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THE EXISTENCE OF THE STABILIZING SOLUTION OF THE RICCATI EQUATION ARISING IN DISCRETE-TIME STOCHASTIC ZERO SUM LQ DYNAMIC GAMES WITH PERIODIC COEFFICIENTS*

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Abstract

We investigate the problem for solving a discrete-time periodic generalized Riccati equation with an indefinite sign of the quadratic term. A necessary condition for the existence of bounded and stabilizing solution of the discrete-time Riccati equation with an indefinite quadratic term is derived. The stabilizing solution is positive semidefinite and satisfies the introduced sign conditions. The proposed condition is illustrated via a numerical example.

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1 Introduction

The aim of the present paper is to study the global solutions for a Riccati equation arising in discrete-time stochastic zero sum LQ dynamic games with periodic coefficients. The theory of linear quadratic differential games based on Riccati equations has been received much attention [15]. The increasing interest in investigation of control problems for systems with periodic coefficients can be viewed in [1, 2, 4, 13]. The infinite time horizon nonzero-sum linear quadratic differential games of stochastic systems is discussed in [17].

The investigations presented in this paper can be considered as an extension of the results published in a paper from the same journal [9]. In this previous work some iterative methods for computing the stabilizing solution of the discrete-time generalized Riccati equations with definite quadratic part were developed. Here, we consider the discrete-time Riccati equation with an indefinite quadratic term and moreover, we will discuss some details reporting the role of the stabilizing solution of the considered Riccati type equation satisfying the appropriate sign conditions in derivation of the equilibrium strategy in the case of a discrete time zero sum game with periodic coefficients.

2 The problem

2.1 Model description

On the space $\mathcal{S}_n$ of the symmetric matrices of size $n \times n$ we consider the discrete time backward nonlinear equation

$$\begin{aligned} X(t) &= \Pi_1(t)[X(t+1)] + M(t) - (L(t) + \Pi_2(t)[X(t+1)]) \\ &\quad \times (R(t) + \Pi_3(t)[X(t+1)])^{-1}(L(t) + \Pi_2(t)[X(t+1)])^T \end{aligned} \tag{1}$$

$t \in \mathbb{Z}_+ = \{0, 1, ...\}$ where the operators $X \to \Pi_j(t)[X], j = 1, 2, 3$ are described by

$$\Pi_1(t)[X] = \sum_{k=0}^{r} A_k^T(t) X A_k(t)\,, \quad \Pi_2(t)[X] = \sum_{k=0}^{r} A_k^T(t) X B_k(t)\,, \tag{2}$$

$$\Pi_3(t)[X] = \sum_{k=0}^{r} B_k^T(t) X B_k(t)$$

for all $X \in \mathcal{S}_n$. This kind of discrete-time nonlinear equations occurs in connection with the solution of a linear quadratic control problem described by a controlled system of the form:

$$x(t+1) = A_0(t)x(t) + B_0(t)u(t) + \sum_{k=1}^{r} w_k(t)(A_k(t)x(t) + B_k(t)u(t)),$$
$$t \geq 0, \quad x(0) = x_0 \tag{3}$$

$x_0 \in \mathbb{R}^n$ and the quadratic functional

$$J(x_0, u) = E[\sum_{t=0}^{\infty} \begin{pmatrix} x_u^T(t) & u^T(t) \end{pmatrix} \mathbb{Q}(t) \begin{pmatrix} x_u^T(t) & u^T(t) \end{pmatrix}^T] \tag{4}$$

where

$$\mathbb{Q}(t) = \begin{pmatrix} M(t) & L(t) \\ L^T(t) & R(t) \end{pmatrix}. \tag{5}$$

In (3) $\{w(t)\}_{t\geq 0}$ $(w(t) = (w_1(t), ..., w_r(t))^T)$ is a sequence of r-dimensional independent random vectors defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and having the properties that

$$E[w(t)] = 0, \ \ E[w(t)w^T(t)] = I_r, \quad t \in \mathbb{Z}_+.$$

Throughout the presentation $E[\cdot]$ denotes the mathematical expectation and the superscript T stands for the transpose of a vector or a matrix.
In (4) $x_u(t)$, $t \geq 0$ is the solution of the problem with given initial value (3) corresponding to the input $u = \{u(t)\}_{t\geq 0} \subset \mathbb{R}^m$.
We recall that $\ell_w^2\{\mathbb{Z}_+, \mathbb{R}^m\}$ denotes the linear space of the stochastic processes $u = \{u(t)\}_{t\geq 0}$ with the properties:

a) for each $t \geq 0$, $u(t) : \Omega \to \mathbb{R}^m$ is measurable with respect to the sigma algebra $\mathcal{F}_t$ where $\mathcal{F}_t$ is the sigma algebra generated by the random vectors $w(s)$, $0 \leq s \leq t-1$, if $t \geq 1$ and $\mathcal{F}_0 = \{\Omega, \emptyset\}$;

b) $\sum_{t=0}^{\infty} E[|u(t)|^2] < \infty$.

