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ON A DIFFERENTIAL INCLUSION WITH CERTAIN NONLOCAL CONDITIONS*

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Abstract

We consider a first-order nonconvex multivalued differential equation suject to some nonlocal conditions. We establish a Filippov type existence theorem and we prove the arcwise connectedness of the solution set of the problem considered.

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keywords: differential inclusion, boundary value problem, arcwise connectedness.

1 Introduction

This paper is concerned with the following problem

$$x' \in F(t,x) \quad a.e.\ [0,1], \tag{1.1}$$

$$x(0) + \sum_{i=1}^{m} a_i x(t_i) = x_0, \tag{1.2}$$

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where $a_i \in \mathbf{R}$, $a_i \neq 0$, $i = \overline{1,m}$, $x_0 \in \mathbf{R}$, $0 < t_1 < t_2 < ... < t_m < 1$ and $F : [0,1] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map.

In the case when F is un upper semicontinuous set-valued map with convex values existence results for problem (1.1)-(1.2) may be found in [2,3]. In [2] the result uses Bohnenblust and Karlin fixed point theorem for set-valued maps and in [3], in the case $x_0 = 0$, the approach is based on topological transversality theory for set-valued maps.

The nonlocal condition (1.2) was used by Byszewski ([4,5]). If $a_i \neq 0$, $i = \overline{1,m}$ the results can be applied in kinematics to determine the evolution $t \to x(t)$ of the location of a physical object for which the positions $x(0), x(t_1), ..., x(t_m)$ are unknown but it is known the condition (1.2). Consequently, to describe some physical phenomena the nonlocal condition may be more useful than the standard initial condition $x(0) = x_0$. Obviously, when $a_i = 0$, $i = \overline{1,m}$, one has the classical initial condition.

Our aim is to study problem (1.1)-(1.2) in the case when the values of F are not convex and is twofold. On one hand, we show that Filippov's ideas ([8]) can be suitably adapted in order to obtain the existence of solutions for problem (1.1)-(1.2). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([8]) consists in proving the existence of a solution starting from a given almost solution. Moreover, the result provides an estimate between the starting almost solution and the solution of the differential inclusion.

On the other hand, following the approach in [10] we prove the arcwise connectedness of the solution set of problem (1.1)-(1.2). The proof is based on a result ([9,10]) concerning the arcwise connectedness of the fixed point set of a class of set-valued contractions.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel, Section 3 is devoted to the Filippov type existence theorem and in Section 4 we obtain the arcwise connectedness of the solution set.

2 Preliminaries

In what follows we denote by I the interval $[0,1]$, $C(I,\mathbf{R})$ is the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x||_C = \sup_{t\in I}|x(t)|$ and $L^1(I,\mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u||_1 = \int_0^T |u(t)|dt$. The Banach space of all absolutely continuous functions $x(.) : I \to \mathbf{R}$ with the norm $||x(.)||_{AC} = |x(0)| + ||x'(.)||_1$ will be denoted by $AC(I,\mathbf{R})$.

Let (X, d) be a metric space. We recall that the Pompeiu-Hausdorff distance of the nonempty closed subsets $A, B \subset X$ is defined by

$$D(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

Next we need the following technical result proved in [2].

Lemma 2.1. *Assume that* $1 + \sum_{i=1}^m a_i \neq 0$. *For a given* $f(.) \in C(I, \mathbf{R})$, *the unique solution* $x(.)$ *of problem* $x' = f(t)$ *with boundary condition (1.2) is given by*

$$x(t) = a(x_0 - \sum_{i=1}^m a_i \int_0^{t_i} f(s)ds) + \int_0^t f(s)ds,$$

where $a = \frac{1}{1+\sum_{i=1}^m a_i}$.

Remark 2.2. If we denote $G(t,s) = \chi_{[0,t]}(s) - \sum_{i=1}^m a a_i \chi_{[0,t_i]}(s)$, where $\chi_S(\cdot)$ is the characteristic function of the set S, then the solution $x(\cdot)$ in Lemma 2.1 may be written as

$$x(t) = a x_0 + \int_0^1 G(t,s) f(s) ds.$$

Obviously, $|G(t,s)| \leq 1 + |a| \sum_{i=1}^m |a_i| =: M \ \forall\, t, s \in I.$.

