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PERTURBATION ESTIMATES FOR THE MAXIMAL SOLUTION OF A NONLINEAR MATRIX EQUATION*

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Abstract

In this paper a nonlinear matrix equation is considered. Perturbation estimations for the maximal solution of the considered equation are obtained. The results are illustrated by the use of numerical examples.

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1 Introduction

We consider the nonlinear matrix equation

$$X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q, \tag{1}$$

where A_i, $i = 1, 2, \ldots, m$, Q, are $n \times n$ complex matrices with Q Hermitian positive definite, and A^* is the conjugate transpose of a matrix A.

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Eq. (1) can be reduced to

$$Y + \sum_{i=1}^{m} B_i^* Y^{-1} B_i = I, \tag{2}$$

where I is the identity matrix.

Eq. (1) is investigated by many authors [1, 2, 3, 4, 5, 6, 7] in cases of $m = 1$, and $m = 2$ [8, 9, 10, 11].

In general case of $m > 1$ Eq. (2) is introduced by He and Long [12]. They proposed a basic fixed point iteration and an inversion free variant method for finding the maximal positive definite solution of Eq. (2). Hasanov and Hakkaev [13] considered the Newton's method for Eq. (2) and in [14] give convergence rates of some iterative methods for finding the malximal solution of Eq. (2). Duan et al. [15] give a perturbation estimate for the maximal positive definite solution of Eq. (2) based on the matrix differentiation.

Moreover, Liu and Chen [16] investigated more general equation $Y^s + \sum_{i=1}^{m} A_i^* Y^{-t_i} A_i = Q$, where s is a positive integer, and $0 < t_i \leq 1$, $i = 1, 2, \ldots, m$. By using the denotes $X := Y^s$ and $q_i := t_i/s$ the above equation can be reduced to

$$X + \sum_{i=1}^{m} A_i^* X^{-q_i} A_i = Q, \quad (0 < q_i \leq 1). \tag{3}$$

Al-Dubiban [17] considered Eq. (3) in case of $Q = I$ and q_i are positive integers. Yin et al. [18] obtained perturbation estimate of the maximal positive definite solution studied of Eq. (3) in case of $q_i = q_2 = \cdots = q_m$. Konstantinov et al. [19, 20] made a perturbation analysis of the matrix equation $A_0 + \sum_{i=1}^{k} \sigma_i A_i^* X^{p_i} A_i = 0$, $\sigma_i = \pm 1$, where p_i are real numbers. They obtained local and non-local perturbation estimations which depend on the maximal solution. In [11] we generalized the estimation by Xu [5] in case of $m = 2$, which does not require the solution to the perturbed or the unperturbed equations. Moreover, Eq. (3), in case of $m = 1$ and q_1 is a positive integer, is considered in many papers (see [21] and references therein, for example).

The paper is organized as follows: in Section 2, we give some preliminaries for the perturbation analysis. Generalizations of the estimation by Xu [5] for arbitrary m and the estimation by Duan et al. [15] for $Q > 0$ and with slight improvement are obtained in Section 3. Moreover, in this section we compare conditions which are applicable to our estimation and the estimation by Yin et al. [18]. In Section 4, we give some numerical experiments.

Throughout the paper, the symbol $\|A\|$ denotes the spectral norm of a matrix A. The notation $A > 0$ ($A \geq 0$) means that A is a Hermitian positive definite (semidefinite) matrix. For Hermitian matrices A and B, we write $A > B$ ($A \geq B$) if $A - B > 0$ ($A - B \geq 0$). A Hermitian solution X_L of a matrix equation is called maximal if $X_L \geq X$ for any Hermitian solution X of the equation.

2 Statement of the problem and preliminaries

In this section we present some preliminary results.

It's well-known that, if $\sum_{i=1}^{m} \|B_i\|^2 < \frac{1}{4}$, then Eq. (2) has maximal positive definite solution Y_L. Moreover, $\frac{1}{2}I < Y_L \leq I$ and it's an unique solution with these properties (see [8]).

Therefore, $\|Y_L^{-1}\| < 2$ and $\frac{1}{2} \leq \|Y_L\| \leq 1$.