The class of admissible inputs $\mathcal{U}_{adm}(x_0)$ consists of all stochastic processes $u \in \ell_w^2\{\mathbb{Z}_+, \mathbb{R}^m\}$ with the additional properties

$$\begin{aligned} &\alpha) \qquad -\infty < J(x_0, u) < \infty \\ &\beta) \qquad \lim_{t\to\infty} E[|x_u(t)|^2] = 0. \end{aligned} \tag{6}$$

Regarding the sequences $\{A_k(t)\}_{t\in\mathbb{Z}_+}$, $\{B_k(t)\}_{t\in\mathbb{Z}_+}$, $\{M(t)\}_{t\in\mathbb{Z}_+}$, $\{L(t)\}_{t\in\mathbb{Z}_+}$, $\{R(t)\}_{t\in\mathbb{Z}_+}$ we make the assumptions:

$\mathbf{H_1}$) *(i) there exists an integer $\theta \geq 1$ such that $A_k(t+\theta) = A_k(t)$, $B_k(t+\theta) = B_k(t)$, $0 \leq k \leq r$, $M(t+\theta) = M(t)$, $L(t+\theta) = L(t)$, $R(t+\theta) = R(t)$, $\forall t \in \mathbb{Z}_+$;*

(ii) $M(t) = M^T(t)$, $R(t) = R^T(t)$, $t \in \mathbb{Z}_+$.

Since, in the special case $r = 0$ the nonlinear equation (1) reduces to the well known discrete time Riccati equation involved in the solution of some linear quadratic control problems for the deterministic framework, we shall call equation (1) with $r \geq 1$, discrete-time Riccati equation of stochastic control (SDTRE).

The SDTRE (1) can be written in a compact form as

$$X(t) = \mathcal{R}(t, X(t+1)) \tag{7}$$

where $(t, X) \to \mathcal{R}(t, X) : Dom\mathcal{R} \to \mathcal{S}_n$ is the Riccati operator described by

$$\begin{aligned}\mathcal{R}(t, X) = \Pi_1(t)[X] - (L(t) + \Pi_2(t)[X])\\ \times(R(t) + \Pi_3(t)[X])^{-1}(L(t) + \Pi_2(t)[X])^T + M(t)\end{aligned} \tag{8}$$

and $Dom\mathcal{R} = \{(t, X) \in \mathbb{Z}\times\mathcal{S}_n | det(R(t)+\Pi_3(t)[X]) \neq 0\}$. One sees that if $\tilde{X}$ is such that $(t, \tilde{X}) \in Dom\mathcal{R}$ then the operator valued function $X \to \mathcal{R}(t, X)$ is Frechet differentiable at $X = \tilde{X}$. By direct calculations one obtains that the Frechet derivative $\mathcal{R}_X(t, \tilde{X})$ is given by

$$\mathcal{R}_X(t, \tilde{X})Y = \sum_{k=0}^{r}(A_k(t) + B_k(t)F^{\tilde{X}}(t))^T Y(A_k(t) + B_k(t)F^{\tilde{X}}(t)) \tag{9}$$

for all $Y \in \mathcal{S}_n$, where

$$F^{\tilde{X}}(t) = -(R(t) + \Pi_3(t)[\tilde{X}])^{-1}(L(t) + \Pi_2(t)[\tilde{X}])^T \tag{10}$$

2.2 The stabilizing solution of SDTRE

For a sequence of matrices $\{F(t)\}_{t\in\mathbb{Z}_+} \subset \mathbb{R}^{m\times n}$ we introduce the sequence of linear operators $\{\mathcal{L}_F(t)\}_{t\in\mathbb{Z}_+}$ as:

$$\mathcal{L}_F(t)[X] = \sum_{k=0}^{r}(A_k(t) + B_k(t)F(t))X(A_k(t) + B_k(t)F(t))^T. \tag{11}$$

On the space $\mathcal{S}_n$ we introduce the inner product

$$< X, Y >= Tr[XY] \tag{12}$$

$\forall X, Y \in \mathcal{S}_n$. The space $\mathcal{S}_n$ equipped with the inner product (12) becomes a finite dimensional real Hilbert space.

By direct calculation one obtains that the adjoint of the linear operator $Y \to \mathcal{R}_X(t, \tilde{X})Y$ with respect to the inner product (12) is given by

$$\mathcal{R}_X^*(t, \tilde{X})Y = \sum_{k=0}^{r}(A_k(t) + B_k(t)F^{\tilde{X}}(t))Y(A_k(t) + B_k(t)F^{\tilde{X}}(t))^T \qquad (13)$$

for all $Y \in \mathcal{S}_n$.
Now we are in position to introduce the concept of stabilizing solution of SDTRE (1).

Definition 1. A global solution $\{X_s(t)\}_{t\in\mathbb{Z}}$ of (1) is named **stabilizing solution** if the zero solution of the discrete time linear equation on $\mathcal{S}_n$:

$$Z(t+1) = \mathcal{L}_{F_s}(t)[Z(t)] \qquad (14)$$

is exponentially stable, where $\mathcal{L}_{F_s}(t)$ is the linear operator of type (11) associated to the gain matrix

$$F_s(t) = -(R(t) + \Pi_3(t)[X_s(t+1)])^{-1}(L(t) + \Pi_2(t)[X_s(t+1)])^T, \quad t \in \mathbb{Z}. \qquad (15)$$

The next result provides equivalent definitions of the concept of stabilizing solution of the equation (1).

