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NEAR OPTIMAL LINEAR QUADRATIC REGULATOR FOR CONTROLLED SYSTEMS DESCRIBED BY ITÔ DIFFERENTIAL EQUATIONS WITH TWO FAST TIME SCALES*

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Abstract

In this paper a linear quadratic optimization problem is addressed under the assumption that the state space representation of the controlled system is described by a system of Itô differential equations with two fast time scales. It is well known that under some standard assumptions this optimization problem has a unique optimal control which is in a state feedback form. The gain matrix of the optimal control is computed based on the stabilizing solution of a suitable algebraic Riccati equation. The presence of the small parameters associated to the fast time scales may produce some ill conditioning of the numerical computations of the stabilizing solution. Our goal is to perform a detailed study of the dependence of the stabilizing solution of the Riccati equation with respect to the small parameters which are describing the fast time scales. In this way, we are able to obtain a near optimal control whose gain matrix does not depend upon the small parameters (which may be unknown). An estimation of the loss of the performance of the near optimal control is also done.

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1 Introduction

Often, the mathematical models of various physical phenomena contain several parasitic small parameters such as small time constants, resistances, inductances, capacitances, moments of inertia, small masses, etc. The presence of those parasitic constants in the mathematical model leads to the appearance of some small positive parameters often unknown, multiplying the derivatives of some differential equations which are describing the phenomena under consideration. In these cases the systems of differential equations involved in the mathematical model are known as *singularly perturbed systems of differential equations* or *systems with several fast time scales.*

In order to avoid the ill conditioning of the numerical computation due to the presence of the fast time scales in both the analysis and synthesis of the automatic control laws, the techniques of the singular perturbations were successfully involved.

For an historical perspective on these topics we refer to the monographs [1, 2, 3, 4] and to the recent surveys [5, 6] for the deterministic framework. In the stochastic context the problem of exponential stability in mean square of the zero solution of a system of singularly perturbed Itô differential equations was studied in [9] for the linear case and in [10] for the nonlinear case.

Several aspects regarding the linear quadratic control problem in the case when the controlled system is described by a system of singularly perturbed Itô differential equations were investigated in [11, 12, 13] and, in the case when the controlled system is modeled by a system of singularly perturbed differential equation with Markovian switching we refer to [14, 15].

In the stochastic framework the problem of minimization of a quadratic cost functional along of the trajectories of a system of stochastic linear differential equations was widely investigated. For the readers convenience we refer to [16, 18, 19] and the references therein.

In the present work we consider the optimal control problem asking for the minimization of a quadratic functional along of the trajectories of a controlled system described by a system of linear Itô differential equations with two fast time scales. Sometimes, such a control problem is known as *linear quadratic optimal regulator problem.* It is well known that under some standard assumptions this optimization problem has a unique optimal control which is in a state feedback form. The gain matrix of the optimal control is computed based on the stabilizing solution of a suitable algebraic Riccati equation. The presence of the small parameters associated to the fast time scales may produce some ill conditioning of the numerical computations of the stabilizing solution.

Our goal is to perform a detailed study of the dependence of the stabilizing solution of the Riccati equation with respect to the small parameters which are describing the fast time scales. In this way, we are able to obtain a near optimal control whose gain matrix does not depend upon the small parameters (which may be unknown).

An estimation of the loss of the performance of the near optimal control is also done.

The paper is organized as follows: Section 2 contains the problem formulation together with the motivation of this study. The derivation of the system of reduced algebraic Riccati equations may be found in Section 3, while Section 4 contains necessary and sufficient conditions for the existence of the stabilizing solution of the system of reduced coupled algebraic Riccati equations. The main results of the paper are presented in Section 5. First, in subsection 5.1 the asymptotic structure of the stabilizing solution of the corresponding algebraic Riccati equation is derived and in subsection 5.2 the near optimal linear quadratic regulator is obtained. The paper end with some conclusions in Section 6.

2 The problem

Let us consider the optimal control problem described by the controlled system

$$\begin{aligned} \varepsilon_k dx_k(t) &= [\sum_{j=0}^{2} A_{kj}x_j(t) + B_k u(t)]dt + \varepsilon_k^{\delta}[\sum_{j=0}^{2} C_{kj}x_j(t) + D_k u(t)]dw(t) \\ x_k(0) &= x_{k0}, \qquad 0 \le k \le 2, \end{aligned} \tag{1}$$

and the quadratic cost functional

$$\begin{aligned} J(u) &= E\int_0^\infty [\sum_{k,j=0}^{2} x_k^T(t)M_{kj}x_j(t) \\ &\quad + \sum_{j=0}^{2}(x_j^T(t)L_j u(t) + u^T(t)L_j^T x_j(t)) + u^T(t)Ru(t)]dt \end{aligned} \tag{2}$$

$M_{kj} = M_{jk}^T, \quad 0 \le j \le k \le 2$, $R = R^T$, where $x_j(t) \in \mathbb{R}^{n_j}$, $0 \le j \le 2$, are the state vectors and $u(t) \in \mathbb{R}^m$ are the control parameters; $\{w(t)\}_{t \ge 0}$ is a one dimensional standard Brownian motion defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$. In (1) $\varepsilon_0 = 1$ and for $k = 1$ and $k = 2$, $\varepsilon_k : [0, \varepsilon^*] \to [0, \infty)$ satisfy

$$\lim_{\varepsilon \to 0_+} \varepsilon_k(\varepsilon) = 0, k = 1, 2, \quad \lim_{\varepsilon \to 0_+} \frac{\varepsilon_2(\varepsilon)}{\varepsilon_1(\varepsilon)} = 0. \tag{3}$$

We also assume that in (1) $\delta > \frac{1}{2}$.

Throughout the paper $E[\cdot]$ stands for the mathematical expectation and superscript T denotes the transpose of a vector or a matrix.

The class of admissible controls $\mathcal{U}_{adm}(x_0)$ consists of all measurable stochastic processes $u = \{u(t)\}_{t \ge 0}$ which are adapted to the filtration generated by the stochastic process $\{w(t)\}_{t \ge 0}$ and have the properties:

a) $\int\limits_0^\infty E[|u(t)|^2]dt < \infty$

b) $-\infty < J(u) < \infty$ and

$$\lim_{t \to \infty} \sum_{j=0}^{2} E[|x_j(t; x_0, u)|^2] = 0 \tag{4}$$

where $x_j(t; x_0, u)$, $0 \le j \le 2$ is the solution of the problem with given initial value (1) determined by the input u.

The linear quadratic optimization problem which we want to solve requires to find a control $\tilde{u} \in \mathcal{U}_{adm}(x_0)$ with the property that

$$J(\tilde{u}) = \min_{u \in \mathcal{U}_{adm}(x_0)} J(u).$$

We set

$$
\begin{aligned}
\mathbb{A} &= \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ \frac{1}{\varepsilon_1}A_{10} & \frac{1}{\varepsilon_1}A_{11} & \frac{1}{\varepsilon_1}A_{12} \\ \frac{1}{\varepsilon_2}A_{20} & \frac{1}{\varepsilon_2}A_{21} & \frac{1}{\varepsilon_2}A_{22} \end{pmatrix} \\
\mathbb{C} &= \begin{pmatrix} C_{00} & C_{01} & C_{02} \\ \varepsilon_1^{\delta-1}C_{10} & \varepsilon_1^{\delta-1}C_{11} & \varepsilon_1^{\delta-1}C_{12} \\ \varepsilon_2^{\delta-1}C_{20} & \varepsilon_2^{\delta-1}C_{21} & \varepsilon_2^{\delta-1}C_{22} \end{pmatrix} \\
\mathbb{B} &= \begin{pmatrix} B_0^T & \frac{1}{\varepsilon_1}B_1^T & \frac{1}{\varepsilon_2}B_2^T \end{pmatrix} \in \mathbb{R}^{n\times m} \\
\mathbb{D} &= \begin{pmatrix} D_0^T & \varepsilon_1^{\delta-1}D_1^T & \varepsilon_2^{\delta-1}D_2^T \end{pmatrix}^T \in \mathbb{R}^{n\times m} \\
\mathbb{M} &= \begin{pmatrix} M_{00} & M_{01} & M_{02} \\ M_{01}^T & M_{11} & M_{12} \\ M_{02}^T & M_{12}^T & M_{22} \end{pmatrix} \\
\mathbb{L} &= \begin{pmatrix} L_0^T & L_1^T & L_2^T \end{pmatrix}^T \in \mathbb{R}^{n\times m}
\end{aligned}
\tag{5}
$$

