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ON PERTURBATION ESTIMATES FOR THE EXTREME SOLUTION OF A MATRIX EQUATION*

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Dedicated to Professor Milko Petkov

Abstract

Some perturbation estimates for the unique positive definite solution of a nonlinear matrix equation connected to the interpolation theory are derived. The considered estimations are modification of some existing one. They are obtained by similar transformations of the matrix coefficients with a positive definite matrix. The theoretical results are illustrated by several numerical examples.

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1 Introduction

In this paper we derive new perturbation estimates for the matrix equation

$$X - \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q\,, \tag{1}$$

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where $A_1, A_2, \dots, A_m, Q$ are $n \times n$ complex matrices, Q is a Hermitian positive definite and A^* is the conjugate transpose of a matrix A.

Throughout this paper, $\mathcal{C}^{p\times q}$ denotes the set of $p \times q$ complex matrices and $\mathcal{H}^{n\times n}$ denotes the set of $n \times n$ Hermitian matrices. The notation $A > 0$ ($A \geq 0$) means that A is a Hermitian positive definite (semidefinite) matrix. If $A - B > 0$ (or $A - B \geq 0$) we write $A > B$ (or $A \geq B$). I (or I_n) stands for the identity matrix of order n. The symbols $\|\cdot\|$, $\|\cdot\|_2$, $\|\cdot\|_F$ and $\rho(\cdot)$ denote a unitary invariant norm, the spectral norm, the Frobenius norm, and the spectral radius, respectively. For the matrices $A = (a_{ij})$ and B, $A \otimes B = (a_{ij}B)$ is a Kronecker product. Finally, for a matrix Z, we denote with $\widehat{Z}$ the $m \times m$ block diagonal matrix with Z on its diagonal, i.e. $\widehat{Z} = I_m \otimes Z$.

Eq. (1) can be written as follows:

$$X - A^*\widehat{X}^{-1}A = Q\,, \tag{2}$$

where

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix} \in \mathcal{C}^{mn\times n}.$$

Eq. (2) for $m = 1$ arises in the analysis of stationary Gaussian reciprocal processes over a finite interval [1, 2]. It has been investigated for the existence a positive definite solution in [1, 3, 4], and it has been executed the perturbation analysis in [5, 6, 7, 8, 9].

Ran and Reurings [10] have investigated the equation

$$X - A^*(\widehat{X} - C)^{-1}A = Q\,, \tag{3}$$

where $A \in \mathcal{C}^{mn\times n}$, $C \in \mathcal{H}^{mn}$ and $C \geq 0$. They proved that, under the condition $\widehat{Q} > C$, Eq. (3) has a unique positive definite solution X_+, satisfying $\widehat{X_+} > C$. Eq. (3) is connected with certain interpolation problems [10, 12, Chapter 7]. Liu and Zhang [11, Lemma 2.1] have proved that, under the condition $\widehat{Q} > C$, Eq. (3) is equivalent to an equation in the form of Eq. (2). The perturbation analysis of Eq. (2) and Eq. (3) is executed in Yin and Fang [13], and Sun [14], respectively. In addition, there are many contributions in the literature to the solvability, numerical solutions, and perturbation analysis for the matrix equations $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q$ [15, 16, 17, 18], $X - \sum_{i=1}^{m} A_i^* \mathcal{F}(X) A_i = Q$ [19], $A_0 + \sum_{i=1}^{k} \sigma_i A_i^* X^{p_i} A_i = 0$ [20, 21].

The unique positive definite solution X_+ of Eq. (2) (or Eq. (3)) we will call extreme.

Now we consider the perturbed equation

$$\tilde{X} - \tilde{A}^* \widehat{\tilde{X}}^{-1} \tilde{A} = \tilde{Q}\,, \tag{4}$$

where $\tilde{A}$ and $\tilde{Q}$ ($\tilde{Q} > 0$) are small perturbations of A and Q in (2), respectively. We note that

$$\tilde{A} = \begin{pmatrix} \tilde{A}_1 \\ \vdots \\ \tilde{A}_m \end{pmatrix} \in \mathcal{C}^{mn\times n},$$

where $\tilde{A}_i$, $i = 1, 2, \ldots, m$ are small perturbations of A_i in (1), respectively.

Let $\tilde{X}_+$ be the extreme solution of Eq. (4).