3 A Filippov type existence result

First we recall a selection result ([1]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

Lemma 3.1. *Consider* B *the closed unit ball in* $\mathbf{R}^n$, $H : I \to \mathcal{P}(\mathbf{R}^n)$ *is a set-valued map with nonempty closed values* $g : I \to \mathbf{R}^n$ *and* $L : I \to [0, \infty)$ *are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \emptyset \quad a.e.\ (I),$$

then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ *has a measurable selection.*

In the sequel we assume the following conditions on F.

Hypothesis 3.2. i) $F : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and for every* $x \in \mathbf{R}$ $F(.,x)$ *is measurable.*

ii) *There exists* $L(.) \in L^1(I, \mathbf{R})$ *such that for almost all* $t \in I, F(t, .)$ *is* $L(t)$*-Lipschitz in the sense that*

$$D(F(t,x), F(t,y)) \leq L(t)|x-y| \quad \forall\ x,y \in \mathbf{R}.$$

We are now ready to prove the main result of this section.

Theorem 3.3. *Assume that* $1+\sum_{i=1}^m a_i \neq 0$*, Hypothesis 3.2 is satisfied,* $M||L||_1 < 1$ *and let* $y \in AC(I, \mathbf{R})$ *be such that there exists* $q(.) \in L^1(I, \mathbf{R})$ *with* $d(y'(t), F(t, y(t))) \leq q(t)$ *a.e.* (I)*. Denote* $y(0) + \sum_{i=1}^m a_i y(t_i) = y_0$.

Then there exists $x(.) \in AC(I, \mathbf{R})$ *a solution of problem (1.1)-(1.2) satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{1}{1 - M||L||_1}[|a||x_0 - y_0| + M||q||_1]. \tag{3.1}$$

Proof. The set-valued map $t \to F(t, y(t))$ is measurable with closed values and the hypothesis that $d(y'(t), F(t, y(t))) \leq q(t)$ a.e. (I) is equivalent to

$$F(t, y(t)) \cap \{y'(t) + q(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows from Lemma 3.1 that there exists a measurable selection $f_1(t) \in F(t, y(t))$ *a.e.* (I) such that

$$|f_1(t) - y'(t)| \leq q(t) \quad a.e.\ (I) \tag{3.2}$$

Define $x_1(t) = ax_0 + \int_0^1 G(t,s) f_1(s) ds$ and one has

$$\begin{aligned} &|x_1(t) - y(t)| = |ax_0 - ay_0 + \textstyle\int_0^1 G(t,s)(f_1(s) - y'(s))ds| \leq \\ &|a||x_0 - y_0| + \textstyle\int_0^1 |G(t,s)| q(s) ds \leq |a||x_0 - y_0| + M||q||_1. \end{aligned}$$

We claim that it is enough to construct the sequences $x_n(.) \in AC(I, \mathbf{R})$, $f_n(.) \in L^1(I, \mathbf{R})$, $n \geq 1$ with the following properties

$$x_n(t) = ax_0 + \int_0^1 G(t,s) f_n(s) ds, \quad t \in I, \tag{3.3}$$

$$f_n(t) \in F(t, x_{n-1}(t)) \quad a.e.\ (I),\ n \geq 1, \tag{3.4}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)|x_n(t) - x_{n-1}(t)| \quad a.e.\ (I), n \geq 1. \tag{3.5}$$

If this construction is realized then from (3.2)-(3.5) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq \int_0^1 |G(t,t_1)|.|f_{n+1}(t_1) - f_n(t_1)| dt_1 \leq M \int_0^1 L(t_1)|x_n(t_1) -$$

$$x_{n-1}(t_1)|dt_1 \leq M\int_0^1 L(t_1)\int_0^1 |G(t_1,t_2)|.|f_n(t_2)-f_{n-1}(t_2)|dt_2 \leq M^2\int_0^1 L(t_1)$$
$$\int_0^1 L(t_2)|x_{n-1}(t_2)-x_{n-2}(t_2)|dt_2dt_1 \leq M^n\int_0^1 L(t_1)\int_0^1 L(t_2)...\int_0^1 L(t_n)$$
$$|x_1(t_n)-y(t_n)|dt_n...dt_1 \leq (M||L||_1)^n(|a||x_0-y_0|+M||q||_1).$$

Therefore $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in the Banach space $C(I,\mathbf{R})$, hence converging uniformly to some $x\in C(I,\mathbf{R})$. Therefore, by (3.5), for almost all $t\in I$, the sequence $\{f_n(t)\}_{n\in\mathbf{N}}$ is Cauchy in $\mathbf{R}$. Let f be the pointwise limit of f_n.