where $n = n_0 + n_1 + n_2$. With these notations (1) and (2) may be written in a compact form as follows:

$$dx(t) = (\mathbb{A}x(t) + \mathbb{B}u(t))dt + (\mathbb{C}x(t) + \mathbb{D}u(t))dw(t) \tag{6}$$

$$J(u) = E\int_0^\infty [x^T(t)\mathbb{M}x(t) + 2x^T(t)\mathbb{L}u(t) + u^T(t)Ru(t)]dt \tag{7}$$

where $x(t) = \begin{pmatrix} x_0^T(t) & x_1^T(t) & x_2^T(t) \end{pmatrix}^T$, $x_0 = \begin{pmatrix} x_{00}^T & x_{10}^T & x_{20}^T \end{pmatrix}^T$.

From (6) and (7) one sees that for each fixed value of $\varepsilon > 0$ the optimal control problem stated before is a standard stochastic LQ problem which was investigated in [16] (see also Chapter 6 from [18] for a more general setting). Applying the results derived in the afore mentioned references, it follows that the optimal control in the optimization problem described by (6) and (7) and the class of admissible controls $\mathcal{U}_{adm}(x_0)$ is

$$\tilde{u}(t) = \tilde{\mathbb{F}}\tilde{x}(t) \tag{8}$$

where

$$\tilde{\mathbb{F}} = -(R + \mathbb{D}^T\tilde{X}\mathbb{D})^{-1}(\mathbb{B}^T\tilde{X} + \mathbb{D}^T\tilde{X}\mathbb{C} + \mathbb{L}^T) \tag{9}$$

$\tilde{X}$ being the unique stabilizing solution of the algebraic Riccati equation of stochastic control SARE

$$\begin{aligned}\mathbb{A}^T X + X\mathbb{A} + \mathbb{C}^T X\mathbb{C} + \mathbb{M} - (X\mathbb{B} + \mathbb{C}^T X\mathbb{D} + \mathbb{L})(R + \\ + \mathbb{D}^T X\mathbb{D})^{-1}(X\mathbb{B} + \mathbb{C}^T X\mathbb{D} + \mathbb{L})^T = 0\end{aligned} \tag{10}$$

satisfying the sign condition

$$R + \mathbb{D}^T X\mathbb{D} > 0. \tag{11}$$

In [16] a method based on a semidefinite programming (SDP) for numerical computation of the stabilizing solution of SARE which satisfies (11) was proposed, while in [17, 18], an iterative procedure for computing of this solution was derived.

From (5) one sees that the presence of the small parameters ε_k in the coefficients of the system (6) favors the appearance of the stiffness phenomenon which leads to ill conditioning of the numerical computations performed to obtain the stabilizing solution $\tilde{X}$ in (10). On the other hand, in many applications the precise value of some of the small parameters ε_k involved in the mathematical model of the regulated phenomenon are not exactly known. That is why, in the case of Riccati equations of type (10) with the structure of the coefficients given in (5), is preferable to be done a detailed study of the dependence of the stabilizing solution $\tilde{X}$ of (10) with respect to the small parameters $\varepsilon_1, \varepsilon_2$.

The goal of this work is to perform such an investigation of the asymptotic behavior of the stabilizing solution $\tilde{X}$ of SARE (10) satisfying the sign condition (11) when the small parameters $\varepsilon_k, k = 1, 2$ satisfy the condition (3). This study allows us to point out the dominant part of the stabilizing solution $\tilde{X}$ as well as the dominant part of the optimal feedback gain $\tilde{\mathbb{F}}$ given in (9). Based on the dominant part of the optimal feedback gain, we shall construct a suboptimal control $u_{app}(t) = \mathbb{F}_{app}x(t)$ which stabilizes the given system (1) and achieves a near optimal value $J(u_{app})$ of the cost (2). We shall provide an estimation of the deviation of this suboptimal value of the cost from the optimal value $J(\tilde{u})$. In the deterministic case, that is $C_{kj} = 0$, $D_k = 0$, $0 \le k, j \le 2$, an analogous study was done in [1, 2, 3, 4] in the case of system with only one fast time scale and in [7, 8] in the case of system with several fast time scales.

3 The derivation of the reduced coupled algebraic Riccati equations

Following the approach from [7] we shall investigate the asymptotic behavior of the solution $(X, \mathbb{F})$ of the following Lurie-Yakubovich-Popov type system of stochastic control:

$$\begin{aligned} &\mathbb{B}^T X + \mathbb{D}^T X \mathbb{C} + \mathbb{L}^T = -(R + \mathbb{D}^T X \mathbb{D})\mathbb{F} \\ &\mathbb{A}^T X + X\mathbb{A} + \mathbb{C}^T X \mathbb{C} + \mathbb{M} - \mathbb{F}^T (R + \mathbb{D}^T X \mathbb{D})\mathbb{F} = 0 \end{aligned} \tag{12}$$

Proceeding in this way, we obtain simultaneously both the asymptotic structure of the stabilizing solution of SARE (10)-(11) as well as the asymptotic structure of the optimal stabilizing feedback gain (9).

We take $X = \begin{pmatrix} X_{00} & \varepsilon_1 X_{01} & \varepsilon_2 X_{02} \\ \varepsilon_1 X_{01}^T & \varepsilon_1 X_{11} & \varepsilon_2 X_{12} \\ \varepsilon_2 X_{02}^T & \varepsilon_2 X_{12}^T & \varepsilon_2 X_{22} \end{pmatrix}$ and $\mathbb{F} = (F_0 \quad F_1 \quad F_2)$, $X_{ij} \in \mathbb{R}^{n_i \times n_j}$, $X_{jj} = X_{jj}^T$, $F_j \in \mathbb{R}^{m \times n_j}$, $\ i, j = 0, 1, 2$.
Using (5) we obtain the following partition of the system (12):

$$\begin{aligned} &B_0^T X_{00} + B_1^T X_{01}^T + B_2^T X_{02}^T + D_0^T (X_{00} C_{00} + \varepsilon_1^\delta X_{01} C_{10} + \\ &+\varepsilon_2^\delta X_{02} C_{20}) + D_1^T (\varepsilon_1^\delta X_{01}^T C_{00} + \varepsilon_1^{2\delta-1} X_{11} C_{10} + \\ &+(\frac{\varepsilon_2}{\varepsilon_1})^\delta \varepsilon_1^{2\delta-1} X_{12} C_{20}) + D_2^T (\varepsilon_2^\delta X_{02}^T C_{00} + (\frac{\varepsilon_2}{\varepsilon_1})^\delta \varepsilon_1^{2\delta-1} X_{12}^T C_{10} + \\ &+\varepsilon_2^{2\delta-1} X_{22} C_{20}) + L_0^T = -(R + \mathbb{D}^T X \mathbb{D}) F_0 \end{aligned}$$

$$\begin{aligned} &\varepsilon_1 B_0^T X_{01} + B_1^T X_{11} + B_2^T X_{12}^T + D_0^T (X_{00} C_{01} + \varepsilon_1^\delta X_{01} C_{11} + \varepsilon_2^\delta X_{02} C_{21}) + \\ &+D_1^T (\varepsilon_1^\delta X_{01}^T C_{01} + \varepsilon_1^{2\delta-1} X_{11} C_{11} + (\varepsilon_2/\varepsilon_1)^\delta \varepsilon_1^{2\delta-1} X_{12} C_{21}) + \\ &+D_2^T (\varepsilon_2^\delta X_{02}^T C_{01} + (\varepsilon_2/\varepsilon_1)^\delta \varepsilon_1^{2\delta-1} X_{12}^T C_{11} + \\ &+\varepsilon_2^{2\delta-1} X_{22} C_{21}) + L_1^T = -(R + \mathbb{D}^T X \mathbb{D}) F_1 \end{aligned}$$