Denote $\Delta X_+ = \tilde{X}_+ - X_+$, $\Delta Q = \tilde{Q} - Q$ and $\Delta A = \tilde{A} - A$.

Yin and Fang [13] have generalized the perturbation estimation for the solution X_+ of Eq. (2) in case of $m = 1$ (see [6, Theorem 3.1]). They have derived the following result.

Theorem 1. ([13, Theorem 2.1]) *Let A, Q and $\tilde{A}$, $\tilde{Q}$ with Q, $\tilde{Q}$ positive definite be coefficient matrices for the matrix equations (2) and (4), respectively. Denote*

$$b = 1 - \|A\|_2^2 \|X_+^{-1}\|_2^2 + \|X_+^{-1}\|_2 \|\Delta Q\|_2\,,$$

$$c = \|\Delta Q\|_2 + 2\,\|A\|_2 \|X_+^{-1}\|_2 \|\Delta A\|_2 + \|X_+^{-1}\|_2 \|\Delta A\|_2{}^2\,,$$

$$\textit{and} \quad D = b^2 - 4c\|X_+^{-1}\|_2.$$

If

$$\|A\|_2^2 \|X_+^{-1}\|_2^2 < 1 \quad \textit{and} \quad 2\,\|\Delta A\|_2 + \|\Delta Q\|_2 \le \frac{\left(1 - \|A\|_2 \|X_+^{-1}\|_2\right)^2}{\|X_+^{-1}\|_2}\,, \tag{5}$$

then the extreme solutions X_+ and $\tilde{X}_+$ the respective equations (2) and (4) satisfies

$$\|\Delta X_+\|_2 \,\le\, \frac{b - \sqrt{D}}{2\|X_+^{-1}\|_2} \,\equiv\, S_{err}\,.$$

For obtaining of a perturbation bound for the extreme solution of Eq. (2) in [14] is required $\|(\widehat{X_+} - C)^{-1}A\|_2 < 1$. For Eq. (2) in case of $m = 1$, $\|X_+^{-1}A\|_2 < 1$ is required, also [6].

In this paper we propose new perturbation estimation to the extreme solution to nonlinear matrix equation (1). Our estimations have wader filed of applications than the those derived in Theorem 1.

Applying the technique developed in [8], in the next section we modified Theorem 1. The investigation was motivated by the fact that the condition $\|\widehat{X_+^{-1}A}\|_2 < 1$ are not always satisfied. Moreover,

$$\|\widehat{X_+^{-1}A}\|_2 \leq \|\widehat{X_+^{-1}}\|_2\|A\|_2, \quad \text{and} \quad \|\widehat{X_+^{-1}}\|_2 = \|X_+^{-1}\|_2.$$

2 Perturbation estimates

We start with some preliminary result.

Lemma 1. ([22, Lemma 3.4.1]) *Let $A_1, \ldots, A_m \in \mathcal{C}^{n\times n}$. Then $\rho(\sum_{i=1}^m A_i^T \otimes A_i^*) < 1$ if and only if there exists a positive definite matrix K such that $K - \sum_{i=1}^m A_i^* K A_i > 0$.*

The same result can be found in [23] (see the proof of Theorem 2.2). Moreover, the inequality

$$\rho\Big(\sum_{i=1}^m A_i^T \otimes A_i^*\Big) \leq \Big\|\sum_{i=1}^m A_i^* A_i\Big\|_2$$

is derived in [22, 23] .

Therefore, for the extreme solution X_+ of Eq. (1) (Eq. (2)) we have

$$\rho\Big(\big(X_+^{-T} \otimes X_+^{-1}\big)\sum_{i=1}^m A_i^T \otimes A_i^*\Big) = \rho\Big(\sum_{i=1}^m (X_+^{-1}A_i)^T \otimes (X_+^{-1}A_i)^*\Big) < 1,$$

$$\rho\Big(\sum_{i=1}^m (X_+^{-1}A_i)^T \otimes (X_+^{-1}A_i)^*\Big) \leq \Big\|\sum_{i=1}^m A_i^* X_+^{-2} A_i\Big\|_2 = \big\|\widehat{X_+^{-1}A}\big\|_2^2.$$