Proposition 1. *The following are equivalent:*

(i) $\{X_s(t)\}_{t\in\mathbb{Z}}$ *is stabilizing solution of SDTRE (1) in the sense of Definition 1;*

(ii) the zero solution of the discrete-time linear equation on $\mathcal{S}_n$*:*

$$Y(t+1) = \mathcal{R}_X^*(t, X_s(t+1))Y(t)$$

is exponentially stable;

(iii) the zero solution of the discrete time linear stochastic equation

$$x(t+1) = [A_0(t) + B_0(t)F_s(t) + \sum_{k=1}^{r} w_k(t)(A_k(t) + B_k(t)F_s(t))]x(t) \qquad (16)$$

(obtained plugging $u(t) = F_s(t)x(t)$ *is (3)) is exponentially stable in mean square (ESMS).*

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows from the fact that $\mathcal{R}_X^*(t, X_s(t+1))$ coincides with $\mathcal{L}_{F_s}(t)$. The equivalence $(i) \Leftrightarrow (iii)$ is just the equivalence between the property of ESMS of (16) and the exponential stability of the linear equation defined by the sequence of Lyapunov type operators associated to this linear stochastic equation. □

The next result provides two important properties of the stabilizing solution of a SDTRE of type (1).

Theorem 2. *(i) The SDTRE of type (1) has at most one bounded and stabilizing solution.*

(ii) Under the assumption $\mathbf{H_1}$) *the unique bounded and stabilizing solution of SDTRE (1) (if any) is a periodic sequence with period* θ.

Proof. (i) Let us assume that the SDTRE (1) has two bounded and stabilizing solutions $\{X^k(t)\}_{t\in\mathbb{Z}}$, $k=1,2$ and let $F_s^k(t)$ be the corresponding stabilizing feedback gains associated as in (15) when $X_s(t)$ is replace by $X^k(t)$. Let $\Delta(t) = X^1(t) - X^2(t)$, $t \in \mathbb{Z}$. By direct calculation one obtains that $\Delta(\cdot)$ is a bounded solution of the discrete time backward linear equation:

$$\Delta(t) = \sum_{k=0}^{r}(A_k(t) + B_k(t)F_s^1(t))^T\Delta(t+1)(A_k(t) + B_k(t)F_s^2(t)), \quad t \in \mathbb{Z}. \tag{17}$$

Employing the property of ESMS of the linear stochastic equations

$$x(t+1) = [A_0(t) + B_0(t)F_s^j(t) + \sum_{k=1}^{r} w_k(t)(A_k(t) + B_k(t)F_s^j(t))]x(t)$$

$j = 1,2$ one shows that the equation (17) has a unique bounded solution. This allows us to conclude that $X^1(t) - X^2(t) = 0$, $t \in \mathbb{Z}$, which confirms the uniqueness of the bounded and stabilizing solution of the Riccati equation (1).

(ii) If $\{X_s(t)\}_{t\in\mathbb{Z}}$ is the bounded and stabilizing solution of the equation (1) we set $\hat{X}_s(t) = X_s(t+\theta)$, $t \in \mathbb{Z}$. If the assumption $\mathbf{H_1}$) is fulfilled one shows that $\{\hat{X}_s(t)\}_{t\in\mathbb{Z}}$ is also a bounded and stabilizing solution of SDTRE (1). From the uniqueness of the bounded and stabilizing solution it follows that $\hat{X}_s(t) = X_s(t)$, $t \in \mathbb{Z}$, i.e. $X_s(t+\theta) = X_s(t)$, $t \in \mathbb{Z}$. □

In order to point out the role of the bounded and stabilizing solution of SDTRE (1) in the construction of the optimal control of a linear quadratic control problem described by the control system (3) and the performance index (4), we recall the following:

Proposition 3. *If* $\{X_s(t)\}_{t\in\mathbb{Z}}$ *is the bounded and stabilizing solution of SDTRE (1) and* $F_s(t)$ *is the corresponding stabilizing feedback gain, then the following hold:*

(i) the performance index (4) may be rewritten as:

$$J(x_0,u) = x_0^T X_s(0)x_0 + \sum_{t=0}^{\infty} E[(u(t) - F_s(t)x_u(t))^T \\ \times(R(t) + \Pi_3(t)[X_s(t+1)])(u(t) - F_s(t)x_u(t))] \tag{18}$$

for all $u \in \mathcal{U}_{adm}(x_0)$*;*

(ii) let

$$\tilde{u}(t) = F_s(t)\tilde{x}(t) \tag{19}$$

where $\tilde{x}(t)$, $t \geq 0$ *is the solution of equation (16) satisfying the initial condition* $\tilde{x}(0) = x_0$.

Under these conditions, $\tilde{u} \in \mathcal{U}_{adm}(x_0)$ *and* $x_{\tilde{u}}(t) = \tilde{x}(t)$, $t \geq 0$.

From (18) one sees that a fundamental role in the establishment of the type of the optimal control problem described by the system (3) and the performance index (4) is played by the signature of the matrices

$$\mathbb{R}(t, X_s(t+1)) = \mathbb{R}(t) + \Pi_3(t)[X_s(t+1)], \ \ t \geq 0. \tag{20}$$

We distinguish the following important cases:

1) There exists $\nu > 0$ such that

$$\mathbb{R}(t, X_s(t+1)) \geq \nu I_m, \forall t \geq 0. \tag{21}$$

In this case, (18) yields:

$$J(x_0,u) \geq J(x_0,\tilde{u}) = x_0^T X_s(0)x_0, \ \ \forall u \in \mathcal{U}_{adm}(x_0)$$

and $\forall x_0 \in \mathbb{R}^n$. This means that $\tilde{u}(\cdot)$ defined in (19) is the unique control which is minimizing the cost functional (4).