$$
\begin{aligned}
&\varepsilon_2 B_0^T X_{02} + (\varepsilon_2/\varepsilon_1) B_1^T X_{12} + B_2^T X_{22} + D_0^T (X_{00} C_{02} + \varepsilon_1^\delta X_{01} C_{12} + \\
&+\varepsilon_2^\delta X_{02} C_{22}) + D_1^T (\varepsilon_1^\delta X_{01}^T C_{02} + \varepsilon_1^{2\delta-1} X_{11} C_{12} + \\
&+(\varepsilon_2/\varepsilon_1)^\delta \varepsilon_1^{2\delta-1} X_{12} C_{22}) + D_2^T (\varepsilon_2^\delta X_{02}^T C_{02} + \\
&+(\varepsilon_2/\varepsilon_1)^\delta \varepsilon_1^{2\delta-1} X_{12}^T C_{12} + \varepsilon_2^{2\delta-1} X_{22} C_{22}) + L_2^T = -(R + \mathbb{D}^T X \mathbb{D}) F_2 \\
&A_{00}^T X_{00} + A_{10}^T X_{01}^T + A_{20}^T X_{02}^T + X_{00} A_{00} + X_{01} A_{10} + X_{02} A_{20} + \\
&+[\mathbb{C}^T X \mathbb{C}]_{00} - F_0^T (R + \mathbb{D}^T X \mathbb{D}) F_0 + M_{00} = 0 \\
&\varepsilon_1 A_{00}^T X_{01} + A_{10}^T X_{11} + A_{20}^T X_{12}^T + X_{00} A_{01} + X_{01} A_{11} + X_{02} A_{21} + \\
&+[\mathbb{C}^T X \mathbb{C}]_{01} - F_0^T (R + \mathbb{D}^T X \mathbb{D}) F_1 + M_{01} = 0 \qquad (13) \\
&\varepsilon_2 A_{00}^T X_{02} + (\varepsilon_2/\varepsilon_1) A_{10}^T X_{12} + A_{20}^T X_{22} + X_{00} A_{02} + \\
&+X_{01} A_{12} + X_{02} A_{22} + [\mathbb{C}^T X \mathbb{C}]_{02} - F_0^T (R + \mathbb{D}^T X \mathbb{D}) F_2 + M_{02} = 0 \\
&\varepsilon_1 A_{01}^T X_{01} + A_{11}^T X_{11} + A_{21}^T X_{12}^T + \varepsilon_1 X_{01}^T A_{01} + X_{11} A_{11} + \\
&+X_{12} A_{21} + [\mathbb{C}^T X \mathbb{C}]_{11} - F_1^T (R + \mathbb{D}^T X \mathbb{D}) F_1 + M_{11} = 0 \\
&\varepsilon_2 A_{01}^T X_{02} + (\varepsilon_2/\varepsilon_1) A_{11}^T X_{12} + A_{21}^T X_{22} + \varepsilon_1 X_{01}^T A_{02} + \\
&+X_{11} A_{12} + X_{12} A_{22} + [\mathbb{C}^T X \mathbb{C}]_{12} - F_1^T (R + \mathbb{D}^T X \mathbb{D}) F_2 + M_{12} = 0 \\
&\varepsilon_2 A_{02}^T X_{02} + (\varepsilon_2/\varepsilon_1) A_{12}^T X_{12} + A_{22}^T X_{22} + \varepsilon_2 X_{02}^T A_{02} + \\
&+\varepsilon_2/\varepsilon_1 X_{12}^T A_{12} + X_{22} A_{22} + [\mathbb{C}^T X \mathbb{C}]_{22} - F_2^T (R + \mathbb{D}^T X \mathbb{D}) F_2 + M_{22} = 0
\end{aligned}
$$

where $[\mathbb{C}^T X \mathbb{C}]_{ij}$ is the ij-block of the matrix $\mathbb{C}^T X \mathbb{C}$, $0 \le i, j \le 2$. Setting formally $\varepsilon = 0$ in (13) and taking into account (3) and $\delta > \frac{1}{2}$ we obtain:

$$
\begin{aligned}
&B_0^T X_{00} + B_1^T X_{01}^T + B_2^T X_{02}^T + D_0^T X_{00} C_{00} + L_0^T = -(R + D_0^T X_{00} D_0) F_0 \\
&B_1^T X_{11} + B_2^T X_{12}^T + D_0^T X_{00} C_{01} + L_1^T = -(R + D_0^T X_{00} D_0) F_1 \\
&B_2^T X_{22} + D_0^T X_{00} C_{02} + L_2^T = -(R + D_0^T X_{00} D_0) F_2
\end{aligned}
$$

$$
\begin{aligned}
A_{00}^T X_{00} + A_{10}^T X_{01}^T + A_{20}^T X_{02}^T + X_{00} A_{00} + X_{01} A_{10} + X_{02} A_{20} + & \\
+ C_{00}^T X_{00} C_{00} - F_0^T (R + D_0^T X_{00} D_0) F_0 + M_{00} &= 0 \\
A_{10}^T X_{11} + A_{20}^T X_{12}^T + X_{00} A_{01} + X_{01} A_{11} + X_{02} A_{22} + & \\
+ C_{00}^T X_{00} C_{01} - F_0^T (R + D_0^T X_{00} D_0) F_1 + M_{01} &= 0 \\
A_{20}^T X_{22} + X_{00} A_{02} + X_{01} A_{12} + X_{02} A_{22} + & \qquad (14) \\
+ C_{00}^T X_{00} C_{02} - F_0^T (R + D_0^T X_{00} D_0) F_2 + M_{02} &= 0 \\
A_{11}^T X_{11} + A_{21}^T X_{12}^T + X_{11} A_{11} + X_{12} X_{21} + & \\
+ C_{01}^T X_{00} C_{01} - F_1^T (R + D_0^T X_{00} D_0) F_1 + M_{11} &= 0 \\
A_{21}^T X_{12} + X_{11} A_{12} + X_{12} A_{22} + & \\
+ C_{01}^T X_{00} C_{02} - F_1^T (R + D_0^T X_{00} D_0) F_2 + M_{12} &= 0 \\
A_{22}^T X_{22} + X_{22} A_{22} + C_{02}^T X_{00} C_{02} - F_2^T (R + D_0^T X_{00} D_0) F_2 + M_{22} &= 0.
\end{aligned}
$$

Assuming that A_{22} is invertible, we may introduce the notations:

$$\begin{aligned} A_{ij}^1 &= A_{ij} - A_{i2}A_{22}^{-1}A_{2j} \\ B_j^1 &= B_j - A_{j2}A_{22}^{-1}B_2, \quad i,j=0,1 \end{aligned} \tag{15}$$

$$\begin{aligned} C_{0k}^1 &= C_{0k} - C_{02}A_{22}^{-1}A_{2k} \\ L_k^1 &= L_k - A_{2k}^T A_{22}^{-1}L_2 - (M_{k2} - A_{2k}^T A_{22}^{-1}M_{22})A_{22}^{-1}B_2, \quad k=0,1 \end{aligned} \tag{16}$$

$$\begin{aligned} R^1 &= R - L_2^T A_{22}^{-1}B_2 - B_2^T A_{22}^{-T}L_2 + B_2^T A_{22}^{-T}M_{22}A_{21}^{-1}B_2 \\ D_0^1 &= D_0 - C_{02}A_{22}^{-1}B_2 \end{aligned} \tag{17}$$

$$\begin{pmatrix} M_{00}^1 & M_{01}^1 \\ (M_{01}^1)^T & M_{11}^1 \end{pmatrix} = \tag{18}$$

$$\begin{pmatrix} I_{n_0} & 0 \\ 0 & I_{n_1} \\ H_{20} & H_{21} \end{pmatrix}^T \begin{pmatrix} M_{00} & M_{01} & M_{02} \\ M_{01}^T & M_{11} & M_{12} \\ M_{02}^T & M_{12}^T & M_{22} \end{pmatrix} \begin{pmatrix} I_{n_0} & 0 \\ 0 & I_{n_1} \\ H_{20} & H_{21} \end{pmatrix}$$

where $H_{2j} = -A_{22}^{-1}A_{2j}$, $j=0,1$.