In [8], we have introduced an example of Eq. (2) with $m = 1$, for which the condition $\|X_+^{-1}A\| < 1$ is not satisfied. Moreover, Theorem 3.1 from [6] is modified in [8] by replacing this condition with $\|PX_+^{-1}AP^{-1}\| < 1$, where P is a positive definite matrix. For general case of Eq. (2) this situation appears, also. We consider:

Example 1. ([13, Example 4.1], [10, Example 3.1]) *Consider Eq. (2) with $m = 2$ and matrices A and Q as follows:*

$$A = \begin{pmatrix} -0.4326 & -1.1465 \\ -1.6665 & 1.1909 \\ 0.1253 & 1.1892 \\ 0.2877 & -0.0376 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0.1376 & 0.0656 \\ 0.0656 & 0.5616 \end{pmatrix}.$$

The approximation of the extreme solution

$$X_+ \approx \begin{pmatrix} 1.1572575 & 0.01971555 \\ 0.01971555 & 3.3569583 \end{pmatrix}$$

we obtain after 200 iteration by formula

$$X_{k+1} = Q + A^* \widehat{X_k^{-1}} A, \quad X_0 = Q. \tag{6}$$

In [13] it is shown that, for Example 1 the first condition of Theorem 1 is not satisfied, therefore the bound S_{err} is not applicable. Indeed $\|X_+^{-1}\|_2 \|A\|_2 \approx 1.9443 > 1$. Moreover $\|\widehat{X_+^{-1}A}\|_2 = 1.4926 > 1$, also, but $\|\widehat{PX_+^{-1}}AP^{-1}\|_2 = 0.9621$, where $P = \sqrt{Q} + 2\sqrt[4]{Q}$.

Applying the technique developed in [8], we obtain the following result.

Theorem 2. *Let A, Q and $\tilde{A}$, $\tilde{Q}$ with Q, $\tilde{Q}$ positive definite be coefficient matrices for the matrix equations (2) and (4), respectively, P is a positive definite matrix. Denote $\alpha = \|\widehat{PX_+^{-1}}AP^{-1}\|_2$, $\beta = \|PX_+^{-1}P\|_2$, where X_+ is the extreme solution of Eq. (2),*

$$\begin{aligned} b &= 1 - \alpha^2 + \beta \|P^{-1}\Delta Q P^{-1}\|_2 \,, \\ c &= \|P^{-1}\Delta Q P^{-1}\|_2 + 2\alpha \|\widehat{P^{-1}}\Delta A P^{-1}\|_2 + \beta \|\widehat{P^{-1}}\Delta A P^{-1}\|_2^{\,2} \,. \end{aligned}$$

If $\alpha < 1$ and

$$2\|\widehat{P^{-1}}\Delta A P^{-1}\|_2 + \|P^{-1}\Delta Q P^{-1}\|_2 \le \frac{(1-\alpha)^2}{\beta} \,, \tag{7}$$

then $D = b^2 - 4c\beta \ge 0$ and

$$\|\Delta X_+\|_2 \;\le\; \|P\|_2^2 \frac{b - \sqrt{D}}{2\beta} \;\equiv\; S_{err}^P \,.$$

Proof: The proof is like to the proof of Theorem 1 ([13, Theorem 2.1]) and [8, Theorem 2.4]. Let $\tilde{X}_+$ be the extreme solution of Eq. (4). By subtraction of Eq. (2) from Eq. (4), we obtain

$$\Delta X_+ + A^* \widehat{\tilde{X}_+^{-1}} \widehat{\Delta X_+} \widehat{X_+^{-1}} A - A^* \widehat{\tilde{X}_+^{-1}} \Delta A - \Delta A^* \widehat{\tilde{X}_+^{-1}} \tilde{A} = \Delta Q \,. \tag{8}$$

From (8), using the equalities

$$\tilde{X}_+^{-1} = X_+^{-1}\left(I + \Delta X_+ X_+^{-1}\right)^{-1} = \left(I + X_+^{-1}\Delta X_+\right)^{-1} X_+^{-1},$$

we get

$$\Delta X_+ = \Delta Q \,+\, \Delta A^*\big(I_{n^2} + \widehat{X_+^{-1}}\widehat{\Delta X_+}\big)^{-1}\widehat{X_+^{-1}}(A+\Delta A) \\ -\, A^*\widehat{X_+^{-1}}\big(I_{n^2} + \widehat{\Delta X_+}\widehat{X_+^{-1}}\big)^{-1}\big(\widehat{\Delta X_+}\widehat{X_+^{-1}}A - \Delta A\big). \tag{9}$$