2) There exists $\nu > 0$ such that

$$\mathbb{R}(t, X_s(t+1)) \leq -\nu I_m, \ \ \forall t \geq 0. \tag{22}$$

In this case (18) leads to $J(x_0,u) \leq J(x_0,\tilde{u}) = x_0^T X_s(0)x_0$, $\forall u \in \mathcal{U}_{adm}(x_0)$, $x_0 \in \mathbb{R}^n$. In this case, $\tilde{u}$ provides the maximal value of the cost functional (4).

3) For each $t \in \mathbb{Z}_+$ the matrix $\mathbb{R}(t, X_s(t+1))$ has m_1 negative eigenvalues and m_2 positive eigenvalues where $m_k \geq 1$, $k = 1, 2$, do not depend upon t and $m_1 + m_2 = m$ the number of the inputs of (3). In this case, the linear quadratic control problem described by the controlled system (3) and the performance index (4) is a problem of discrete-time linear quadratic

dynamic games. This kind of optimal control problem will be analysed in detail in the next section.

Remark 1. a) It must be noted that we do not know apriori the initial values of some time instance t_0 of the bounded and stabilizing solution of the Riccati equation (1). That is why the problem of the existence of such a solution is a difficult problem. Unlike the problem of the uniqueness of the bounded and stabilizing solution of a SDTRE of type (1), the problem of the existence of this kind of solution can be solved taking into account the sign conditions which we want to be satisfied by the matrices from (20). The necessary and sufficient conditions for the existence of the bounded and stabilizing solution of a Riccati equation of type (1) satisfying sign conditions of type (21) may be found, for example, in Theorem 5.12 in [7]. Necessary and sufficient conditions for the existence of the bounded and stabilizing solution of (1) satisfying sign conditions of type (22) may be deduced from those obtained in the case when the conditions of type (21) are satisfied. In Section 4, we will provide a set of sufficient conditions to guarantee the existence of the bounded and stabilizing solution $X_s(\cdot)$ of a Riccati equation of type (1) with the property that the matrices $\mathbb{R}(t, X_s(t+1))$ have indefinite sign.

b) From the uniqueness of the bounded and stabilizing solution one deduces that in the case of a linear quadratic control problem described by a controlled system of type (3) and a performance index of type (4) the bounded and stabilizing solution SDTRE (1) (if any) will satisfy at most one of the sign conditions discussed above. Hence, for a given controlled system and a given performance index only one of the optimal control problems displayed above (minimization, maximization or dynamic game) may be well posed.

3 Discrete-time zero sum LQ dynamic games

Let us assume that based on the some practical considerations, the inputs of the system (3) are partitioned as follows:

$$u(t) = (u_1(t), u_2(t)), \tag{23}$$

where $u_j(t) \in \mathbb{R}^{m_j}$, $j = 1, 2$. Correspondingly we have the partitions:

$$B_k(t) = \begin{pmatrix} B_{k1}(t) & B_{k2}(t) \end{pmatrix}, \quad L(t) = \begin{pmatrix} L_1(t) & L_2(t) \end{pmatrix} \tag{24}$$

$B_{kj}(t), L_j(t) \in \mathbb{R}^{n\times m_j}$, $j = 1, 2, \quad 0 \leq k \leq r$,

$$R(t) = \begin{pmatrix} R_{11}(t) & R_{12}(t) \\ R_{12}^T(t) & R_{22}(t) \end{pmatrix} \tag{25}$$

where $R_{ij}(t) \in \mathbb{R}^{m_i\times m_j}$, $i, j = 1, 2$.
The partitions (24), (25) lead to the following partitions of the operators defined in (2):

$$\begin{aligned} \Pi_2(t)[X] &= \begin{pmatrix} \Pi_{2,1}(t)[X] & \Pi_{2,2}(t)[X] \end{pmatrix} \\ \Pi_3(t)[X] &= \begin{pmatrix} \Pi_{3,11}(t)[X] & \Pi_{3,12}(t)[X] \\ \Pi_{3,21}(t)[X] & \Pi_{3,22}(t)[X] \end{pmatrix} \end{aligned} \tag{26}$$

where $\Pi_{2,j}(t)[X] = \sum\limits_{k=0}^{r} A_k^T(t)XB_{kj}(t)$, $\Pi_{3,ij}(t)[X] = \sum\limits_{k=0}^{r} B_{ki}^T(t)XB_{kj}(t), \quad i, j = 1, 2$.
Employing (23), (24) we rewrite the system (3) as

$$\begin{aligned} x(t+1) &= A_0(t)x(t) + B_{01}(t)u_1(t) + B_{02}(t)u_2(t) + \\ &+ \sum_{k=1}^{r} w_k(t)(A_k(t)x(t) + B_{k1}(t)u_1(t) + B_{k2}(t)u_2(t)). \end{aligned} \tag{27}$$

In order to introduce the terminology used in the game theory, the input $u_k(t)$ will be called STRATEGY (or POLICY) of the k-th player, $k = 1, 2$.