The next result allows us to reduce the number of equations and the number of unknowns of the systems (14).

Lemma 1. *If A_{22} is invertible, the following hold:*

(i) If $(X_{00}, X_{01}, X_{11}, X_{02}, X_{12}, X_{22}, F_0, F_1, F_2)$ is a solution of the system (14) with the property that the matrix $A_{22} + B_2F_2$ is invertible, then $(X_{00}, X_{01}, X_{11}, X_{22}, F_0^1, F_1^1, F_2)$ is a solution of the following system:

$$\begin{aligned} &(B_0^1)^T X_{00} + (B_1^1)^T X_{01}^T + (D_0^1)^T X_{00} C_{00}^1 + (L_0^1)^T = -(R^1 + (D_0^1)^T X_{00} D_0^1) F_0^1 \\ &(B_1^1)^T X_{11} + (D_0^1)^T X_{00} C_{01}^1 + (L_1^1)^T = -(R^1 + (D_0^1)^T X_{00} D_0^1) F_1^1 \\ &B_2^T X_{22} + D_0^T X_{00} C_{02} + L_2^T = -(R + D_0^T X_{00} D_0) F_2 \end{aligned}$$

$$\begin{aligned} (A_{00}^1)^T X_{00} + (A_{10}^1)^T X_{01}^T + X_{00}A_{00}^1 + X_{01}A_{10}^1 + & \\ +(C_{00}^1)^T X_{00} C_{00}^1 - (F_0^1)^T (R^1 + (D_0^1)^T X_{00} D_0^1) F_0^1 + M_{00}^1 &= 0 \\ (A_{10}^1)^T X_{11} + X_{00}A_{01}^1 + X_{01}A_{11}^1 + & \\ +(C_{00}^1)^T X_{00} C_{01}^1 - (F_0^1)^T (R^1 + (D_0^1)^T X_{00} D_0^1) F_1^1 + M_{01}^1 &= 0 \\ (A_{11}^1)^T X_{11} + X_{11}A_{11}^1 + & \\ +(C_{01}^1)^T X_{00} C_{01}^1 - (F_1^1)^T (R^1 + (D_0^1)^T X_{00} D_0^1) F_1^1 + M_{11}^1 &= 0 \\ A_{22}^T X_{22} + X_{22}A_{22} + C_{02}^T X_{00} C_{02} - F_2^T (R + D_0^T X_{00} D_0) F_2 + M_{22} &= 0. \end{aligned} \tag{19}$$

where $A^1_{ij}, B^1_j, C^1_{0k}, L^1_k, R^1, D^1_0, M^1_{ij}$ *are defined in (15)-(18) and*

$$F^1_j = (I_m + F_2A_{22}^{-1}B_2)^{-1}(F_j - F_2A_{22}^{-1}A_{2j}), \quad j = 0, 1.$$

(ii) If $(X_{00}, X_{01}, X_{11}, X_{22}, F^1_0, F^1_1, F_2)$ *is a solution of the system (19) with the property that* $A_{22}+B_2F_2$ *is invertible, then* $(X_{00}, X_{01}, X_{11}, X_{02}, X_{12}, X_{22}, F_0, F_1, F_2)$ *is a solution of the system (14) if*

$$F_j = (I_m + F_2A_{22}^{-1}B_2)F^1_j + F_2A_{22}^{-1}A_{2j}, \quad j = 0, 1 \tag{20}$$

and

$$\begin{aligned} X_{02} &= -[A^T_{20}X_{22} + X_{00}A_{02} + X_{01}A_{11} + C^T_{00}X_{00}C_{02} \\ &\quad -F^T_0(R + D^T_0X_{00}D_0)F_2 + M_{02}]A_{22}^{-1} \end{aligned} \tag{21}$$

$$\begin{aligned} X_{12} &= -[A^T_{21}X_{22} + X_{11}A_{12} + C^T_{01}X_{00}C_{02} \\ &\quad -F^T_1(R + D^T_0X_{00}D_0)F_2 + M_{12}]A_{22}^{-1}. \end{aligned} \tag{22}$$

Proof is done by direct calculation. The details are omitted.

We assume now that A^1_{11} is invertible and introduce the new notations:

$$\begin{aligned} A^0_{00} &= A^1_{00} - A^1_{01}(A^1_{11})^{-1}A^1_{10} \\ C^0_{00} &= C^1_{00} - C^1_{01}(A^1_{11})^{-1}A^1_{10} \\ B^0_0 &= B^1_0 - A^1_{01}(A^1_{11})^{-1}B^1_1 \\ D^0_0 &= D^1_0 - C^1_{01}(A^1_{11})^{-1}B^1_1 \end{aligned} \tag{23}$$

$$\begin{aligned} L^0_0 &= L^1_0 - (A^1_{10})^T(A^1_{11})^{-T}L^1_1 - (M^1_{01} - (A^1_{10})^T(A^1_{11})^{-T}M^1_{11})(A^1_{11})^{-1}B^1_1 \\ R^0 &= R^1 - (B^1_1)^T(A^1_{11})^{-T}L^1_1 - (L^1_1)^T(A^1_{11})^{-1}B^1_1 \\ &\quad +(B^1_1)^T(A^1_{11})^{-T}M^1_{11}(A^1_{11})^{-1}B^1_1 \\ M^0 &= \begin{pmatrix} I_{n_0} \\ -(A^1_{11})^{-1}A^1_{10} \end{pmatrix}^T \begin{pmatrix} M^1_{00} & M^1_{01} \\ (M^1_{01})^T & M^1_{11} \end{pmatrix} \begin{pmatrix} I_{n_0} \\ -(A^1_{11})^{-1}A^1_{10} \end{pmatrix}. \end{aligned} \tag{24}$$

The next result allows us to reduce the number of equations and unknowns of the system (19).

Lemma 2. *Assume that A_{11}^1 is invertible, then the following hold:*

(i) If $(X_{00}, X_{01}, X_{11}, X_{22}, F_0^1, F_1^1, F_2)$ is a solution of the system (19) with the property that $A_{11}^1 + B_1^1 F_1^1$ is invertible, then $(X_{00}, X_{11}, X_{22}, F_0^0, F_1^1, F_2)$ is a solution of the system:

$$
\begin{aligned}
&(B_0^0)^T X_{00} + (D_0^0)^T X_{00} C_{00}^0 + L_0^0 = -(R^0 + (D_0^0)^T X_{00} D_0^0) F_0^0 \\
&(B_1^1)^T X_{11} + (D_0^1)^T X_{00} C_{01}^1 + (L_1^1)^T = -(R^1 + (D_0^1)^T X_{00} D_0^1) F_1^1 \\
&B_2^T X_{22} + D_0^T X_{00} C_{02} + L_2^T = -(R + D_0^T X_{00} D_0) F_2
\end{aligned}
$$

$$
\begin{array}{rcl}
(A_{00}^0)^T X_{00} + X_{00} A_{00}^0 + & & \\
+(C_{00}^0)^T X_{00} C_{00}^0 - (F_0^0)^T (R^0 + (D_0^0)^T X_{00} D_0^0) F_0^0 + M^0 & = & 0 \\
(A_{11}^1)^T X_{11} + X_{11} A_{11^1} + & & \\
+(C_{01}^1)^T X_{00} C_{01}^1 - (F_1^1)^T (R^1 + (D_0^1)^T X_{00} D_0^1) F_1^1 + M_{11}^1 & = & 0 \\
A_{22}^T X_{22} + X_{22} A_{22} + & & \\
+(C_{02})^T X_{00} C_{02} - F_2^T (R + D_0^T X_{00} D_0) F_2 + M_{22} & = & 0
\end{array}
\qquad (25)
$$

where $A_{00}^0, B_0^0, C_0^0, D_0^0, L_0^0, R^0, M^0$ are defined as in (23)-(24) and