Let P be a positive definite matrix such that $\big\|\widehat{PX_+^{-1}}AP^{-1}\big\|_2 < 1$. From (9), we have

$$\begin{aligned} P^{-1}\Delta X_+ P^{-1} =& P^{-1}\big[\Delta Q + \Delta A^*\big(I_{n^2} + \widehat{X_+^{-1}}\widehat{\Delta X_+}\big)^{-1}\widehat{X_+^{-1}}(A+\Delta A)\big]P^{-1} \\ & - P^{-1}A^*\widehat{X_+^{-1}}\big(I_{n^2} + \widehat{\Delta X_+}\widehat{X_+^{-1}}\big)^{-1}\big(\widehat{\Delta X_+}\widehat{X_+^{-1}}A - \Delta A\big)P^{-1} \\ =& P^{-1}\Delta Q P^{-1} + P^{-1}\Delta A^*\widehat{P^{-1}}\big(I_{n^2} + \widehat{PX_+^{-1}}\widehat{\Delta X_+}\widehat{P^{-1}}\big)^{-1} \\ & \times \widehat{PX_+^{-1}}(A+\Delta A)P^{-1} - P^{-1}A^*\widehat{X_+^{-1}P} \\ & \times \big(I_{n^2} + \widehat{P^{-1}}\widehat{\Delta X_+}\widehat{X_+^{-1}P}\big)^{-1}\widehat{P^{-1}}\big(\widehat{\Delta X_+}\widehat{X_+^{-1}}A - \Delta A\big)P^{-1}. \end{aligned}$$

Let $\mathbf{P} : \mathcal{H}^{n\times n} \to \mathcal{H}^{n\times n}$ be the operator defined by $\mathbf{P}(B) = P^{-1}BP^{-1}$. Then

$$\begin{aligned} \mathbf{P}(\Delta X_+) =& \mathbf{P}(\Delta Q) + P^{-1}\Delta A^*\widehat{P^{-1}}\big(I_{n^2} + \widehat{PX_+^{-1}P}\,\widehat{\mathbf{P}(\Delta X_+)}\big)^{-1} \\ & \times \big(\widehat{PX_+^{-1}}\,AP^{-1} + \widehat{PX_+^{-1}P}\,\widehat{P^{-1}}\Delta AP^{-1}\big) \\ & - P^{-1}A^*\,\widehat{X_+^{-1}P}\big(I_{n^2} + \widehat{\mathbf{P}(\Delta X_+)}\,\widehat{PX_+^{-1}P}\big)^{-1} \\ & \times \big(\widehat{\mathbf{P}(\Delta X_+)}\,\widehat{PX_+^{-1}}AP^{-1} - \widehat{P^{-1}}\Delta AP^{-1}\big). \end{aligned} \tag{10}$$

Consider a map $\mu : \mathcal{H}^{n\times n} \to \mathcal{H}^{n\times n}$ defined by the following way

$$\begin{aligned} \mu(M) =& \mathbf{P}(\Delta Q) + P^{-1}\Delta A^*\widehat{P^{-1}}\big(I_{n^2} + \widehat{PX_+^{-1}P}\,\widehat{M}\big)^{-1} \\ & \times \big(\widehat{PX_+^{-1}}\,AP^{-1} + \widehat{PX_+^{-1}P}\,\widehat{P^{-1}}\Delta AP^{-1}\big) \\ & - P^{-1}A^*\,\widehat{X_+^{-1}P}\big(I_{n^2} + \widehat{M}\,\widehat{PX_+^{-1}P}\big)^{-1} \\ & \times \big(\widehat{M}\,\widehat{PX_+^{-1}}AP^{-1} - \widehat{P^{-1}}\Delta AP^{-1}\big). \end{aligned}$$

From (7), we have

$$2\beta\|\widehat{P^{-1}}\Delta AP^{-1}\|_2 + \beta\|\mathbf{P}(\Delta Q)\|_2 \le 1 + \alpha^2 - 2\alpha\,,$$

from which follows that

$$b \le 2 - 2\big(\alpha + \beta\|\widehat{P^{-1}}\Delta AP^{-1}\|_2\big)\,. \tag{11}$$

From the definitions of D, b, c and by inequality (11) we obtain

$$D = b^2 - 4c\beta = b^2 - 4b + 4 - 4\big(\alpha + \beta\|\widehat{P^{-1}}\Delta AP^{-1}\|_2\big)^2 \geq 0\,.$$

Since $D \geq 0$, the quadratic equation

$$\beta S^2 - bS + c = 0 \tag{12}$$

has two positive real roots. If $D > 0$, then the smaller root is

$$S_1 = \frac{b - \sqrt{D}}{2\beta}\,.$$

If $D = 0$, $S_1 = \frac{b}{2\beta}$ is the double root of (12).