Yin et al. [18] generalized above result for Eq. (3) with $q_i = q$, which is for $q = 1$ as follows:

Lemma 1. *If* $\|Q^{-1}\|^2 \sum_{i=1}^{m} \|A_i\|^2 < \frac{1}{4}$*, then the maximal solution* X_L *of Eq.* (1) *exists and satisfies*

$$\|X_L^{-1}\| < 2\|Q^{-1}\| \quad \text{and} \quad \frac{1}{2}\|Q\| \leq \|X_L\| \leq \|Q\|.$$

Let the matrix coefficients A_i, $i = 1, 2, \ldots, m$ and Q in Eq. (1) be slightly perturbed as $\tilde{A}_i := A_i + \Delta A_i$, $i = 1, 2, \ldots, m$ and $\tilde{Q} := Q + \Delta Q$, where ΔA_i and ΔQ are the perturbations in the matrix coefficients, respectively. We consider the perturbed equation

$$\tilde{X} + \sum_{i=1}^{m} \tilde{A}_i^* \tilde{X}^{-1} \tilde{A}_i = \tilde{Q}, \tag{4}$$

with maximal solution $\tilde{X}_L = X_L + \Delta X_L$, where ΔX_L is the perturbation of the maximal solution X_L of Eq. (1).

In this way Duan et al. [15] considered Eq. (2) and its perturbed equation

$$\tilde{Y} + \sum_{i=1}^{m} \tilde{B}_i^* \tilde{Y}^{-1} \tilde{B}_i = I, \tag{5}$$

where $\tilde{B}_i = B_i + \Delta B_i$, $i = 1, 2, \ldots, m$, and obtained the following result:

Theorem 1. [15, Theorem 3.1] *If*

(i) $\sum_{i=1}^{m} \|B_i\|^2 < 1/4$,

(ii) $2\sum_{i=1}^{m} \|B_i\|\|\Delta B_i\| + \sum_{i=1}^{m} \|\Delta B_i\|^2 < \frac{1}{4} - \sum_{i=1}^{m} \|B_i\|^2$,

then equations (2) and (5) have maximal positive definite solutions Y_L and $\tilde{Y}_L$, respectively, and

$$\|\tilde{Y}_L - Y_L\| \leq \frac{4\sum_{i=1}^{m}(\|B_i\| + \|\Delta B_i\|)\|\Delta B_i\|}{1 - 4\sum_{i=1}^{m}(\|B_i\| + \|\Delta B_i\|)^2} =: S_{err}.$$

In [18] Yin et al. considered Eq. (3) with $q_i = q$ and made perturbation analysis. They result rewritten for $q = 1$ as follows

Theorem 2. [18, Theorem 3.1] *Let*

(i) $\theta := \frac{1}{4} - \|Q^{-1}\|^2 \sum_{i=1}^{m} \|A_i\|^2 > 0$,

(ii) $\|\Delta Q\| \leq \|Q^{-1}\|^{-1}(1 - \sqrt{1-\theta})$,

(iii) $\sum_{i=1}^{m}(\|\tilde{A}_i\|^2 - \|A_i\|^2) < \frac{3}{4}\theta\, \|Q^{-1}\|^{-2}$.

Then the equations (1) and (4) have maximal positive definite solutions X_L and $\tilde{X}_L$, respectively. Moreover,

$$\|\Delta X_L\| \leq \frac{1}{\xi}\left[\|\Delta Q\| + 2\sum_{i=1}^{m} \|X_L^{-1}A_i\|\|\Delta A_i\| + \|X_L^{-1}\| \sum_{i=1}^{m} \|\Delta A_i\|^2\right] =: E_Y,$$

where

$$\xi = 1 - c^2 \sum_{i=1}^{m} \|\tilde{A}_i\|^2, \quad c = 2\max\{\|Q^{-1}\|, \|\tilde{Q}^{-1}\|\}.$$

3 Main results

In this section we generalize two results obtained in [11] for Eq. (1) in case of $m = 2$. The first is similar to the Duan's result (see Theorem 1), but it is for arbitrary right hand $Q > 0$ of Eq. (1).

Theorem 3. *Let*

(i) $\|Q^{-1}\|^2 \sum_{i=1}^{m} \|A_i\|^2 < \frac{1}{4}$;

(ii) $\|\Delta Q\| \leq \left[\frac{1}{2} - \|Q^{-1}\| \Big(\sum_{i=1}^{m} \|A_i\|^2 \Big)^{\frac{1}{2}} \right] \|Q^{-1}\|^{-1}$,

(iii) $\sum_{i=1}^{m} \|\Delta A_i\|^2 + 2 \sum_{i=1}^{m} \|A_i\| \|\Delta A_i\| < \left[\frac{1}{4} - \|\tilde{Q}^{-1}\|^2 \sum_{i=1}^{m} \|A_i\|^2 \right] \|\tilde{Q}^{-1}\|^{-2}$.