$$F_0^0 = (I_m + F_1^1 (A_{11}^1)^{-1} B_1^1)^{-1} (F_0^1 - F_1^1 (A_{11}^1)^{-1} A_{10}^1).$$

(ii) If $(X_{00}, X_{11}, X_{22}, F_0^0, F_1^1, F_2)$ is a solution of the system (25) with the property that $A_{11}^1 + B_1^1 F_1^1$ is an invertible matrix, then $(X_{00}, X_{01}, X_{11}, X_{22}, F_0^1, F_1^1, F_2)$ is a solution of the system (19) if

$$F_0^1 = (I_m + F_1^1 (A_{11}^1)^{-1} B_1^1) F_0^0 + F_1^1 (A_{11}^1)^{-1} A_{10}^1 \qquad (26)$$

and

$$
\begin{aligned}
X_{01} \;=\; & -[(A_{10}^1)^T X_{11} + X_{00} A_{01}^1 + (C_{00}^1)^T X_{00} C_{01}^1 \\
& -(F_0^1)^T (R^1 + (D_0^1)^T X_{00} D_0^1) F_1^1 + M_{01}^1](A_{11}^1)^{-1}. \qquad (27)
\end{aligned}
$$

The proof may be done by a laborious calculation.

Assuming that X_{00} are such that the matrices $R^0 + (D_0^0)^T X_{00} D_0^0$, $R^1 + (D_0^1)^T X_{00} D_0^1$, $R + D_0^T X_{00} D_0$ are invertible we may eliminate the unknowns F_0^0, F_1^1, F_2 from (25) and obtain the following system of nonlinear equations with the unknowns X_{00}, X_{11}, X_{22}:

$$
\begin{array}{rcl}
(A_{00}^0)^T X_{00} + X_{00}A_{00}^0 + (C_{00}^0)^T X_{00}C_{00}^0 - & & \\
-(X_{00}B_0^0 + (C_{00}^0)^T X_{00}D_0^0 + L_0^0)(R^0 + (D_0^0)^T X_{00}D_0^0)^{-1}\times & & \\
\times((B_0^0)^T X_{00} + (D_0^0)^T X_{00}C_{00}^0 + (L_0^0)^T) + M^0 & = & 0 \\
(A_{11}^1)^T X_{11} + X_{11}A_{11}^1 + (C_{01}^1)^T X_{00}C_{01}^1 - & & \\
-(X_{11}B_1^1 + (C_{01}^1)^T X_{00}D_0^1 + L_1^1)(R^1 + (D_0^1)^T X_{00}D_0^1)^{-1}\times & & \\
\times((B_1^1)^T X_{11} + (D_0^1)^T X_{00}C_{01}^1 + (L_1^1)^T) + M_{11}^1 & = & 0 \\
A_{22}^T X_{22} + X_{22}A_{22} + C_{02}^T X_{00}C_{02} - & & \\
-(X_{22}B_2 + C_{02}^T X_{00}D_0 + L_2)(R + D_0^T X_{00}D_0)^{-1}\times & & \\
\times(B_2^T X_{22} + D_0^T X_{00}C_{02} + L_2^T) + M_{22} & = & 0.
\end{array}
\tag{28}
$$

In the special case $C_{jk} = 0$, $D_k = 0$, $0 \leq j,k \leq 2$, the system of nonlinear equations (28) reduces to three uncoupled algebraic Riccati equations arising in the investigations of the asymptotic structure of the stabilizing solution of an algebraic Riccati equation associated to a linear quadratic optimization problem for a deterministic controlled system with several time scales (see e.q. [7]).

In the stochastic context considered on this work, the system of nonlinear equation (28) will play the same role that, in the deterministic case, is played by the algebraic Riccati equations of lower dimension obtained neglecting the small parameters. That is why, in the following, the system (28) will be called **the system of coupled reduced algebraic Riccati equations** (SCRARE).

In the next section we shall introduce the concept of stabilizing solution of the system (28) and we shall provide a set of necessary and sufficient conditions for the existence of the stabilizing solution.

4 Stabilizing solution of SCRARE (28).

Let $\mathcal{S}_{n_k} \subset \mathbb{R}^{n_k \times n_k}$ be the linear space of symmetric matrices of size $n_k \times n_k$, $0 \leq k \leq 2$. We set $\mathfrak{X} = \mathcal{S}_{n_0} \times \mathcal{S}_{n_1} \times \mathcal{S}_{n_2}$. Hence, $\mathbf{X} \in \mathfrak{X}$ if and only if $\mathbf{X} = (X_0, X_1, X_2)$. $\mathfrak{X}$ is a real Hilbert space with respect to the inner product:

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{j=0}^{2} Tr[X_j Y_j] \tag{29}$$

for all $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}$. On $\mathfrak{X}$ we consider the ordering relation "$\geq$" induced by the convex closed cone with not empty interior

$$\mathfrak{X}_+ = \{\mathbf{X} \in \mathfrak{X} | \mathbf{X} = (X_0, X_1, X_2),\ X_j \,\text{positive semidefinite matrix}, j = 0, 1, 2\}$$

The system of nonlinear equations (28) can be regarded as a generalized algebraic Riccati equation on $\mathfrak{X}$ of the form:

$$\begin{aligned} \mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} + \mathbf{\Pi}_1[\mathbf{X}] - (\mathbf{X}\mathbf{B} + \\ +\mathbf{\Pi}_2[\mathbf{X}] + \mathbf{L})(\mathbf{R} + \mathbf{\Pi}_3[\mathbf{X}])^{-1}(\mathbf{X}\mathbf{B} + \mathbf{\Pi}_2[\mathbf{X}] + \mathbf{L})^T + \mathbf{M} &= 0 \end{aligned} \tag{30}$$

where

$$\mathbf{A} = (A_{00}^0, A_{11}^1, A_{22}) \in \mathbb{R}^{n_0 \times n_0} \times \mathbb{R}^{n_1 \times n_1} \times \mathbb{R}^{n_2 \times n_2}$$
$$\mathbf{B} = (B_0^0, B_1^1, B_2),\ \mathbf{L} = (L_0^0, L_1^1, L_2), \mathbf{B}, \mathbf{L} \in \mathbb{R}^{n_0 \times m} \times \mathbb{R}^{n_1 \times m} \times \mathbb{R}^{n_2 \times m}$$
$$\mathbf{M} = (M^0, M_{11}^1, M_{22}) \in \mathfrak{X},$$
$$\mathbf{R} = (R^0, R^1, R) \in \mathcal{S}_m \times \mathcal{S}_m \times \mathcal{S}_m,$$
$$\mathbf{\Pi}_1[\mathbf{X}] = ((C_0^0)^T X_0 C_0^0, (C_{01}^1)^T X_0 C_{01}^1, C_{02}^T X_0 C_{02})$$
$$\mathbf{\Pi}_2[\mathbf{X}] = ((C_{00}^0)^T X_0 D_0^0, (C_{01}^1)^T X_0 D_0^1, C_{02}^T X_0 D_0)$$
$$\mathbf{\Pi}_3[\mathbf{X}] = ((D_0^0)^T X_0 D_0^0, (D_0^1)^T X_0 D_0^1, D_0^T X_0 D_0)$$

for all $\mathbf{X} = (X_0, X_1, X_2) \in \mathfrak{X}$.