In this section we assume that each player has access to the strategy in a state feedback form, i.e.

$$u_k(t) = F_k(t)x(t) \tag{28}$$

where $F_k(t) \in \mathbb{R}^{m_k\times n}$, $k = 1, 2$. Substituting $u_k(t)$ in (4) using (28) we obtain a new version of the performance index:

$$\begin{aligned} J(x_0, \mathbb{F}_1, \mathbb{F}_2) = \sum_{t=0}^{\infty} E[x_F^T(t)(M(t) + L_1(t)F_1(t) + L_2(t)F_2(t) \\ +F_1^T(t)L_1^T(t) + F_2^T(t)L_2^T(t) + \sum_{i,j=1}^{2} F_i^T(t)R_{ij}(t)F_j(t))x_F(t)] \end{aligned} \tag{29}$$

where $x_F(t)$, $t \geq 0$ is the solution of the closed loop system:

$$\begin{aligned} x(t+1) &= (A_0(t) + B_{01}(t)F_1(t) + B_{02}(t)F_2(t))x(t + \\ &+ (\sum_{k=1}^{r} w_k(t)(A_k(t) + B_{k1}(t)F_1(t) + B_{k2}(t)F_2(t)))x(t) \end{aligned} \tag{30}$$

satisfying $x_F(0) = x_0$ and $\mathbb{F}_k = \{F_k(t)\}_{t\in\mathbb{Z}_+}$, $k = 1, 2$.
By $\mathcal{F}_{adm}$ we denote the set of the θ-periodic sequences $(\mathbb{F}_1, \mathbb{F}_2) = \{(F_1(t), F_2(t))\}_{t\in\mathbb{Z}}$ with $F_k(t) \in \mathbb{R}^{m_k\times n}$, $k = 1, 2$ and having the property that the corresponding system (30) is ESMS.

Definition 2. We say that $(\tilde{\mathbb{F}}_1, \tilde{\mathbb{F}}_2) \in \mathcal{F}_{adm}$ or equivalently

$$(\tilde{u}_1(t), \tilde{u}_2(t)) = (\tilde{F}_1(t)\tilde{x}(t), \tilde{F}_2(t)\tilde{x}(t))$$

is a closed-loop equilibrium strategy of the discrete-time zero-sum dynamic games described by the dynamic system (27) and the performance index (29) if

$$J(x_0, \mathbb{F}_1, \tilde{\mathbb{F}}_2) \le J(x_0, \tilde{\mathbb{F}}_1, \tilde{\mathbb{F}}_2) \le J(x_0, \tilde{\mathbb{F}}_1, \mathbb{F}_2) \tag{31}$$

for all $(\mathbb{F}_1, \tilde{\mathbb{F}}_2)$ and $(\tilde{\mathbb{F}}_1, \mathbb{F}_2)$ lie in $\mathcal{F}_{adm}$.

From (31) one sees that the aim of the player which computes the strategy $u_2(t)$ is to minimize the performance index (29), while the player which computes the strategy $u_1(t)$ wants to maximize the same performance index.

The next result points out the role of the bounded and stabilizing solution of SDTRE (1) in the construction of a closed-loop equilibrium strategy for a discrete time zero sum dynamic game.

Theorem 4. *Assume: a) the assumption* $\mathbf{H_1}$) *is fulfilled.*

b) the SDTRE (1) has a bounded and stabilizing solution $\{X_s(t)\}_{t\in\mathbb{Z}}$ *satisfying the following sign conditions:*

$$R_{11}(t) + \sum_{k=0}^{r} B_{k1}^T(t)X_s(t+1)B_{k1}(t) < 0 \tag{32}$$

$$R_{22}(t) + \sum_{k=0}^{r} B_{k2}^T(t)X_s(t+1)B_{k2}(t) > 0 \tag{33}$$

$0 \le t \le \theta - 1.$
Let $F_s(t)$ *be the corresponding stabilizing feedback gain associated via (15). We set*

$$F_{s1}(t) = \begin{pmatrix} I_{m_1} & 0 \end{pmatrix} F_s(t), \quad F_{s2}(t) = \begin{pmatrix} 0 & I_{m_2} \end{pmatrix} F_s(t).$$

Under these conditions

$$(\tilde{u}_1(t), \tilde{u}_2(t)) = (F_{s1}(t)\tilde{x}(t), F_{s2}(t)\tilde{x}(t)) \tag{34}$$

is a closed-loop equilibrium strategy for the zero-sum LQ dynamic game described by the system (27) and the performance index (29).

Proof. Employing (18), (25), (26), (32)-(34) we get

$$\begin{aligned} & J(x_0,\mathbb{F}_1,\tilde{\mathbb{F}}_2) = \sum_{t=0}^{\infty} E\left[((F_1(t)-F_{s1}(t))\hat{x}(t))^T \right.\\ & \left.\times(R_{11}(t)+\Pi_{3,11}(t)[X_s(t+1)])((F_1(t)-F_{s1}(t))\hat{x}(t))\right] + x_0^T X_s(0)x_0 \\ \le\ & x_0^T X_s(0)x_0 \ = \ J(x_0,\tilde{\mathbb{F}}_1,\tilde{\mathbb{F}}_2) \\ \le\ & x_0^T X_s(0)x_0 + \sum_{t=0}^{\infty} E[((F_2(t)-F_{s2}(t))\check{x}(t))^T \\ & \times(R_{22}(t)+\Pi_{3,22}(t)[X_s(t+1)])((F_2(t)-F_{s2}(t))\check{x}(t))] = J(x_0,\tilde{\mathbb{F}}_1,\mathbb{F}_2) \end{aligned}$$

for all $(\mathbb{F}_1,\mathbb{F}_2)\in\mathcal{F}_{adm}$ such that $(\mathbb{F}_1,\tilde{\mathbb{F}}_2)$ and $(\tilde{\mathbb{F}}_1,\mathbb{F}_2)$ lie in $\mathcal{F}_{adm}$, where $\tilde{\mathbb{F}}_k=\{F_{sk}(t)\}_{t\in\mathbb{Z}}$, $k=1,2$, $\hat{x}(t)$ is the solution of (30) when $F_2(t)$ is replaced by $F_{s2}(t)$ and $\check{x}(t)$ is the solution of (30) when $F_1(t)$ is replaced by $F_{s1}(t)$. □

From Theorem 4, it follows that it is useful to know conditions which guarantee the existence of the bounded and stabilizing solution of a Riccati equation of type (1) satisfying sign conditions of type (32) and (33). Such conditions will be provided in the next section.