Then the equations (1) and (4) have maximal solutions X_L and $\tilde{X}_L$, respectively. Moreover,

$$\|\Delta X_L\| \leq \frac{1}{c_1} \left[\|\Delta Q\| + 2\|\tilde{Q}^{-1}\| \sum_{i=1}^{m} \|\Delta A_i\| (2\|A_i\| + \|\Delta A_i\|) \right] =: E_1,$$

where

$$c_1 = 1 - 4\|Q^{-1}\| \|\tilde{Q}^{-1}\| \sum_{i=1}^{m} \|A_i\|^2.$$

Proof: From condition *(i)* and Lemma 1 it follows that the Eq. (1) has maximal solution X_L and

$$\|X_L^{-1}\| < 2\|Q^{-1}\|, \quad \frac{1}{2}\|Q\| \leq \|X_L\| \leq \|Q\|, \tag{6}$$

Now, we will show that

$$\|\tilde{Q}^{-1}\|^2 \sum_{i=1}^{m} \|A_i\|^2 < \frac{1}{4}, \quad \|\tilde{Q}^{-1}\|^2 \sum_{i=1}^{m} \|\tilde{A}_i\|^2 < \frac{1}{4}. \tag{7}$$

From identity $\tilde{Q}^{-1} = Q^{-1} - Q^{-1} \Delta Q \, \tilde{Q}^{-1}$ and condition *(ii)* we have

$$\begin{aligned} \|\tilde{Q}^{-1}\| &\leq \|Q^{-1}\| + \|Q^{-1}\| \|\Delta Q\| \|\tilde{Q}^{-1}\| \\ &\leq \|Q^{-1}\| + \left[\frac{1}{2} - \|Q^{-1}\| \Big(\sum_{i=1}^{m} \|A_i\|^2 \Big)^{\frac{1}{2}} \right] \|\tilde{Q}^{-1}\|. \end{aligned}$$

Hence

$$\|\tilde{Q}^{-1}\| \leq \frac{\|Q^{-1}\|}{\frac{1}{2} + \big(\sum_{i=1}^{m} \|A_i\|^2 \big)^{\frac{1}{2}} \|Q^{-1}\|}.$$

Thus from condition *(i)* we receive

$$\|\tilde{Q}^{-1}\|^2 \sum_{i=1}^{m} \|A_i\|^2 \leq \frac{\|Q^{-1}\|^2 \sum_{i=1}^{m} \|A_i\|^2}{\left[\frac{1}{2} + \|Q^{-1}\| \left(\sum_{i=1}^{m} \|A_i\|^2 \right)^{\frac{1}{2}} \right]^2} < \frac{1}{4}.$$

Therefore, the right hand in condition *(iii)* is a positive quantity. The second inequality in (7) follows from *(iii)*:

$$\|\tilde{Q}^{-1}\|^2 \sum_{i=1}^{m} \|\tilde{A}_i\|^2 \leq \|\tilde{Q}^{-1}\|^2 \sum_{i=1}^{m} (\|\Delta A_i\| + \|A_i\|)^2 < \frac{1}{4}.$$

Therefore, from Lemma 1 it follows that the Eq. (4) has maximal solution $\tilde{X}_L$ and

$$\|\tilde{X}_L^{-1}\| < 2\|\tilde{Q}^{-1}\|, \quad \frac{1}{2}\|\tilde{Q}\| \leq \|\tilde{X}_L\| \leq \|\tilde{Q}\|. \tag{8}$$

After substraction of the equalities

$$X_L = Q - \sum_{i=1}^{m} A_i^* X_L^{-1} A_i, \qquad \tilde{X}_L = \tilde{Q} - \sum_{i=1}^{m} \tilde{A}_i{}^* \tilde{X}_L^{-1} \tilde{A}_i$$

we receive

$$\Delta X_L - \sum_{i=1}^{m} A_i^* \tilde{X}_L^{-1} \Delta X_L X_L^{-1} A_i = \Delta Q - \sum_{i=1}^{m} \Delta A_i^* \tilde{X}_L^{-1} \tilde{A}_i - \sum_{i=1}^{m} A_i^* \tilde{X}_L^{-1} \Delta A_i. \tag{9}$$