We set $\mathbf{\Pi}[\mathbf{X}] = \begin{pmatrix} \mathbf{\Pi}_1[\mathbf{X}] & \mathbf{\Pi}_2[\mathbf{X}] \\ \mathbf{\Pi}_2^T[\mathbf{X}] & \mathbf{\Pi}_3[\mathbf{X}] \end{pmatrix}$. To the triple $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$ and the feedback gain $\mathbf{F} = (F_0, F_1, F_2) \in \mathbb{R}^{m \times n_0} \times \mathbb{R}^{m \times n_1} \times \mathbb{R}^{m \times n_2}$ we associate the linear operator $\mathbb{L}_F : \mathfrak{X} \to \mathfrak{X}$ by

$$\mathbb{L}_F[\mathbf{X}] = (\mathbb{L}_{F0}[\mathbf{X}], \mathbb{L}_{F1}[\mathbf{X}], \mathbb{L}_{F2}[\mathbf{X}])$$

with

$$\begin{aligned} \mathbb{L}_{F0}[\mathbf{X}] &= (A_{00}^0 + B_0^0 F_0)^T X_0 + X_0 (A_{00}^0 + B_0^0 F_0) \\ &\quad + (C_{00}^0 + D_0^0 F_0)^T X_0 (C_{00}^0 + D_0^0 F_0) \\ \mathbb{L}_{F1}[\mathbf{X}] &= (A_{11}^1 + B_1^1 F_1)^T X_1 + X_1 (A_{11}^1 + B_1^1 F_1) \\ &\quad + (C_{01}^1 + D_0^1 F_1)^T X_0 (C_{01}^1 + D_0 F_1) \\ \mathbb{L}_{F2}[\mathbf{X}] &= (A_{22} + B_2 F_2)^T X_2 + X_2 (A_{22} + B_2 F_2) \\ &\quad + (C_{02} + D_0 F_2)^T X_0 (C_{02} + D_0 F_2) \end{aligned} \tag{31}$$

for all $\mathbf{X} = (X_0, X_1, X_2)$. Several properties of the operator of type (31) are summarized in the following proposition:

Proposition 1. *(i) For each feedback gain the corresponding operator* $\mathbb{L}_F$ *generates a positive evolution on* $\mathfrak{X}$*, that is:* $e^{\mathbb{L}_F t}\mathfrak{X}_+ \subset \mathfrak{X}_+$ *for all* $t \geq 0$*.*

(ii) The spectrum of the operator $\mathbb{L}_F$ *is in the half plane* $\mathbb{C}_- = \{\lambda \in \mathbb{C}, Re\lambda < 0\}$ *if and only if, there exists* $\hat{\mathbf{X}} \in Int\mathfrak{X}_+$ *such that* $\mathbb{L}_F[\hat{\mathbf{X}}] < 0$*.*

Now we are in a position to introduce the concept of stabilizing solution of SCRARE (28).

Definition 1. *A solution* $\tilde{\mathbf{X}} = (\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ *of (28) is called "***stabilizing solution***" if the spectrum of the linear operator* $\mathbb{L}_{\tilde{F}}$ *is located in the half plane* $\mathbb{C}_-$*,* $\mathbb{L}_{\tilde{F}}$ *being the linear operator of type (31) defined for* $\tilde{F} = (\tilde{F}_0, \tilde{F}_1, \tilde{F}_2)$ *where*

$$\begin{aligned}
\tilde{F}_0 &= -(R^0 + (D_0^0)^T \tilde{X}_0 D_0^0)^{-1}((B_0^0)^T \tilde{X}_0 + (D_0^0)^T \tilde{X}_0 C_{00}^0 + (L_0^0)^T) \\
\tilde{F}_1 &= -(R^1 + (D_0^1)^T \tilde{X}_0 D_0^1)^{-1}((B_1^1)^T \tilde{X}_1 + (D_0^1)^T \tilde{X}_0 C_{01}^1 + (L_1^1)^T) \\
\tilde{F}_2 &= -(R + D_0^T \tilde{X}_0 D_0)^{-1}(B_2^T \tilde{X}_2 + D_0^T \tilde{X}_0 C_{02} + L_2^T)
\end{aligned} \tag{32}$$

Before to state the result providing the conditions for the existence of the stabilizing solution of SCRARE (28) we introduce the concept of stabilizability of the triple $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$.

Definition 2. *We say that the triple* $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$ *is stabilizable if there exists a feedback gain* $F = (F_0, F_1, F_2)$ *with the property that the spectrum of the corresponding linear operator* $\mathbb{L}_F$ *is located in the half plane* $\mathbb{C}_-$*.*

Necessary and sufficient conditions for the stabilizability of the triple $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$ can be derived employing part (ii) of Proposition 1 combined with Schur complement technique.

The next result provides a set of conditions equivalent to the existence of the stabilizing solution $\tilde{\mathbf{X}} = (\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ of SCRARE (28) satisfying the sign conditions:

$$\begin{aligned}
R^0 + (D_0^0)^T \tilde{X}_0 D_0^0 &> 0 \\
R^1 + (D_0^1)^T \tilde{X}_0 D_0^1 &> 0 \\
R + D_0^T \tilde{X}_0 D_0 &> 0.
\end{aligned} \tag{33}$$

Theorem 1. *The following are equivalent:*

(i) the SCRARE (28) has a unique stabilizing solution $\tilde{\mathbf{X}} = (\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ *satisfying the sign conditions (33);*

(ii) the triple $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$ *is stabilizable and there exists* $\mathbf{Y} = (Y_0, Y_1, Y_2) \in \mathfrak{X}$ *solving the following system of LMIs:*

$$\begin{pmatrix} (A_0^0)^T Y_0 + Y_0 A_0^0 + (C_{00}^0)^T Y_0 C_{00}^0 + M^0 & Y_0 B_0^0 + (C_{00}^0)^T Y_0 D_0^0 + L_0^0 \\ (B_0^0)^T Y_0 + (D_0^0)^T Y_0 C_{00}^0 + (L_0^0)^T & R^0 + (D_0^0)^T Y_0 D_0^0 \end{pmatrix} > 0$$

$$\begin{pmatrix} (A_{11}^1)^T Y_1 + Y_1 A_{11}^1 + (C_{01}^1)^T Y_0 C_{01}^1 + M_{11}^1 & Y_1 B_1^1 + (C_{01}^1)^T Y_0 D_0^1 + L_1^1 \\ (B_1^1)^T Y_1 + (D_0^1)^T Y_0 C_{01}^1 + (L_1^1)^T & R^1 + (D_0^1)^T Y_0 D_0^1 \end{pmatrix} > 0$$

$$\begin{pmatrix} A_{22}^T Y_2 + Y_2 A_{22} + C_{02}^T Y_0 C_{02} + M_{22} & Y_2 B_2 + C_{02}^T Y_0 D_0 + L_2 \\ B_2^T Y_2 + D_0^T Y_0 C_{02} + L_2^T & R + D_0^T Y_0 D_0 \end{pmatrix} > 0.$$

The proof can be done adapting to the special case of equation (30) the proof of Theorem 5.4.6 from [18].

5 Main results

5.1 The asymptotic structure of the stabilizing solution of SARE (10)-(11).

Let us assume that conditions from Theorem 1 (ii) are fulfilled. Then SCRARE (28) has a unique stabilizing solution $\tilde{\mathbf{X}} = (\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ satisfying the sign conditions (33).

Let $(\tilde{F}_0, \tilde{F}_1, \tilde{F}_2)$ be the corresponding feedback gains defined in (32). Employing Proposition 1 (ii) in the case of operator $\mathbb{L}_{\tilde{F}}$ we may infer that the matrices $A_{jj}^j + B_j^j \tilde{F}_j$ are Hurwitz matrices $0 \le j \le 2$, with $A_{22}^2 = A_{22}$ and $B_2^2 = B_2$. Taking $\tilde{F}_1^1 \triangleq \tilde{F}_1$ we compute $\tilde{F}_0^1$ by

$$\tilde{F}_0^1 = (I_m + \tilde{F}_1^1 (A_{11}^1)^{-1} B_1^1) \tilde{F}_0 + \tilde{F}_1^1 (A_{11}^1)^{-1} A_{10}^1. \tag{34}$$

Further, we compute $\tilde{\tilde{F}}_j$ by

$$\tilde{\tilde{F}}_j = (I_m + \tilde{F}_2 A_{22}^{-1} B_2) \tilde{F}_j^1 + \tilde{F}_2 A_{22}^{-1} A_{2j}, \ \ j = 0, 1 \tag{35}$$

and

$$\tilde{\tilde{F}}_2 \triangleq \tilde{F}_2. \tag{36}$$

We compute $\tilde{X}_{01}$ by

$$\begin{aligned} \tilde{X}_{01} \ = \ & -[(A_{10}^1)^T \tilde{X}_1 + \tilde{X}_0 A_{01}^1 + (C_{00}^1)^T \tilde{X}_0 C_{01}^1 \\ & -(\tilde{F}_0^1)^T (R^1 + (D_0^1)^T \tilde{X}_0 D_0^1) \tilde{F}_1^1 + M_{01}^1](A_{11}^1)^{-1}. \end{aligned} \tag{37}$$

