2} x_{e+2} x_{f+1}$ (in (43))and $f_2 = x_a x_d x_{b+1} x_{c+3} x_{e+1} x_{f+1}$ (in (44)) do not vanish. Then $in_<((43) - (44)) = f_1$, that we write as

$$\begin{bmatrix} a & b & e \\ d & c+1 & f-1 \end{bmatrix} = (45).$$

Now f_2 does not vanish with any product of (43) and (44). Then $in_<((43) - (44) + (45)) = f_2$ that we write as

$$+\begin{bmatrix} a & b & c+1 \\ d & e & f-1 \end{bmatrix}.$$

(VC1) $d \leq b,\, c+3 = f, \quad a < d \leq b < c \ll e < f$

$$\begin{bmatrix} a & b & c \\ d & c+2 & c+3 \end{bmatrix} - \begin{bmatrix} a & b & c+11 \\ d & c+1 & c+3 \end{bmatrix} = (46) - (47).$$

We consider the monomials with $\{a, d\}$ in the support: in (46) $x_a x_d x_{b+1}$ $x_{c+2} x_{c+3} x_{c+5}$ vanishes with a product of (47), while $f_1 = x_a x_d x_{b+1}$ $x_{c+2} x_{c+4} x_{c+4}$ (in (46))and $f_2 = x_a x_d x_{b+1} x_{c+3} x_{c+3} x_{c+4}$ (in (47)) do not vanish. Then $in_<((46) - (47)) = f_1$ that we write as

$$\begin{bmatrix} a & b & c+2 \\ d & c+1 & c+2 \end{bmatrix} = (48).$$

Now f_1 and f_2 vanish with products of (48) and we consider the monomials containing x_a, x_d, x_{b+2}. The monomial $f_3 = x_a x_d x_{b+2} x_{c+1} x_{c+3}$ x_{c+5} does not vanish. Then $in_<((46) - (47) + (48)) = f_3$ that we write as

$$\begin{bmatrix} a & b+1 & c+1 \\ d & c & c+3 \end{bmatrix}.$$

(VIA) $c < d,\, c+3 < f, \quad a < b < c < d < e < f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & e-1 \\ b+1 & c+1 & f \end{bmatrix} = (49) - (50).$$

Consider the monomials with $\{a, b+1\}$ (since there are not monomials containing x_a, x_b): in (49) we have $x_a x_{b+1} x_{c+2} x_d x_{e+1} x_{f+2}$ (that vanishes with a product of (50)), $f_1 = x_a x_{b+1} x_{c+2} x_d x_{e+2} x_{f+1}$ and $x_a x_{b+1} x_{c+3} x_d x_{e+1} x_{f+1}$. Then $in_<((49) - (50)) = f_1$ that gives:

$$\begin{bmatrix} a & d-1 & e \\ b+1 & c+1 & f-1 \end{bmatrix} = (51).$$

Considering the monomials containing variables x_a, x_{b+1} starting by x_a, x_{b+1}, x_{c+2}, we observe that the monomials f_1 and $x_a x_{b+1} x_{c+2} x_{d+1} x_e$ x_{f+2} of (49) vanish with products of (51) and (50), while $f_2 = x_a x_{b+1}$ $x_{c+2} x_{d+1} x_{e+2} x_f$ of (49) do not vanish. Then $in_<((49) - (50) + (51)) =$ f_2 that gives

$$\begin{bmatrix} a & d & e \\ b+1 & c+1 & f-2 \end{bmatrix}.$$

(VIA1) $c < d$, $c + 3 = f$, $\quad a < b < c < c+1 < c+2 < c+3$

$$\begin{bmatrix} a & b & c \\ c+1 & c+2 & c+3 \end{bmatrix} - \begin{bmatrix} a & c & c+1 \\ b+1 & c+1 & c3+ \end{bmatrix} = (52) - (53).$$

We consider the monomials with $\{a, b+1\}$ in the support (since there are not monomials containing x_a, x_b).The products containing x_{c+1} in (52) are $x_a x_{b+1} x_{c+1} x_{c+2} x_{c+3} x_{c+5}$ that vanishes with a product of (53) and $f_1 = x_a x_{b+1} x_{c+1} x_{c+2} x_{c+4} x_{c+4}$. In (53) we have $f_2 = x_a x_{b+1} x_{c+1}$ $x_{c+3} x_{c+3} x_{c+4}$. Then $in_<((52) - (53)) = f_1$, hence

$$\begin{bmatrix} a & c & c+2 \\ b+1 & c+1 & c+2 \end{bmatrix} = (54).$$

Now the monomials containing x_a, x_{b+1}, x_{c+1} are: $x_a x_{b+1} x_{c+1} x_{c+2} x_{c+4}$ x_{c+4}, that vanishes with f_1, and $x_a x_{b+1} x_{c+1} x_{c+3} x_{c+3} x_{c+4}$,that vanishes with f_2. Then we have to consider the products containing x_a, x_{b+1}, x_{c+2} in (52), (53) and (54): in (52), $x_a x_{b+1} x_{c+2} x_{c+2} x_{c+3} x_{c+4}$ and $x_a x_{b+1} x_{c+2} x_{c+2} x_{c+2} x_{c+5}$ vanish with two products of (53), $x_a x_{b+1}$