We define

$$\mathcal{L}_{S_1} = \{M \in \mathcal{H}^{n\times n} : \|M\|_2 \leq S_1\}.$$

For each $M \in \mathcal{L}_{S_1}$, we have

$$\|PX_+^{-1}PM\|_2 \leq \|PX_+^{-1}P\|_2\|M\|_2 \leq \beta S_1 \leq \frac{b}{2} < 1\,.$$

Thus, the matrices $I + PX_+^{-1}PM$ and $I_{n^2} + \widehat{PX_+^{-1}P}\,\widehat{M}$ are nonsingular, $\mu(M)$ is a continuous map and

$$\big\|\big(I_{n^2} + \widehat{PX_+^{-1}P}\,\widehat{M}\big)^{-1}\big\|_2 \leq \frac{1}{1 - \big\|PX_+^{-1}PM\big\|} \leq \frac{1}{1 - \beta S_1}\,.$$

For each $M \in \mathcal{L}_{S_1}$, we obtain

$$\begin{aligned}\|\mu(M)\|_2 &\leq \|\mathbf{P}(\Delta Q)\|_2 + \|P^{-1}\Delta A^*\widehat{P^{-1}}\|_2\frac{\alpha + \beta\|\widehat{P^{-1}}\,\Delta AP^{-1}\|_2}{1 - \beta S_1}\\ &\quad + \alpha\frac{\alpha S_1 + \|\widehat{P^{-1}}\,\Delta AP^{-1}\|_2}{1 - \beta S_1}\\ &= \frac{(1-b)S_1 + c}{1 - \beta S_1} = S_1\,.\end{aligned}$$

Thus $\mu(M) \in \mathcal{L}_{S_1}$ for every $M \in \mathcal{L}_{S_1}$, which means that $\mu(\mathcal{L}_{S_1}) \subset \mathcal{L}_{S_1}$. By the Schauder's fixed point theorem, there exists an $M_* \in \mathcal{L}_{S_1}$ such that $\mu(M_*) = M_*$. Hence there exists a Hermitian solution ΔX_* of the equation (10) for which

$$\|\mathbf{P}(\Delta X_*)\|_2 = \|P^{-1}\Delta X_*P^{-1}\|_2 \leq S_1$$

and

$$\|\Delta X_*\|_2 \leq \|P\|_2^2 \, S_1 \, .$$

Let

$$\tilde{X}_* = X_+ + \Delta X_* \, . \tag{13}$$

Since X_+ is the positive definite solution of (2) and ΔX_* is a Hermitian solution of (10), then $\tilde{X}_*$ is a Hermitian solution of the perturbed equation (4).

Finally, we have prove that $\tilde{X}_*$ is a positive definite matrix, from which follows that $\tilde{X}_* \equiv \tilde{X}_+$ and $\Delta X_* \equiv \Delta X_+$.

Since X_+ is a positive definite matrix then there exists a positive definite matrix square root of $PX_+^{-1}P$. From (13), we receive

$$\sqrt{PX_+^{-1}P}\; \mathbf{P}(\tilde{X}_*) \sqrt{PX_+^{-1}P} = I + \sqrt{PX_+^{-1}P}\; \mathbf{P}(\Delta X_*) \sqrt{PX_+^{-1}P} \, .$$

Since

$$\left\| \sqrt{PX_+^{-1}P}\, \mathbf{P}(\Delta X_*) \sqrt{PX_L^{-1}P} \right\|_2 \leq \|PX_+^{-1}P\|_2 \, \|\mathbf{P}(\Delta X_*)\| \leq \beta S_1 < 1 \, ,$$

then $\sqrt{PX_+^{-1}P}\; \mathbf{P}(\tilde{X}_*) \sqrt{PX_+^{-1}P} > 0$. Thus $\tilde{X}_*$ is a positive definite matrix.