Denote by

$$\varphi := \Delta X_L - \sum_{i=1}^{m} A_i^* \tilde{X}_L^{-1} \Delta X_L X_L^{-1} A_i,$$
$$\psi := \Delta Q - \sum_{i=1}^{m} \Delta A_i^* \tilde{X}_L^{-1} \tilde{A}_i - \sum_{i=1}^{m} A_i^* \tilde{X}_L^{-1} \Delta A_i.$$

Using (6) and (8) we obtain

$$\begin{aligned}\|\varphi\| &\geq \|\Delta X_L\|\Big(1 - \|X_L^{-1}\|\|\tilde{X}_L^{-1}\|\sum_{i=1}^{m}\|A_i\|^2\Big)\\ &\geq \|\Delta X_L\|\Big(1 - 4\|Q^{-1}\|\|\tilde{Q}^{-1}\|\sum_{i=1}^{m}\|A_i\|^2\Big) = c_1\|\Delta X_L\|\,,\\ \|\psi\| &\leq \|\Delta Q\| + \sum_{i=1}^{m}\|\Delta A_i\|\|\tilde{X}_L^{-1}\|\big(\|\tilde{A}_i\| + \|A_i\|\big)\\ &\leq \|\Delta Q\| + 2\sum_{i=1}^{m}\|\Delta A_i\|\|\tilde{Q}^{-1}\|\big(2\|A_i\| + \|\Delta A_i\|\big) = c_1 E_1\,,\end{aligned}$$

hence by (9) we have $\|\Delta X_L\| \leq \frac{1}{c_1}\|\varphi\| = \frac{1}{c_1}\|\psi\| \leq E_1$. □

We will prove the following theorem which can be considered as an extension of Xu's result derived in [5] (The Xu's result is for Eq. (1) in case of $m = 1$).

Theorem 4. *Let*

(i) $\|A_j\|\|Q^{-1}\| < \dfrac{\sqrt{m}}{2\,m}, \quad j = 1, 2, \ldots, m;$

(ii) $\|\Delta Q\| \leq \left[\dfrac{1}{2} - \Big(\sum_{i=1}^{m}\|A_i\|^2\Big)^{\frac{1}{2}}\|Q^{-1}\|\right]\|Q^{-1}\|^{-1};$

(iii) $\|\Delta A_j\| < \dfrac{\sqrt{m}}{2\,m}\left\{\dfrac{1}{2} + \left[\Big(\sum_{i=1}^{m}\|A_i\|^2\Big)^{\frac{1}{2}} - 2\sqrt{m}\,\|A_j\|\right]\|Q^{-1}\|\right\}\|Q^{-1}\|^{-1},$
$j = 1, 2, \ldots, m.$

Then the equations (1) and (4) have maximal solutions X_L and $\tilde{X}_L$, respectively. Moreover,

$$\|\Delta X_L\| \leq \frac{1}{c_2}\left(\|\Delta Q\| + \frac{2\sqrt{m}}{m}\sum_{i=1}^{m}\|\Delta A_i\|\right) =: E_2,$$

$$\frac{\|\Delta X_L\|}{\|X_L\|} \leq \frac{2}{c_2}\left(\frac{\|\Delta Q\|}{\|Q\|} + \frac{1}{m}\sum_{i=1}^{m}\frac{\|\Delta A_i\|}{\|A_i\|}\right) =: RE_2,$$

where $c_2 = 1 - 2\|Q^{-1}\|\big(\sum_{i=1}^{m}\|A_i\|^2\big)^{\frac{1}{2}}$.

Proof: From condition *(i)* we have

$$\|Q^{-1}\|^2 \sum_{i=1}^{m} \|A_i\|^2 < \frac{1}{4},$$

hence by Lemma 1 it follows that the maximal solution X_L of Eq. (1) exists and $\|X_L^{-1}\| < 2\|Q^{-1}\|$.