Moreover, one has

$$\begin{aligned}&|x_n(t)-y(t)| \leq |x_1(t)-y(t)| + \textstyle\sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)| \leq |a||x_0-y_0|\\ &+M||q||_1 + \textstyle\sum_{i=1}^{n-1}|a||x_0-y_0|+M||q||_1)(M||L||_1)^i = \frac{|a||x_0-y_0|+M||q||_1}{1-M||L||_1}.\end{aligned} \tag{3.6}$$

On the other hand, from (3.2), (3.5) and (3.6) we obtain for almost all $t\in I$

$$\begin{aligned}&|f_n(t)-y'(t)| \leq \textstyle\sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)|+\\ &|f_1(t)-y'(t)| \leq L(t)\frac{|a||x_0-y_0|+M||q||_1}{1-M||L||_1}+q(t).\end{aligned}$$

Hence the sequence f_n is integrably bounded and therefore $f\in L^1(I,\mathbf{R})$.

Using Lebesque's dominated convergence theorem and taking the limit in (3.3), (3.4) we deduce that x is a solution of (1.1). Finally, passing to the limit in (3.6) we obtained the desired estimate on x.

It remains to construct the sequences x_n, f_n with the properties in (3.3)-(3.5). The construction will be done by induction.

Since the first step is already realized, assume that for some $N\geq 1$ we already constructed $x_n\in C(I,\mathbf{R})$ and $f_n\in L^1(I,\mathbf{R}), n=1,2,...N$ satisfying (3.3),(3.5) for $n=1,2,...N$ and (3.4) for $n=1,2,...N-1$. The set-valued map $t\to F(t,x_N(t))$ is measurable. Moreover, the map $t\to L(t)|x_N(t)-x_{N-1}(t)|$ is measurable. By the lipschitzianity of $F(t,.)$ we have that for almost all $t\in I$

$$F(t,x_N(t))\cap\{f_N(t)+L(t)|x_N(t)-x_{N-1}(t)|[-1,1]\}\neq\emptyset.$$

From Lemma 3.1 there exists a measurable selection $f_{N+1}(.)$ of $F(.,x_N(.))$ such that

$$|f_{N+1}(t)-f_N(t)| \leq L(t)|x_N(t)-x_{N-1}(t)| \quad a.e.\ (I).$$

We define x_{N+1} as in (3.3) with $n=N+1$. Thus f_{N+1} satisfies (3.4) and (3.5) and the proof is complete.

Remark 3.4. We note that similar existence results for other classes of differential inclusions may be found in our previous papers [6,7].

4 Arcwise connectedness of the solution set

In this section we are concerned with the more general problem

$$x' \in F(t, x, H(t, x)) \quad a.e. \ ([0, 1]), \tag{4.1}$$

$$x(0) + \sum_{i=1}^{m} a_i x(t_i) = c, \tag{4.2}$$

where $c \in \mathbf{R}$, $F : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$, $H : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ and by $F(t, x, H(t, x))$ we understand $\cup_{y \in H(t,x)} F(t, x, y)$.

We assume that F and H are closed-valued multifunctions Lipschitzian with respect to the second variable and F is contractive in the third variable. Obviously, the right-hand side of the differential inclusion in (4.1) is in general neither convex nor closed. We prove the arcwise connectedness of the solution set to (4.1)-(4.2). When F does not depend on the last variable (4.1) reduces to (1.1) and the result remains valid for problem (1.1)-(1.2).

Let Z be a metric space with the distance d_Z. In what follows, when the product $Z = Z_1 \times Z_2$ of metric spaces $Z_i, i = 1, 2$, is considered, it is assumed that Z is equipped with the distance $d_Z((z_1, z_2), (z_1', z_2')) = \sum_{i=1}^{2} d_{Z_i}(z_i, z_i')$.