The theorem is completely proved. □

In applying of the above theorems difficulties arise in the choice of a matrix P. Therefore, the question arises: **How to choose the matrix P, such that $\|\widehat{PX_+^{-1}}AP^{-1}\|_2 < 1$ in Theorem 2?** This question is an open problem.

Now, we modify the proof of Theorem 2 and obtain a perturbation estimate about an unitary invariant norm. We rewrite the perturbed equation (4) in form (1):

$$\tilde{X} - \sum_{i=1}^{m} \tilde{A}_i^* \tilde{X}^{-1} \tilde{A}_i = \tilde{Q} \, . \tag{14}$$

Theorem 3. *Let A_i, Q and $\tilde{A}_i$, $\tilde{Q}$ ($Q > 0$, $\tilde{Q} > 0$) be coefficient matrices for the matrix equations (1) and (14), respectively, P is a positive definite matrix. Denote $\alpha_i = \|PX_+^{-1}A_iP^{-1}\|_2$, $\beta = \|PX_+^{-1}P\|_2$, where X_+ is the extreme solution of Eq. (2),*

$$b_1 = 1 - \sum_{i=1}^{m} \alpha_i^2 + \beta \|P^{-1}\Delta QP^{-1}\| \, ,$$

$$c_1 = \|P^{-1}\Delta QP^{-1}\| + 2\sum_{i=1}^{m} \alpha_i \|P^{-1}\Delta A_iP^{-1}\| + \beta \sum_{i=1}^{m} \|P^{-1}\Delta A_iP^{-1}\|^2 \, .$$

If

$$\sum_{i=1}^{m} \alpha_i^2 < 1$$

and

$$\beta\|P^{-1}\Delta Q P^{-1}\| + 2\sqrt{\sum_{i=1}^{m}\left(\beta\|P^{-1}\Delta A_i P^{-1}\| + \alpha_i\right)^2} \leq 1 + \sum_{i=1}^{m}\alpha_i^2\,, \qquad (15)$$

then $D_1 = b_1^2 - 4c_1\beta \geq 0$ *and*

$$\|\Delta X_+\| \;\leq\; \|P\|_2^2 \frac{b_1 - \sqrt{D_1}}{2\beta} \;\equiv\; U_1^P\,.$$

Proof: Let $\tilde{X}_+$ be the extreme solution of Eq. (14). By subtraction of Eq. (1) from Eq. (14), we obtain

$$\Delta X_+ + \sum_{i=1}^{m}\left(A_i^*\tilde{X}_+^{-1}\Delta X_+ X_+^{-1}A_i - A_i^*\tilde{X}_+^{-1}\Delta A_i - \Delta A_i^*\tilde{X}_+^{-1}\tilde{A}_i\right) = \Delta Q\,,$$

and

$$\Delta X_+ = \Delta Q + \sum_{i=1}^{m}\Delta A_i^*\left(I + X_+^{-1}\Delta X_+\right)^{-1}X_+^{-1}(A_i + \Delta A_i)$$
$$+ \sum_{i=1}^{m} A_i^* X_+^{-1}\left(I + \Delta X_+ X_+^{-1}\right)^{-1}\left(\Delta A_i - \Delta X_+ X_+^{-1}A_i\right). \qquad (16)$$

Let P be a positive definite matrix, $\alpha_i = \|PX_+^{-1}A_iP^{-1}\|$ and

$$\sum_{i=1}^{m}\alpha_i^2 < 1.$$

Define the operator $\mathbf{P}(B) = P^{-1}BP^{-1}$. Then from (16), we have

$$\mathbf{P}(\Delta X_+) = \mathbf{P}(\Delta Q) + \sum_{i=1}^{m}\Big[P^{-1}\Delta {A_i}^* P^{-1}\left(I + PX_+^{-1}P\mathbf{P}(\Delta X_+)\right)^{-1}$$
$$\times\left(PX_+^{-1}A_iP^{-1} + PX_+^{-1}P\mathbf{P}(\Delta A_i)\right)\Big]$$
$$+ \sum_{i=1}^{m}\Big[P^{-1}A_i^*X_+^{-1}P\left(I + \mathbf{P}(\Delta X_+)\,PX_+^{-1}P\right)^{-1}$$
$$\times\left(\mathbf{P}(\Delta A_i) - \mathbf{P}(\Delta X_+)PX_+^{-1}A_iP^{-1}\right)\Big]. \qquad (17)$$