Then, we compute $\tilde{X}_{02}$, $\tilde{X}_{12}$ by:

$$\begin{aligned}\tilde{X}_{02} &= -[A_{20}^T\tilde{X}_2 + \tilde{X}_0 A_{02} + \tilde{X}_{01}A_{12} + C_{00}^T\tilde{X}_0 C_{02} \\ &\quad -\tilde{\tilde{F}}_0^T(R + D_0^T\tilde{X}_0 D_0)\tilde{\tilde{F}}_2 + M_{02}]A_{22}^{-1} \end{aligned} \tag{38}$$

$$\begin{aligned}\tilde{X}_{12} &= -[A_{21}^T\tilde{X}_2 + \tilde{X}_1 A_{12} + C_{01}^T\tilde{X}_1 A_{12} + C_{01}^T\tilde{X}_0 C_{02} \\ &\quad -\tilde{\tilde{F}}_1^T(R + D_0^T\tilde{X}_0 D_0)\tilde{\tilde{F}}_2 + M_{12}]A_{22}^{-1}. \end{aligned} \tag{39}$$

Applying Lemma 1 (ii), Lemma 2 (ii) together with (34)-(39) we obtain that $(\tilde{X}_0, \tilde{X}_{01}, \tilde{X}_1, \tilde{X}_{02}, \tilde{X}_{12}, \tilde{X}_2, \tilde{\tilde{F}}_0, \tilde{\tilde{F}}_1, \tilde{\tilde{F}}_2)$ is a solution of the system (14) obtained starting from the stabilizing solution of SCRARE (28).

In order to obtain the existence of the stabilizing solution of SARE (10) satisfying the sign condition (11), we shall use the implicit function theorem applied to the system (13). To this end, we regard (13) as an equation of type

$$\Phi(\mathbb{W}, \eta) = 0 \tag{40}$$

on the finite dimensional Banach space $\mathcal{W} = \mathcal{S}_{n_0} \times \mathbb{R}^{n_0\times n_1} \times \mathcal{S}_{n_1} \times \mathbb{R}^{n_0\times n_1} \times \mathbb{R}^{n_0\times n_2} \times \mathcal{S}_{n_2} \times \mathbb{R}^{m\times n_0} \times \mathbb{R}^{m\times n_1} \times \mathbb{R}^{m\times n_2}$.

In (40) η stands for $(\varepsilon_1, \varepsilon_2, \frac{\varepsilon_2}{\varepsilon_1}, \varepsilon_1^\delta, \varepsilon_1^{2\delta-1}, \varepsilon_2^\delta, \varepsilon_2^{2\delta-1}, (\frac{\varepsilon_2}{\varepsilon_1})^\delta)$. One shows that the assumptions of the implicit function theorem are fulfilled for equation (40) around the solution $\tilde{\mathbb{W}} = (\tilde{X}_0, \tilde{X}_{01}, \tilde{X}_1, \tilde{X}_{02}, \tilde{X}_{12}, \tilde{X}_2, \tilde{\tilde{F}}_0, \tilde{\tilde{F}}_1, \tilde{\tilde{F}}_2)$ and $\eta = 0$.

Thus we have obtained the main result of this work:

Theorem 2. *Assume:*

a) the matrices A_{22} and A_{11}^1 are invertible;

b) conditions from (ii) of Theorem 1 are fulfilled.

Then the following hold: there exists $\varepsilon_k^* > 0$, $\rho^* > 0$, with the property that for any $0 < \varepsilon_k < \varepsilon_k^*$, $k = 1, 2$ and $0 < \varepsilon_2/\varepsilon_1 < \rho^*$ the SARE (10) has a stabilizing solution $\tilde{X}(\varepsilon_1, \varepsilon_2)$ satisfying the sign condition (11). Furthermore the solution $\tilde{X}(\varepsilon_1, \varepsilon_2)$ has the asymptotic structure:

$$\tilde{X}(\varepsilon_1, \varepsilon_2) = \begin{pmatrix} \tilde{X}_0 + O(\eta) & \varepsilon_1(\tilde{X}_{01} + O(\eta)) & \varepsilon_2(\tilde{X}_{02} + O(\eta)) \\ \varepsilon_1(\tilde{X}_{01}^T + O(\eta)) & \varepsilon_1(\tilde{X}_1 + O(\eta)) & \varepsilon_2(\tilde{X}_{12} + O(\eta)) \\ \varepsilon_2(\tilde{X}_{02}^T + O(\eta)) & \varepsilon_2(\tilde{X}_{12}^T + O(\eta)) & \varepsilon_2(\tilde{X}_2 + O(\eta)) \end{pmatrix}$$

and the corresponding feedback gain (9) of the optimal control has the asymptotic structure:

$$\tilde{F}(\varepsilon_1, \varepsilon_2) = \begin{pmatrix} \tilde{\tilde{F}}_0 & \tilde{\tilde{F}}_1 & \tilde{\tilde{F}}_2 \end{pmatrix} + O(\eta). \tag{41}$$

5.2 Near optimal linear quadratic regulator

The asymptotic structure (41) of the optimal feedback allows us to design a near optimal control whose gain matrices do not depend upon small parameters ε_k, $k = 1, 2$.

Theorem 3. *Assume that the assumptions of Theorem 2 are fulfilled. Consider the control*

$$u_{app}(t) = \tilde{\tilde{F}}_0 x_0(t) + \tilde{\tilde{F}}_1 x_1(t) + \tilde{\tilde{F}}_2 x_2(t) \tag{42}$$

whose gain matrices $\tilde{\tilde{F}}_j$, $0 \leq j \leq 2$ *are computed via (34)-(36) based on the stabilizing feedback gains* $(\tilde{F}_0, \tilde{F}_1, \tilde{F}_2)$ *defined in (32) corresponding to the stabilizing solution* $\tilde{\mathbf{X}}$ *of SCRARE (28). Under the considered assumptions the control (42) stabilizes the system (1) for* $\varepsilon_1 > 0, \varepsilon_2 > 0$ *small enough and an upper bound of the loss of performance is given by:*

$$0 \leq J(u_{app}) - J(\tilde{u}) \leq \gamma \|\eta\|^2 |x_0|^2 \tag{43}$$

where η *is the vector of small parameters introduced in connection with the equation (40).*

Proof Setting $\mathbb{F}_{app} = \begin{pmatrix} \tilde{\tilde{F}}_0 & \tilde{\tilde{F}}_1 & \tilde{\tilde{F}}_2 \end{pmatrix}$ we rewrite the control (42) as:

$$u_{app}(t) = \mathbb{F}_{app} x(t). \tag{44}$$

Substituting (44) in (6) we obtain the following closed-loop system:

$$dx(t) = (\mathbb{A} + \mathbb{B}\mathbb{F}_{app})x(t)dt + (\mathbb{C} + \mathbb{D}\mathbb{F}_{app})x(t)dw(t). \tag{45}$$

Let $\Gamma(t; \varepsilon_1, \varepsilon_2), t \geq 0$ be the fundamental matrix solution of the Itô differential equation (45). This means that the solutions of the differential equation (45) satisfy:

$$x(t; x_0, \varepsilon_1, \varepsilon_2) = \Gamma(t; \varepsilon_1, \varepsilon_2)x_0. \tag{46}$$

Even if in (45) the dependence of the coefficients with respect to the small parameters $\varepsilon_1, \varepsilon_2$ is not displayed we emphasize the dependence of the solutions by these parameters. Let $(\Gamma_0(t; \varepsilon_1, \varepsilon_2) \quad \Gamma_1(t; \varepsilon_1, \varepsilon_2) \quad \Gamma_2(t; \varepsilon_1, \varepsilon_2))$ be the partition of the fundamental matrix solution $\Gamma(t; \varepsilon_1, \varepsilon_2)$ such that $\Gamma_k(t; \varepsilon_1, \varepsilon_2) \in \mathbb{R}^{n \times n_k}, k = 0, 1, 2$ are the block columns of this matrix.