Let X be a nonempty set and let $F : X \to \mathcal{P}(Z)$ be a set-valued map with nonempty closed values. In what follows d_X stands for the distance on X induced by the normed on X and Pompeiu-Hausdorff distance on nonempty closed subsets of X is denoted by D_X..

The range of F is the set $F(X) = \cup_{x \in X} F(x)$. The multifunction F is called Hausdorff continuous if for any $x_0 \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in X, d_X(x, x_0) < \delta$ implies $D_Z(F(x), F(x_0)) < \epsilon$.

Let $(T, \mathcal{F}, \mu)$ be a finite, positive, nonatomic measure space and let $(X, |.|_X)$ be a Banach space. We recall that a set $A \in \mathcal{F}$ is called atom of μ if $\mu(A) \neq 0$ and for any $B \in \mathcal{F}$, $B \subset A$ one has $\mu(B) = 0$ or $\mu(B) = \mu(A)$. μ is called nonatomic measure if $\mathcal{F}$ does not contains atoms of μ. For example, Lebesgue's measure on a given interval in $\mathbf{R}^n$ is a nonatomic measure.

We denote by $L^1(T, X)$ the Banach space of all (equivalence classes of) Bochner integrable functions $u : T \to X$ endowed with the norm

$$|u|_{L^1(T,X)} = \int_T |u(t)|_X d\mu$$

A nonempty set $K \subset L^1(T, X)$ is called decomposable if, for every $u, v \in K$ and every $A \in \mathcal{F}$, one has

$$\chi_A.u + \chi_{T\setminus A}.v \in K$$

where $\chi_B, B \in \mathcal{F}$ indicates the characteristic function of B.

Next we recall some preliminary results that are the main tools in the proof of our result.

To simplify the notation we write E in place of $L^1(T, X)$.

The next two lemmas are proved in [10].

Lemma 4.1. *Assume that $\phi : S \times E \to \mathcal{P}(E)$ and $\psi : S \times E \times E \to \mathcal{P}(E)$ are Hausdorff continuous multifunctions with nonempty, closed, decomposable values, satisfying the following conditions*

a) There exists $L \in [0, 1)$ such that, for every $s \in S$ and every $u, u' \in E$,

$$D_E(\phi(s, u), \phi(s, u')) \leq L|u - u'|_E.$$

b) There exists $M \in [0, 1)$ such that $L + M < 1$ and for every $s \in S$ and every $(u, v), (u', v') \in E \times E$,

$$D_E(\psi(s, u, v), \psi(s, u', v')) \leq M(|u - u'|_E + |v - v'|_E).$$

Set $Fix(\Gamma(s, .)) = \{u \in E; u \in \Gamma(s, u)\}$, where $\Gamma(s, u) = \psi(s, u, \phi(s, u))$, $(s, u) \in S \times E$. Then

1) For every $s \in S$ the set $Fix(\Gamma(s, .))$ is nonempty and arcwise connected.

2) For any $s_i \in S$, and any $u_i \in Fix(\Gamma(s, .)), i = 1, ..., p$ there exists a continuous function $\gamma : S \to E$ such that $\gamma(s) \in Fix(\Gamma(s, .))$ for all $s \in S$ and $\gamma(s_i) = u_i, i = 1, ..., p$.

Lemma 4.2. *Let $U : T \to \mathcal{P}(X)$ and $V : T \times X \to \mathcal{P}(X)$ be two nonempty closed-valued multifunctions satisfying the following conditions*

a) U is measurable and there exists $r \in L^1(T)$ such that $D_X(U(t), \{0\}) \leq r(t)$ for almost all $t \in T$.

b) The multifunction $t \to V(t, x)$ is measurable for every $x \in X$.

c) The multifunction $x \to V(t, x)$ is Hausdorff continuous for all $t \in T$.

Let $v : T \to X$ be a measurable selection from $t \to V(t, U(t))$.

Then there exists a selection $u \in L^1(T, X)$ of $U(.)$ such that $v(t) \in V(t, u(t))$, $t \in T$.