Define the map μ by

$$\mu(M) = \mathbf{P}(\Delta Q) + \sum_{i=1}^{m} \Big[P^{-1} \Delta A_i{}^{*} P^{-1} \big(I + P X_{+}^{-1} P M \big)^{-1} \times \big(P X_{+}^{-1} A_i P^{-1} + P X_{+}^{-1} P \mathbf{P}(\Delta A_i) \big) \Big] + \sum_{i=1}^{m} \Big[P^{-1} A_i^{*} X_{+}^{-1} P \big(I + M P X_{+}^{-1} P \big)^{-1} \times \big(\mathbf{P}(\Delta A_i) - M P X_{+}^{-1} A_i P^{-1} \big) \Big].$$

From (15), we have

$$b_1 \leq 2 - 2 \sqrt{ \sum_{i=1}^{m} \big(\beta \| P^{-1} \Delta A_i P^{-1} \| + \alpha_i \big)^2 } \, .$$

Thus, from the definitions of D_1, b_1, c_1, we obtain

$$D_1 = b_1^2 - 4 c_1 \beta = b_1^2 - 4 b_1 + 4 - 4 \sum_{i=1}^{m} \big(\beta \| P^{-1} \Delta A_i P^{-1} \| + \alpha_i \big)^2 \geq 0 \, .$$

Since $D \geq 0$, the quadratic equation

$$\beta U^2 - b_1 U + c_1 = 0 \tag{18}$$

has positive roots. For $D_1 > 0$, the smaller root of the equation (18) is

$$U_1 = \frac{b_1 - \sqrt{D_1}}{2\beta}. \quad \Big(U_1 = \frac{b_1}{2\beta}, \text{ for } D_1 = 0 \Big)$$

.

We define the set

$$\mathcal{L}_{U_1} = \{ M \in \mathcal{H}^{n \times n} : \| M \| \leq U_1 \}.$$

For each $M \in \mathcal{L}_{U_1}$, we have

$$\| P X_{+}^{-1} P M \| \leq \| P X_{+}^{-1} P \|_2 \| M \| \leq \beta U_1 \leq \frac{b_1}{2} < 1 \, .$$

Thus, the matrix $I + P X_{+}^{-1} P M$ is nonsingular, $\mu(M)$ is a continuous map and

$$\big\| \big(I + P X_{+}^{-1} P M \big)^{-1} \big\|_2 \leq \frac{1}{1 - \| P X_{+}^{-1} P M \|_2} \leq \frac{1}{1 - \beta U_1} \, .$$

For each $M \in \mathcal{L}_{U_1}$, we obtain

$$\|\mu(M)\| \leq \|\mathbf{P}(\Delta Q)\| + \frac{1}{1-\beta U_1} \sum_{i=1}^{m} 2\alpha_i \|\mathbf{P}(\Delta A_i)\| + \beta \|\mathbf{P}(\Delta A_i)\|^2 + \alpha_i^2 U_1$$
$$= U_1 .$$

Thus $\mu(M) \in \mathcal{L}_{U_1}$ for every $M \in \mathcal{L}_{U_1}$, which means that $\mu(\mathcal{L}_{U_1}) \subset \mathcal{L}_{U_1}$. By the Schauder's fixed point theorem, there exists a $M_* \in \mathcal{L}_{U_1}$ such that $\mu(M_*) = M_*$. Hence there exists a Hermitian solution ΔX_* of the equation (17) for which

$$\|\mathbf{P}(\Delta X_*)\| = \|P^{-1} \Delta X_* P^{-1}\| \leq U_1$$

and

$$\|\Delta X_*\| \leq \|P\|_2^2 \, U_1 \equiv U_1^P .$$

Let

$$\tilde{X}_* = X_+ + \Delta X_* .$$

The proof of $\tilde{X}_* \equiv \tilde{X}_+$ and $\Delta X_* \equiv \Delta X_+$ as above theorem. □

Remark 1. *The conditions of Theorem 1 is more restrictive than the conditions of Theorem 2, because*

$$\left\| \widehat{PX_+^{-1}} A P^{-1} \right\|_2^2 \leq \sum_{i=1}^{m} \left\| P X_+^{-1} A_i P^{-1} \right\|_2^2 .$$

3 Numerical examples

We consider some numerical examples for illustration of the theoretical results.