$x_{c+2}x_{c+2}x_{c+3}x_{c+4}$ and $x_a x_{b+1} x_{c+2} x_{c+3} x_{c+3} x_{c+3}$ vanish with two products of (54). Then we consider the products containing x_a, x_{b+2}, beginning with x_a, x_{b+2}, x_{c+1}: in (52) $x_a x_{b+2} x_{c+1} x_{c+1} x_{c+3} x_{c+5}$, $x_a x_{b+2} x_{c+1} x_{c+3} x_{c+3} x_{c+3}$, $x_a x_{b+2} x_{c+1} x_{c+2} x_{c+2} x_{c+5}$ vanish with pro - ducts of (53), $x_a x_{b+2} x_{c+1} x_{c+1} x_{c+4} x_{c+4}$, $2x_a x_{b+2} x_{c+1} x_{c+2} x_{c+3} x_{c+4}$ vanish with products of (53). In (53) $x_a x_{b+2} x_{c+2} x_{c+2} x_{c+3} x_{c+3}$ vanishes with a term of (54). Finally we consider the products containing x_{a+1}, x_b. They appear only in (52) and so

$$in_<((52) - (53) + (54)) = x_{a+1} x_b x_{c+1} x_{c+2} x_{c+3} x_{c+5}$$

that gives:

$$\begin{bmatrix} a+1 & c & c+1 \\ b & c+1 & c+3 \end{bmatrix}.$$

(VIB) $d \le c$, $c+3 < f \quad a < b \ll d \le c \ll e < f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & e-1 \\ b+1 & c+1 & f \end{bmatrix} = (55) - (56).$$

We consider the monomials with $\{a, b+1\}$ in the support (since there are not monomials with x_a, x_b): in (55) $x_a x_{b+1} x_d x_{c+2} x_{e+1} x_{f+2}$ vanishes with a product of (56), while $f_1 = x_a x_{b+1} x_d x_{c+2} x_{e+4} x_{f+1}$ of (55) does not vanish. Then $in_<((55) - (56)) = f_1$ that gives

$$\begin{bmatrix} a & d-1 & e \\ b+1 & c+1 & f-1 \end{bmatrix} = (57).$$

Now f_1 vanishes with a monomial of (57), $f_2 = x_a x_{b+1} x_d x_{c+3} x_{e+1} x_{f+1}$ does not vanish. Then $in_<((55) - (56) + (57)) = f_2$ that gives

$$+\begin{bmatrix} a & d-1 & e-1 \\ b+1 & c+2 & f-1 \end{bmatrix}.$$

(VIB1) $d \le c \quad a < b \ll d \le c \ll c+2 < c+3$

$$\begin{bmatrix} a & b & c \\ d & c+2 & c+3 \end{bmatrix} - \begin{bmatrix} a & d-1 & c+1 \\ b+1 & c+1 & c+3 \end{bmatrix} = (58) - (59).$$

Consider the monomials with the variables x_a, x_b+1 (since there are not monomials with x_a, x_b): we have in (58) $x_a x_{b+1} x_d x_{c+2} x_{c+3} x_{c+5}$, that vanishes with a product of (59), $f_1 = x_a x_{b+1} x_d x_{c+2} x_{c+4} x_{c+4}$ and $f_2 = x_a x_{b+1} x_d x_{c+3} x_{c+3} x_{c+4}$. Then $in_<((58)-(59)) = f_1$ that we write as

$$\begin{bmatrix} a & d-1 & c+2 \\ b+1 & c+1 & c+2 \end{bmatrix} = (60).$$

Here the products containing x_a, x_{b+1}, x_d vanishes with products of (58) and (59). So we consider all products containing x_a, x_{b+1}, x_{d+1}. In (58) the products $x_a x_{b+1} x_d x_{c+2} x_{c+4} x_{c+4}$, $x_a x_{b+1} x_{d+1} x_{c+2} x_{c+3} x_{c+4}$, $x_a x_{b+1} x_{d+1} x_{c+2} x_{c+2} x_{c+5}$, $x_a x_{b+1} x_{d+1} x_{c+2} x_{c+2} x_{c+5}$ vanish with products of (58) or (59). The remaining products are:
$f_2 = x_a x_{b+1} x_{d+1} x_{c+2} x_{c+3} x_{c+4}$ and $x_a x_{b+1} x_{d+1} x_{c+3} x_{c+3} x_{c+3}$. Then

$$in_<((58) - (59) + (60)) = f_2$$

that we write as

$$\begin{bmatrix} a & d & c+1 \\ b+1 & c+1 & c+2 \end{bmatrix}$$

□

Remark 1. *The property of normality of $H(2,n)$ has been studied in [3], by using the partial liftings given in [8].*

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