Hypothesis 4.3. *Let* $F : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ *and* $H : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *be two set-valued maps with nonempty closed values, satisfying the following assumptions*

i) The set-valued maps $t \to F(t,u,v)$ *and* $t \to H(t,u)$ *are measurable for all* $u, v \in \mathbf{R}$.

ii) There exists $l \in L^1(I, \mathbf{R}_+)$ *such that, for every* $u, u' \in \mathbf{R}$,

$$D(H(t,u), H(t,u')) \leq l(t)|u - u'| \quad a.e.\ (I).$$

iii) There exist $m \in L^1(I, \mathbf{R}_+)$ *and* $\theta \in [0,1)$ *such that, for every* u, v, u', $v' \in \mathbf{R}$,

$$D(F(t,u,v), F(t,u',v')) \leq m(t)|u - u'| + \theta|v - v'| \quad a.e.(I).$$

iv) There exist $f, g \in L^1(I, \mathbf{R}_+)$ *such that*

$$d(0, F(t,0,0)) \leq l(t), \quad d(0, H(t,0)) \leq g(t) \quad a.e.(I).$$

For $c \in \mathbf{R}$ we denote by $S(c)$ the solution set of (4.1)-(4.2).

In what follows $N(t) := \max\{l(t), m(t)\}$, $t \in I$.

Theorem 4.4. *Assume that* $1 + \sum_{i=1}^{m} a_i \neq 0$, *Hypothesis 4.3 is satisfied and* $2M \int_0^1 N(s)ds + \theta < 1$. *Then*

1) For every $c \in \mathbf{R}$, *the solution set* $S(c)$ *of (4.1)-(4.2) is nonempty and arcwise connected in the space* $C(I, \mathbf{R})$.

2) For any $c_i \in \mathbf{R}$ *and any* $u_i \in S(c_i)$, $i = 1, ..., p$, *there exists a continuous function* $s : \mathbf{R} \to C(I, \mathbf{R})$ *such that* $s(c) \in S(c)$ *for any* $c \in \mathbf{R}$ *and* $s(c_i) = u_i, i = 1, ..., p$.

3) The set $S = \cup_{c \in \mathbf{R}} S(c)$ *is arcwise connected in* $C(I, \mathbf{R})$.

Proof. 1) For $c \in \mathbf{R}$ and $u \in L^1(I, \mathbf{R})$, set

$$u_c(t) = ac + \int_0^1 G(t,s)u(s)ds, \quad t \in I.$$

We prove that the multifunctions $\phi : \mathbf{R} \times L^1(I, \mathbf{R}) \to \mathcal{P}(L^1(I, \mathbf{R}))$ and $\psi : \mathbf{R} \times L^1(I, \mathbf{R}) \times L^1(I, \mathbf{R}) \to \mathcal{P}(L^1(I, \mathbf{R}))$ given by

$$\phi(c,u) = \{v \in L^1(I, \mathbf{R}); \quad v(t) \in H(t, u_c(t)) \quad a.e.\ (I)\},$$

$$\psi(c,u,v) = \{w \in L^1(I, \mathbf{R}); \quad w(t) \in F(t, u_c(t), v(t)) \quad a.e.\ (I)\},$$

$c \in \mathbf{R}$, $u, v \in L^1(I, \mathbf{R})$ satisfy the hypotheses of Lemma 4.1.

Since u_c is measurable and H satisfies Hypothesis 4.3 i) and ii), the multifunction $t \to H(t, u_c(t))$ is measurable and nonempty closed valued, hence it has a measurable selection. Therefore due to Hypothesis 4.3 iv), the set $\phi(c, u)$ is nonempty. The fact that the set $\phi(c, u)$ is closed and decomposable follows by simple computation. In the same way we obtain that $\psi(c, u, v)$ is a nonempty closed decomposable set.

Pick $(c, u), (c_1, u_1) \in \mathbf{R} \times L^1(I, \mathbf{R})$ and choose $v \in \phi(c, u)$. For each $\varepsilon > 0$ there exists $v_1 \in \phi(c_1, u_1)$ such that, for every $t \in I$, one has

$$|v(t) - v_1(t)| \leq D(H(t, u_c(t)), H(t, u_{c_1}(t))) + \varepsilon \leq \\ N(t)[|a||c - c_1| + \int_0^1 |G(t, s)|.|u(s) - u_1(s)|ds] + \varepsilon.$$

Therefore,

$$||v - v_1||_1 \leq |a||c - c_1|. \int_0^1 N(t)dt + M \int_0^1 N(t)dt||u - u_1||_1 + \varepsilon T$$

for any $\varepsilon > 0$.