4 A set of sufficient conditions for the existence of the bounded and stabilizing solution of a SDTRE with indefinite sign

Let us denote by $\mathfrak{K}$ the set of sequences $\mathbb{K}=\{K(t)\}_{t\in\mathbb{Z}}$, $K(t)\in\mathbb{R}^{m_2\times n}$, which are periodic of period θ and have the properties:

a) the discrete-time system

$$x(t+1) = [A_0(t)+B_{02}(t)K(t)+\sum_{k=1}^{r} w_k(t)(A_k(t)+B_{k2}(t)K(t))] \qquad (35)$$

is ESMS,

b) the SDTRE

$$\begin{aligned} Y(t) = &\sum_{k=0}^{r} (A_k(t) + B_{k2}(t)K(t))^T Y(t+1)(A_k(t) + B_{k2}(t)K(t)) \\ &-[L_K(t) + \sum_{k=0}^{r} (A_k(t) + B_{k2}(t)K(t))^T Y(t+1)B_{k1}(t)] \\ &\times[R_{11}(t) + \sum_{k=0}^{r} B_{k1}^T(t)Y(t+1)B_{k1}(t)]^{-1} \\ &\times[L_K(t) + \sum_{k=0}^{r} (A_k(t) + B_{k2}(t)K(t))Y(t+1)B_{k1}(t)]^T + M_K(t) \end{aligned} \tag{36}$$

has a bounded and stabilizing solution $\{\tilde{Y}_K(t)\}_{t\in\mathbb{Z}}$ satisfying the sign conditions

$$R_{11}(t) + \sum_{j=0}^{r} B_{j1}^T(t)\tilde{Y}_K(t+1)B_{j1}(t) < 0 \tag{37}$$

$0 \leq t \leq \theta - 1$, with

$$\begin{aligned} L_K(t) &= L_1(t) + K^T(t)R_{12}^T(t) \\ M_K(t) &= \begin{pmatrix} I_n \\ K(t) \end{pmatrix}^T \begin{pmatrix} M(t) & L_2(t) \\ L_2^T(t) & R_{22}(t) \end{pmatrix} \begin{pmatrix} I_n \\ K(t) \end{pmatrix}. \end{aligned} \tag{38}$$

Remark 2. According with Theorem 2 applied in the case of equation (36) we deduce that if the assumption $\mathbf{H_1}$) is fulfilled, then the unique bounded and stabilizing solution of this Riccati equation is a periodic sequence of period θ. That is why, in (37) we have taken $0 \leq t \leq \theta - 1$ instead of $t \in \mathbb{Z}$. For the developments from this section we need the following assumption:

$\mathbf{H_2}$) $R_{22}(t) > 0$ and $M(t) - L_2(t)R_{22}^{-1}(t)L_2^T(t) \geq 0, \quad t \in \mathbb{Z}$.

We associate the following auxiliary system:

$$\begin{aligned} x(t+1) &= (\check{A}_0(t) + \sum_{k=1}^{r} w_k(t)\check{A}_k(t))x(t) \\ y(t) &= \check{C}(t)x(t), \end{aligned} \tag{39}$$

$t \geq 0$, where we have denoted

$$\check{A}_k(t) = A_k(t) - B_{k2}(t)R_{22}^{-1}(t)L_2^T(t)$$

$0 \le k \le r$ and $\check{C}(t)$ are obtained from the factorization

$$(\check{C}(t))^T \check{C}(t) = M(t) - L_2(t) R_{22}^{-1}(t) L_2^T(t).$$

Definition 3. We say that the system (39) is stochastically detectable if there exists a θ periodic sequence $\{H(t)\}_{t\ge 0}$ such that the system

$$x(t+1) = (\check{A}_0(t) + H(t)\check{C}(t) + \sum_{k=1}^{r} w_k(t)\check{A}_k(t))x(t)$$

is ESMS.

Criteria for testing the property of stochastic detectability of the system (39) may be expressed in terms of solvability of some systems of LMIs.

The main result of this section is:

Theorem 5. *Assume: a) the assumptions* $\mathbf{H_1}$) *and* $\mathbf{H_2}$) *are fulfilled,*
b) the system (39) is stochastic detectable,
c) the set $\mathfrak{K}$ *is not empty.*

Under these conditions, the SDTRE (1) has a bounded and stabilizing solution $X_s(\cdot)$ *satisfying the sign conditions (32), (33). Furthermore, we have:* $0 \le X_s(t) \le \hat{X}(t)$, $t \in \mathbb{Z}$, *where* $\hat{X}(\cdot)$ *is an arbitrary positive semi-definite solution of (1) satisfying the sign conditions of type (32), (33).*

Proof may be done following step by step the proof of the main result from [5].

Remark 3. One checks that if the SDTRE (1) has a bounded and stabilizing solution $X_s(\cdot)$ which is positive semidefinite and satisfies the sign conditions (32), (33), then the sequence of feedback gains $\tilde{\mathbb{F}}_2 = \{F_{s2}(t)\}_{t\in\mathbb{Z}}$, lies in $\mathfrak{K}$. So, one sees that the condition c) from the statement of Theorem 5 is also a necessary condition for the existence of bounded and stabilizing solution of the Riccati equation (1) which is positive semidefinite and satisfies the sign conditions of type (32), (33).