Now, we show that

$$\|\tilde{A}_j\|\|\tilde{Q}^{-1}\| < \frac{\sqrt{m}}{2\,m}, \quad \|A_j\|\|\tilde{Q}^{-1}\| < \frac{\sqrt{m}}{2\,m}, \quad j = 1, 2, \ldots, m, \tag{10}$$

hence we have

$$\|\tilde{Q}^{-1}\|^2 \sum_{i=1}^{m} \|\tilde{A}_i\|^2 < \frac{1}{4}, \qquad \|\tilde{Q}^{-1}\|^2 \sum_{i=1}^{m} \|A_i\|^2 < \frac{1}{4}. \tag{11}$$

From condition *(ii)* (see the proof of Theorem 3) we have

$$\|\tilde{Q}^{-1}\| \le \frac{\|Q^{-1}\|}{\frac{1}{2} + \left(\sum_{i=1}^{m} \|A_i\|^2\right)^{\frac{1}{2}} \|Q^{-1}\|}. \tag{12}$$

Thus, by *(iii)* we obtain

$$\max\{\|\tilde{A}_j\|\|\tilde{Q}^{-1}\|, \|A_j\|\|\tilde{Q}^{-1}\|\} \le (\|A_j\| + \|\Delta A_j\|)\, \|\tilde{Q}^{-1}\|$$
$$< \frac{\|A_j\|\|Q^{-1}\| + \frac{\sqrt{m}}{2\,m}\left\{\frac{1}{2} + \left[\left(\sum_{i=1}^{m} \|A_i\|^2\right)^{\frac{1}{2}} - 2\sqrt{m}\|A_j\|\right]\|Q^{-1}\|\right\}}{\frac{1}{2} + \left(\sum_{i=1}^{m} \|A_i\|^2\right)^{\frac{1}{2}} \|Q^{-1}\|} = \frac{\sqrt{m}}{2\,m}.$$

Therefore, by (11) and Lemma 1 it follows that the maximal solution $\tilde{X}_L$ of Eq. (4) exist and $\|\tilde{X}_L^{-1}\| < 2\|\tilde{Q}^{-1}\|$. Moreover, from (10) and (11) we have

$$\left(\sum_{i=1}^{m} \|A_i\|^2\right)^{\frac{1}{2}} \|\tilde{X}_L^{-1}\| < 1, \tag{13}$$

$$\|\tilde{A}_j\|\|\tilde{X}_L^{-1}\| < \frac{\sqrt{m}}{m}, \qquad \|A_j\|\|\tilde{X}_L^{-1}\| < \frac{\sqrt{m}}{m}, \qquad j = 1, 2, \ldots, m. \tag{14}$$

We consider equation (9) and the notations φ and ψ in the proof of

Theorem 3. For φ and ψ by (13) and (14) we have

$$\|\varphi\| \geq \|\Delta X_L\|\Big(1 - \|X_L^{-1}\|\|\tilde{X}_L^{-1}\|\sum_{i=1}^{m}\|A_i\|^2\Big)$$

$$\geq \|\Delta X_L\|\Big[1 - 2\|Q^{-1}\|\Big(\sum_{i=1}^{m}\|A_i\|^2\Big)^{\frac{1}{2}}\Big] = c_2\|\Delta X_L\|\,,$$

$$\|\psi\| \leq \|\Delta Q\| + \sum_{i=1}^{m}\|\Delta A_i\|\|\tilde{X}_L^{-1}\|\big(\|\tilde{A}_i\| + \|A_i\|\big)$$

$$\leq \|\Delta Q\| + \frac{2\sqrt{m}}{m}\sum_{i=1}^{m}\|\Delta A_i\| = c_2 E_2\,.$$

Hence $\|\Delta X_L\| \leq \frac{1}{c_2}\|\varphi\| = \frac{1}{c_2}\|\psi\| \leq E_2$.

From *(i)* and Lemma 1 we have

$$\frac{\|Q\|}{\|X_L\|} < 2, \qquad \frac{\|A_j\|}{\|X_L\|} < \frac{\sqrt{m}}{m}, \quad j = 1, 2, \ldots, m.$$

Thus $\|\Delta X_L\|/\|X_L\| \leq RE_2$. □

Now, we will compare our result (Theorem 3) with the preliminary ones (Theorem 1 and Theorem 2). We consider different cases.

In case of $Q = I$ and $\tilde{Q} = I$ the conditions (i), (ii) in Theorem 1 and (i), (iii) in Theorem 3, respectively, are the same and the condition (ii) in Theorem 3 is satisfied. But $E_1 < S_{err}$.