This implies

$$d_{L^1(I,\mathbf{R})}(v, \phi(c_1, u_1)) \leq |a||c - c_1|. \int_0^1 N(t)dt + M \int_0^1 N(t)dt||u - u_1||_1$$

for all $v \in \phi(c, u)$. Consequently,

$$D_{L^1(I,\mathbf{R})}(\phi(c, u), \phi(c_1, u_1)) \leq |a||c - c_1|. \int_0^1 N(t)dt + M \int_0^1 N(t)dt||u - u_1||_1$$

which shows that ϕ is Hausdorff continuous and satisfies the assumptions of Lemma 4.1.

Pick $(c, u, v), (c_1, u_1, v_1) \in \mathbf{R} \times L^1(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ and choose $w \in \psi(c, u, v)$. Then, as before, for each $\varepsilon > 0$ there exists $w_1 \in \psi(c_1, u_1, v_1)$ such that for every $t \in I$

$$|w(t) - w_1(t)| \leq D(F(t, u_c(t), v(t)), F(t, u_{c_1}(t), v_1(t))) + \varepsilon \leq$$

$$\leq N(t)|u_c(t) - u_{c_1}(t)| + \theta|v(t) - v_1(t)| + \varepsilon \leq$$

$$N(t)[|ac - ac_1| + \int_0^1 |G(t, s)|.|u(s) - u_1(s)|ds] + \theta|v(t) - v_1(t)| + \varepsilon$$

$$\leq N(t)[|a||c - c_1| + M||u - u_1||_1] + \theta|v(t) - v_1(t)| + \varepsilon.$$

Hence

$$||w-w_1||_1 \le |a||c-c_1|.\int_0^1 N(t)dt + M\int_0^1 N(t)dt||u-u_1||_1$$
$$+\theta||v-v_1||_1+\varepsilon T \le |a||c-c_1|.\int_0^1 N(t)dt + (M\int_0^1 N(t)dt+$$
$$\theta)d_{L^1(I,\mathbf{R})\times L^1(I,\mathbf{R})}((u,v),(u_1,v_1))+\varepsilon T.$$

As above, we deduce that

$$D_{L^1(I,\mathbf{R})}(\psi(c,u,v),\psi(c_1,u_1,v_1)) \le |a||c-c_1|.\int_0^1 N(t)dt+$$

$$(M\int_0^1 N(t)dt+\theta)d_{L^1(I,\mathbf{R})\times L^1(I,\mathbf{R})}((u,v),(u_1,v_1)),$$

namely, the multifunction ψ is Hausdorff continuous and satisfies the hypothesis of Lemma 4.1.

Define $\Gamma(c,u)=\psi(c,u,\phi(c,u)),(c,u)\in\mathbf{R}\times L^1(I,\mathbf{R})$. According to Lemma 4.1, the set $Fix(\Gamma(c,.))=\{u\in L^1(I,\mathbf{R}); u\in\Gamma(c,u)\}$ is nonempty and arcwise connected in $L^1(I,\mathbf{R})$. Moreover, for fixed $c_i\in\mathbf{R}$ and $v_i\in Fix(\Gamma(c_i,.)), i=1,...,p$, there exists a continuous function $\gamma:\mathbf{R}\to L^1(I,\mathbf{R})$ such that

$$\gamma(c)\in Fix(\Gamma(c,.)),\quad \forall c\in\mathbf{R}, \tag{4.3}$$

$$\gamma(c_i)=v_i,\quad i=1,...,p. \tag{4.4}$$

We shall prove that

$$Fix(\Gamma(c,.))=\{u\in L^1(I,\mathbf{R});\quad u(t)\in F(t,u_c(t),H(t,u_c(t)))\quad a.e.\ (I)\}. \tag{4.5}$$

Denote by $A(c)$ the right-hand side of (4.5). If $u\in Fix(\Gamma(c,.))$ then there is $v\in\phi(c,v)$ such that $u\in\psi(c,u,v)$. Therefore, $v(t)\in H(t,u_c(t))$ and

$$u(t)\in F(t,u_c(t),v(t))\subset F(t,u_c(t),H(t,u_c(t)))\quad a.e.\ (I),$$

so that $Fix(\Gamma(c,.))\subset A(c)$.