The next result allows us to test if the set $\mathfrak{K}$ is not empty.

Proposition 6. *If the assumptions* $\mathbf{H_1}$) *and* $\mathbf{H_2}$) *are fulfilled, then the following are equivalent:*

(i) the set $\mathfrak{K}$ *is not empty,*

(ii) there exist matrices $Z(t) \in \mathcal{S}_n$ *and* $\Gamma(t) \in \mathbb{R}^{m_2\times n}$, $0 \le t \le \theta - 1$

satisfying the following system of LMIs:

$$\begin{pmatrix} \Theta_0(t) & \Psi_0^T(t) & \Psi_1^T(t) & ... & \Psi_r^T(t) & \Psi_{r+1}^T(t) \\ \Psi_0(t) & -Z(t+1) & 0 & ... & 0 & 0 \\ \Psi_1(t) & 0 & -Z(t+1) & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ \Psi_r(t) & 0 & 0 & ... & -Z(t+1) & 0 \\ \Psi_{r+1}(t) & 0 & 0 & ... & 0 & -I_\rho \end{pmatrix} < 0 \quad (40)$$

$0 \le t \le \theta - 1$, *with* $Z(\theta) = Z(0)$, *where*

$$\Theta_0(t) = \begin{pmatrix} -Z(t) & Z(t)L_1(t) + \Gamma^T(t)R_{12}^T(t) \\ L_1^T(t)Z(t) + R_{12}(t)\Gamma(t) & R_{11}(t) \end{pmatrix},$$

$$\Psi_k(t) = \begin{pmatrix} A_k(t)Z(t) + B_{k2}(t)\Gamma(t) & B_{k1}(t) \end{pmatrix}, \ 0 \le k \le r$$

$$\Psi_{r+1}(t) = \begin{pmatrix} U_1(t)Z(t) + U_2(t)\Gamma(t) & 0 \end{pmatrix}$$

$U_1(t), U_2(t)$ *are obtained from the factorization*

$$\begin{pmatrix} U_1(t) & U_2(t) \end{pmatrix}^T \begin{pmatrix} U_1(t) & U_2(t) \end{pmatrix} = \begin{pmatrix} M(t) & L_2(t) \\ L_2^T(t) & R_{22}(t) \end{pmatrix} \triangleq \mathbb{Q}_2(t)$$

and $\rho = \max\limits_{0 \le t \le \theta-1} rank\mathbb{Q}_2(t)$. *Furthermore, if* $Z(t)$, $\Gamma(t)$, $0 \le t \le \theta - 1$ *is a solution of (40), then* $K(t) = \Gamma(t)Z^{-1}(t)$ *lies in* $\mathfrak{K}$.

5 An example

Proposition 6 can be considered as a practical test whether the set $\mathfrak{K}$ is not empty.

We consider the discrete-time equation (1) in the special case $r = 1$:

$$\begin{aligned} X(t) &= \textstyle\sum_{j=0}^1 A_j^T(t)X(t+1)A_j(t) - [\sum_{j=0}^1 A_j^T(t)X(t+1)\begin{pmatrix} B_{j1}(t) & B_{j2}(t) \end{pmatrix}] \\ &\times[R(t) + \textstyle\sum_{j=0}^1 \begin{pmatrix} B_{j1}(t) & B_{j2}(t) \end{pmatrix}^T X(t+1)\begin{pmatrix} B_{j1}(t) & B_{j2}(t) \end{pmatrix}]^{-1} \\ &+[\textstyle\sum_{j=0}^1 A_j^T(t)X(t+1)\begin{pmatrix} B_{j1}(t) & B_{j2}(t) \end{pmatrix}]^T + M(t) \end{aligned} \quad (41)$$

where $R(t) = \begin{pmatrix} R_{11}(t) & R_{12}(t) \\ R_{12}^T(t) & R_{22}(t) \end{pmatrix}$. We are looking for the stabilizing solution of (41) satisfying the sign conditions:

$$R_{11}(t) + \sum_{j=0}^{r} B_{j1}^T(t)X_s(t+1)B_{j1}(t) < 0 \quad (42)$$

$$R_{22}(t)+\sum_{j=0}^{r}B_{j2}^{T}(t)X_s(t+1)B_{j2}(t)>0 \tag{43}$$

$t\in\mathbb{Z}$.

Assuming $\theta=3$ we construct the matrix coefficients:

$$A_0(0)=\begin{pmatrix}-0.12 & 0\\ -0.4 & -0.2\end{pmatrix}, A_0(1)=\begin{pmatrix}-0.02 & -0.8\\ -0.12 & 0\end{pmatrix},$$

$$A_0(2)=\begin{pmatrix}-0.06 & -0.8\\ -0.14 & 0\end{pmatrix},\quad A_1(0)=\begin{pmatrix}-0.02 & -0.8\\ -0.16 & -0.12\end{pmatrix},$$

$$A_1(1)=\begin{pmatrix}-0.08 & -0.04\\ -0.16 & 0.06\end{pmatrix}, A_1(2)=\begin{pmatrix}-0.09 & 0.02\\ -0.03 & -0.05\end{pmatrix},$$