Example 2. *Consider Eq. (2) with coefficient matrices defined in Example 1 and perturbations on the matrices A and Q*

$$\Delta A = 10^{-j} \begin{pmatrix} 1 & 0.6 \\ 0.2 & 0.4 \\ 0.8 & 0.4 \\ 0.6 & 0.1 \end{pmatrix}, \quad \Delta Q = 10^{-j} \begin{pmatrix} 0.4 & 0.7 \\ 0.7 & 0.3 \end{pmatrix},$$

respectively.

The solution $\tilde{X}_+$ to the perturbed equation we compute iteratively by formula (6) as $\tilde{X}_+ \approx \tilde{X}_{200}$.

We remain that for Example 1

$$\|X_+^{-1}\|_2\|A\|_2 \approx 1.9443 > 1, \quad \|\widehat{X_+^{-1}}A\|_2 \approx 1.4926 > 1,$$

but $\|\widehat{PX_+^{-1}}AP^{-1}\|_2 \approx 0.9621$, where $P = \sqrt{Q} + 2\sqrt[4]{Q}$. Hence, Theorem 1 is not applicable. In Table 1 the perturbation estimates by using Theorem 2 with different values of j are given.

Table 1: Numerical results of Example 2.

j	$j = 4$	$j = 5$	$j = 6$
$\|\Delta X_+\|_2$	$3.0193e-04$	$3.0159e-05$	$2.9752e-06$
S_{err}^P	$1.1394e-02$	$1.0814e-03$	$1.0763e-04$

Perturbation estimates for Eq. (2) have been obtained in [12, 20, 21], also. Here we don't compare these estimates. Our main intention is to show that Theorem 2 is applicable, while Theorem 1 is not. The detailed comparison for the existing estimates is an upcoming work.

Example 3. ([13, Example 4.1]) *Consider Eq. (2) with* $m = 2$, $n = 4$ *and matrices* A *and* Q *as follows:*

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad Q := X_+ - A^*\widehat{X_+^{-1}}A,$$

where

$$A_1 = \frac{1}{100}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 22.5 & 12 & 2 \\ 2 & 9 & 7 & 3 \\ 12 & 1 & 2 & 19 \end{pmatrix}, \quad A_2 = \frac{2\sqrt{3}}{45}\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

and

$$X_+ = \begin{pmatrix} 2,5 & 1 & 1 & 1 \\ 1 & 2,5 & 1 & 1 \\ 1 & 1 & 2,5 & 1 \\ 1 & 1 & 1 & 2,5 \end{pmatrix}.$$

Consider perturbation on the matrices A and Q:

$$\Delta A = 10^{-2j}\begin{pmatrix} \frac{C_1+C_1^*}{\|C_1+C_1^*\|} \\ \frac{C_2+C_2^*}{\|C_2+C_2^*\|} \end{pmatrix}, \quad \Delta Q := \tilde{X}_+ - \tilde{A}^*\widehat{\tilde{X}_+^{-1}}\tilde{A} - Q,$$

with $\tilde{X}_+ = X_+ + 10^{-2j}(I - E)$, E being the 4×4 matrix with all entries equal to 1, C_1, C_2 random matrices generated by MATLAB function **randn**.

For Example 3 we have

$$\|X_+^{-1}\|_2\|A\|_2 \approx 0.2119 < 1, \quad \|\widehat{X_+^{-1}A}\|_2 \approx 0,1667 < 1.$$

Hence, both theorems 1 and 2 are applicable, as $P = I$. In Table 2 the perturbation estimates for different values of j are given.

Table 2: Numerical results of Example 3 ($P = I$).

j	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$\|\Delta X_+\|_2$	$3.0000e-04$	$3.0000e-06$	$3.0000e-08$	$3.0000e-10$
S_{err}	$3.8874e-04$	$3.8333e-06$	$3.8837e-08$	$3.8443e-10$
S_{err}^P	$3.6882e-04$	$3.6362e-06$	$3.6847e-08$	$3.6458e-10$

Usually, when the conditions of both theorems (Theorem 1 and Theorem 2) are satisfied, the estimate S_{err}^P derived in Theorem 2 is shaper than the estimate S_{err} given in Theorem 1.

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