Let now $u\in A(c)$. By Lemma 4.2, there exists a selection $v\in L^1(I,\mathbf{R})$ of the multifunction $t\to H(t,u_c(t)))$ satisfying

$$u(t)\in F(t,u_c(t),v(t))\quad a.e.\ (I).$$

Hence, $v\in\phi(c,v)$, $u\in\psi(c,u,v)$ and thus $u\in\Gamma(c,u)$, which completes the proof of (4.5).

We next note that the function $\mathcal{T}: L^1(I,\mathbf{R}) \to C(I,\mathbf{R})$,

$$\mathcal{T}(u)(t) := \int_0^1 G(t,s)u(s)ds, \quad t \in I$$

is continuous and one has

$$S(c) = ac + \mathcal{T}(Fix(\Gamma(c,.))), \quad c \in \mathbf{R}. \tag{4.6}$$

Since $Fix(\Gamma(c,.))$ is nonempty and arcwise connected in $L^1(I,\mathbf{R})$, the set $S(c)$ has the same properties in $C(I,\mathbf{R})$.

2) Let $c_i \in \mathbf{R}$ and let $u_i \in S(c_i), i = 1,...,p$ be fixed. By (4.6) there exists $v_i \in Fix(\Gamma(c_i,.))$ such that

$$u_i = ac_i + \mathcal{T}(v_i), \quad i = 1,...,p.$$

If $\gamma : \mathbf{R} \to L^1(I,\mathbf{R})$ is a continuous function satisfying (4.3) and (4.4) we define, for every $c \in \mathbf{R}$,

$$s(c) = ac + \mathcal{T}(\gamma(c)).$$

Obviously, the function $s : \mathbf{R} \to C(I,\mathbf{R})$ is continuous, $s(c) \in S(c)$ for all $c \in \mathbf{R}$, and

$$s(c_i) = ac_i + \mathcal{T}(\gamma(c_i)) = ac_i + \mathcal{T}(v_i) = u_i, \quad i = 1,...,p.$$

3) Let $u_1, u_2 \in S = \cup_{c\in\mathbf{R}} S(c)$ and choose $c_i \in \mathbf{R}$, $i = 1,2$ such that $u_i \in S(c_i)$, $i = 1,2$. From the conclusion of 2) we deduce the existence of a continuous function $s : \mathbf{R} \to C(I,\mathbf{R})$ satisfying $s(c_i) = u_i$, $i = 1,2$ and $s(c) \in S(c)$, $c \in \mathbf{R}$. Let $h : [0,1] \to \mathbf{R}$ be a continuous mapping such that $h(0) = c_1$ and $h(1) = c_2$. Then the function $s \circ h : [0,1] \to C(I,\mathbf{R})$ is continuous and verifies

$$s \circ h(0) = u_1, \quad s \circ h(1) = u_2, \quad s \circ h(\tau) \in S(h(\tau)) \subset S, \quad \tau \in [0,1].$$

As an example we consider problem (4.1) defined by $F(.,.,.) : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$, $H(.,.) : [0,1] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ with

$$F(t,x,y) = [-\frac{1}{4M}\frac{|x|}{1+|x|}, 0] \cup [0, \frac{1}{4}\frac{|y|}{1+|y|}],$$

$H(t,x) = \{\frac{1}{4Me^{t+1}(1+|x|)}\}$ and with any nonlocal conditions (4.2). A straightforward computation shows that $m(t) \equiv \frac{1}{4M}$, $\theta = \frac{1}{4}$, $l(t) \equiv \frac{1}{4Me}$. In this case, $2M\int_0^1 N(s)ds + \theta = 2M\int_0^1 \frac{1}{4M}ds + \frac{1}{4} < 1$. Then, if for every $c \in \mathbf{R}$ we denote by $S(c)$ the solution set of (4.1)-(4.2), by Theorem 4.4, $S(c)$ is arcwise connected in the space $C(I,\mathbf{R})$.

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