Proceeding as in the proof of Theorem 4.7 from [9] one may obtain the following upper bounds

$$\begin{array}{lcl} E[\|\Gamma_0(t;\varepsilon_1,\varepsilon_2)\|^2] & \leq & \beta_0 e^{-\alpha_0 t} \\ E[\|\Gamma_1(t;\varepsilon_1,\varepsilon_2)\|^2] & \leq & \beta_1(e^{-\alpha_1 \frac{t}{\varepsilon_1}} + \varepsilon_1 e^{-\alpha_0 t}) \\ E[\|\Gamma_2(t;\varepsilon_1,\varepsilon_2)\|^2] & \leq & \beta_2(e^{-\alpha_2 \frac{t}{\varepsilon_2}} + \dfrac{\varepsilon_2}{\varepsilon_1} e^{-\alpha_1 \frac{t}{\varepsilon_1}} + \varepsilon_2 e^{-\alpha_0 t}) \end{array} \tag{47}$$

for all $t \geq 0$ and for all $\varepsilon_1 > 0, \varepsilon_2 > 0, \frac{\varepsilon_2}{\varepsilon_1} > 0$ small enough, where $\beta_k \geq 1, \alpha_k > 0, k = 0,1,2$ are constants not depending upon ε_k. Employing (46) and (47) we may conclude that the control (42) stabilizes the full system (1). Plugging (44) in (7) we obtain:

$$J(u_{app}) = x_0^T V(\varepsilon_1,\varepsilon_2) x_0 \tag{48}$$

where

$$V(\varepsilon_1,\varepsilon_2) = E\int_0^\infty \Gamma^T(t;\varepsilon_1,\varepsilon_2)\mathbb{M}_{\mathbb{F}_{app}}\Gamma(t;\varepsilon_1,\varepsilon_2)dt \tag{49}$$

with $\mathbb{M}_{\mathbb{F}_{app}} = \begin{pmatrix} I_n \\ \mathbb{F}_{app} \end{pmatrix}^T \begin{pmatrix} \mathbb{M} & \mathbb{L} \\ \mathbb{L}^T & R \end{pmatrix} \begin{pmatrix} I_n \\ \mathbb{F}_{app} \end{pmatrix}$. By direct calculation involving (49) one shows that $V(\varepsilon_1,\varepsilon_2)$ coincides with the unique solution of the Lyapunov type equation

$$\begin{array}{rcl} (\mathbb{A}+\mathbb{B}\mathbb{F}_{app})^T V + V(\mathbb{A}+\mathbb{B}\mathbb{F}_{app}) + & & \\ (\mathbb{C}+\mathbb{D}\mathbb{F}_{app})^T V(\mathbb{C}+\mathbb{D}\mathbb{F}_{app}) + \mathbb{M}_{\mathbb{F}_{app}} & = & 0. \end{array} \tag{50}$$

Taking into account that the optimal value of the performance is $J(\tilde{u}) = x_0^T \tilde{X}(\varepsilon_1,\varepsilon_2)x_0$ we obtain via (48) that

$$J(u_{app}) - J(\tilde{u}) = x_0^T(V(\varepsilon_1,\varepsilon_2) - \tilde{X}(\varepsilon_1,\varepsilon_2))x_0. \tag{51}$$

Hence, in order to obtain the upper bound of the loss of performance given in (43) is sufficient to obtain an upper bound of $\|V(\varepsilon_1,\varepsilon_2) - \tilde{X}(\varepsilon_1,\varepsilon_2)\|$. To this end let us remark that the SARE (10) satisfied by the stabilizing solution $\tilde{X}(\varepsilon_1,\varepsilon_2)$ may be rewritten in the form:

$$\begin{array}{rcl} (\mathbb{A}+\mathbb{B}\mathbb{F}_{app})^T \tilde{X}(\varepsilon_1,\varepsilon_2) + \tilde{X}(\varepsilon_1,\varepsilon_2)(\mathbb{A}+\mathbb{B}\mathbb{F}_{app}) + & & \\ (\mathbb{C}+\mathbb{D}\mathbb{F}_{app})^T \tilde{X}(\varepsilon_1,\varepsilon_2)(\mathbb{C}+\mathbb{D}\mathbb{F}_{app}) - (\tilde{\mathbb{F}}(\varepsilon_1,\varepsilon_2) - \mathbb{F}_{app})^T \times & & \\ (R + \mathbb{D}^T \tilde{X}(\varepsilon_1,\varepsilon_2)\mathbb{D})(\tilde{\mathbb{F}}(\varepsilon_1,\varepsilon_2) - \mathbb{F}_{app}) + \mathbb{M}_{\mathbb{F}_{app}} & = & 0. \end{array} \tag{52}$$

Subtracting (52) from (50) we deduce that $V(\varepsilon_1,\varepsilon_2)-\tilde{X}(\varepsilon_1,\varepsilon_2)$ satisfies the following Lyapunov type equation:

$$(\mathbb{A}+\mathbb{B}\mathbb{F}_{app})^T(V(\varepsilon_1,\varepsilon_2)-\tilde{X}(\varepsilon_1,\varepsilon_2))+(V(\varepsilon_1,\varepsilon_2)-\tilde{X}(\varepsilon_1,\varepsilon_2))(\mathbb{A}+\mathbb{B}\mathbb{F}_{app})+ \\ (\mathbb{C}+\mathbb{D}\mathbb{F}_{app})^T(V(\varepsilon_1,\varepsilon_2)-\tilde{X}(\varepsilon_1,\varepsilon_2))(\mathbb{C}+\mathbb{D}\mathbb{F}_{app})+\Delta(\varepsilon_1,\varepsilon_2)=0 \quad (53)$$

where $\Delta(\varepsilon_1,\varepsilon_2)=(\tilde{\mathbb{F}}(\varepsilon_1,\varepsilon_2)-\mathbb{F}_{app})^T(R+\mathbb{D}^T\tilde{X}(\varepsilon_1,\varepsilon_2)\mathbb{D})(\tilde{\mathbb{F}}(\varepsilon_1,\varepsilon_2)-\mathbb{F}_{app})$. Employing (41) we infer that

$$\|\Delta(\varepsilon_1,\varepsilon_2)\|\leq c\|\eta\|^2.$$

Thus, (53) yields

$$\|V(\varepsilon_1,\varepsilon_2)-\tilde{X}(\varepsilon_1,\varepsilon_2)\|\leq\gamma\|\eta\|^2. \quad (54)$$

Finally (51) and (54) allow us to obtain (43). Thus the proof is complete.

Remark. The gain matrices of the control (42) do not depend upon the small parameters ε_1, ε_2 and $\frac{\varepsilon_2}{\varepsilon_1}$. They may be computed using (32), (34)-(36), based on the stabilizing solution of SCRARE (28), satisfying the sign conditions (33).

6 Conclusion

We have studied a stochastic linear quadratic optimal control problem consisting in minimization of a quadratic cost functional along of the trajectories of a controlled system modeled by a system of linear Itô differential equations with two fast time scales. Our goal was to perform a detailed investigation of the dependence (with respect to the small parameters describing the fast time scales) of the stabilizing solution of the Riccati equation involved in the construction of the optimal control of the considered LQ problem. To this end we associated a system of coupled algebraic Riccati equations (SCRARE) not depending upon the small parameters occurring in the original Riccati equation and we have introduced the concept of the stabilizing solution of this SCRARE. Also we have provided a set of conditions which are equivalent to the existence of the stabilizing solution of SCRARE. The asymptotic structure of the stabilizing solution of the algebraic Riccati equation associated to the considered LQ control problem was obtained applying the implicit functions theorem. Based on the dominant part of the gain matrix of the optimal control we have constructed a near optimal control whose gain matrices can be computed without knowing the small parameters occurring in the mathematical model of the given system.

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