$M(0)=0.012{*}eye(n,n);,\quad M(1)=0.014{*}eye(n,n);,\quad M(2)=0.018{*}eye(n,n);,$

$$B_{01}(0)=\begin{pmatrix}-0.2 & -0.4\\ -0.25 & -0.12\end{pmatrix}, B_{01}(1)=\begin{pmatrix}-0.8 & -0.6\\ -0.5 & 0.9\end{pmatrix},$$

$$B_{01}(2)=\begin{pmatrix}-0.04 & -0.3\\ -0.05 & 0.06\end{pmatrix},\quad B_{02}(0)=\begin{pmatrix}-0.3 & -0.9\\ -1.6 & -1.2\end{pmatrix},$$

$$B_{02}(1)=\begin{pmatrix}-0.04 & -0.07\\ -0.15 & -0.09\end{pmatrix}, B_{02}(2)=\begin{pmatrix}-0.3 & -1.3\\ -0.25 & 1.6\end{pmatrix},$$

$B_0(0)=[B_{01}(0)\ \ B_{02}(0)];,\quad B_0(1)=[B_{01}(1)\ \ B_{02}(1)];,\quad B_0(2)=[B_{01}(2)\ \ B_{02}(2)];,$

$$B_{11}(0)=\begin{pmatrix}-0.8 & -1.4\\ -0.5 & 1\end{pmatrix}, B_{11}(1)=\begin{pmatrix}-0.04 & -0.12\\ -0.05 & 0.1\end{pmatrix},$$

$$B_{11}(2)=\begin{pmatrix}-0.8 & -1.1\\ -0.5 & 1\end{pmatrix},\quad B_{12}(0)=\begin{pmatrix}0.05 & -0.02\\ -0.03 & -0.02\end{pmatrix},$$

$$B_{12}(1)=\begin{pmatrix}-1.4 & -0.35\\ -0.28 & -0.32\end{pmatrix}, B_{12}(2)=\begin{pmatrix}-0.17 & -0.04\\ 0.16 & -0.16\end{pmatrix},$$

$B_1(0)=[B_{11}(0)\ \ B_{12}(0)];,\quad B_1(1)=[B_{11}(1)\ \ B_{12}(1)];,\quad B_1(2)=[B_{11}(2)\ \ B_{12}(2)];,$

$$R_{11}(0)=-\begin{pmatrix}0.05 & 0.01\\ 0.01 & 0.28\end{pmatrix}, R_{11}(1)=-\begin{pmatrix}0.45 & 0.18\\ 0.18 & 0.4\end{pmatrix}, R_{11}(2)=-\begin{pmatrix}0.28 & 0.24\\ 0.24 & 0.35\end{pmatrix},$$

$R_{12}(0)=zeros(2,2);,\quad R_{12}(1)=zeros(2,2);,\quad R_{12}(2)=zeros(2,2);,$

$$R_{22}(0)=\begin{pmatrix}0.65 & 0.55\\ 0.55 & 0.8\end{pmatrix}, R_{22}(1)=\begin{pmatrix}0.14 & 0.15\\ 0.15 & 0.4\end{pmatrix}, R_{22}(2)=\begin{pmatrix}0.2 & 0\\ 0 & 0.2\end{pmatrix},$$

$$R(0) = [R_{11}(0)\ 0;\ \ 0\ R_{22}(0)];,$$
$$R(1) = [R_{11}(1)\ 0;\ \ 0\ R_{22}(1)];,$$
$$R(2) = [R_{11}(2)\ 0;\ \ 0\ R_{22}(2)];,$$

$$L_1(0) = zeros(4,2);, \quad L_2(0) = zeros(4,2);,$$
$$L_1(1) = zeros(4,2);, \quad L_2(1) = zeros(4,2);,$$
$$L_1(2) = zeros(4,2);, \quad L_2(2) = zeros(4,2);,$$

$$L(0) = [L_1(0)\ \ L_2(0)];, \quad L(1) = [L_1(1)\ \ L_2(1)];, \quad L(2) = [L_1(2)\ \ L_2(2)];,$$

The inequality (40) has the form for the above example $r = 1, k = 0, 1$

$$\begin{pmatrix} \Theta_0(t) & \Psi_0^T(t) & \Psi_1^T(t) & \Psi_2^T(t) \\ \Psi_0(t) & -Z(t+1) & 0 & 0 \\ \Psi_1(t) & 0 & -Z(t+1) & 0 \\ \Psi_2(t) & 0 & 0 & -I_\rho \end{pmatrix} < 0 \tag{44}$$

Solving the set of inequalities (44) with $t = 0, 1, 2$ we obtain:

$$Z(0) = \begin{pmatrix} 44.6198 & -7.3326 \\ -7.3326 & 24.3876 \end{pmatrix}, \quad Z(1) = \begin{pmatrix} 54.1656 & 1.3032 \\ 1.3032 & 28.5697 \end{pmatrix},$$

$$Z(2) = \begin{pmatrix} 42.8520 & -0.5892 \\ -0.5892 & 23.3102 \end{pmatrix}, \quad \Gamma(0) = \begin{pmatrix} -1.4582 & -0.2774 \\ 0.0822 & -0.0908 \end{pmatrix},$$

$$\Gamma(1) = \begin{pmatrix} -1.8313 & -0.2965 \\ 0.2880 & 0.0471 \end{pmatrix}, \quad \Gamma(2) = \begin{pmatrix} -1.1588 & -0.1498 \\ -0.7251 & -0.1350 \end{pmatrix},$$

6 Conclusion

We present an open problem. Find a set of sufficient conditions which avoid the restrictive assumption $\mathbf{H_2}$) for the existence of a bounded and stabilizing solution of SDTRE (1) satisfying the sign conditions of type (32)-(33).

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