In case of arbitrary $Q > 0$ we compare Theorem 3 and Theorem 2. Obviously the conditions (i), respectively, are the same. For the right hands of the inequalities in (ii), respectively, it is easy to verify that

$$1 - \sqrt{1-\theta} < \frac{1}{2} - \|Q^{-1}\|\Big(\sum_{i=1}^{m}\|A_i\|^2\Big)^{\frac{1}{2}}.$$

Hence there are examples for which the Theorem 2 is inapplicable. The comparison of the conditions (iii) is more difficult. But, it can be found an example such that the condition (iii) in Theorem 2 is satisfied while the condition (iii) in Theorem 3 is violated. For computation of the perturbation estimate E_Y in Theorem 2, we need the maximal solution of equation. But, by Lemma 1 we have

$$E_Y := \frac{1}{\xi}\Big[\|\Delta Q\| + 2\sum_{i=1}^{m}\|X_L^{-1}A_i\|\|\Delta A_i\| + \|X_L^{-1}\|\sum_{i=1}^{m}\|\Delta A_i\|^2\Big]$$
$$\leq \frac{1}{\xi}\Big[\|\Delta Q\| + 2\|Q^{-1}\|\sum_{i=1}^{m}(2\|A_i\| + \|\Delta A_i\|)\|\Delta A_i\|\Big] =: E'_Y.$$

Now, we will compare estimates E_1 (from Theorem 3) and E'_Y. In case of $\sum_{i=1}^{m}\|A_i\| < \sum_{i=1}^{m}\|\tilde{A}_i\|$, we have $\xi < c_1$. Moreover, if $\|\tilde{Q}^{-1}\| < \|Q^{-1}\|$, the value of expression in brackets at E_1 is less than that in E'_Y. Therefore, in case of $\sum_{i=1}^{m}\|A_i\| < \sum_{i=1}^{m}\|\tilde{A}_i\|$ and $\|\tilde{Q}^{-1}\| < \|Q^{-1}\|$ then $E_1 < E'_Y$.

Remark 1. *If*

$$\|\Delta A_j\| < \frac{1}{\mu}\Big(\frac{1}{4} - \|\tilde{Q}^{-1}\|^2\sum_{i=1}^{m}\|A_i\|^2\Big)\|\tilde{Q}^{-1}\|^{-2}, \quad j = 1, 2, \ldots, m,$$

where

$$\mu = \sum_{i=1}^{m}\|A_i\| + \left[\Big(\sum_{i=1}^{m}\|A_i\|\Big)^2 + \Big(\frac{1}{4} - \|\tilde{Q}^{-1}\|^2\sum_{i=1}^{m}\|A_i\|^2\Big)\|\tilde{Q}^{-1}\|^{-2}m\right]^{\frac{1}{2}},$$

then the condition (iii) in Theorem 3 is satisfied.

Now we give a perturbation estimat which does not depend on the coefficients of the perturbed equation (4).

Theorem 5. *Let*

(i) $\eta := \frac{1}{2} - \|Q^{-1}\|\Big(\sum_{i=1}^{m}\|A_i\|^2\Big)^{\frac{1}{2}} > 0,$

(ii) $\|\Delta Q\| \leq \eta\|Q^{-1}\|^{-1},$

(iii) $\sum_{i=1}^{m}\|\Delta A_i\|^2 + 2\sum_{i=1}^{m}\|A_i\|\|\Delta A_i\| < \frac{\eta(2-3\eta)}{4\|Q^{-1}\|^2}.$

Then the equations (1) and (4) have maximal solutions X_L and $\tilde{X}_L$, respectively. Moreover

$$\|\Delta X_L\| \leq \frac{1}{c_3}\left[(1-\eta)\|\Delta Q\| + 2\|Q^{-1}\|\sum_{i=1}^{m}\big(2\|A_i\| + \|\Delta A_i\|\big)\|\Delta A_i\|\right] =: E_3,$$

where $c_3 = \eta(3 - 4\eta)$.

Proof: From conditions *(i)* and *(ii)* in the proof of Theorem 3 we obtained that: the Eq. (1) has maximal solution X_L and

$$\|X_L^{-1}\| < 2\|Q^{-1}\|, \quad \frac{1}{2}\|Q\| \le \|X_L\| \le \|Q\|,$$

$$\|\tilde{Q}^{-1}\| \le \frac{\|Q^{-1}\|}{\frac{1}{2} + \|Q^{-1}\|\Big(\sum_{i=1}^{m}\|A_i\|^2\Big)^{\frac{1}{2}}} = \frac{\|Q^{-1}\|}{1-\eta}.$$

Thus, by condition *(iii)* we have

$$\|\tilde{Q}^{-1}\|^2 \sum_{i=1}^{m}\|\tilde{A}_i\|^2 \le \frac{\|Q^{-1}\|^2}{(1-\eta)^2}\sum_{i=1}^{m}(\|\Delta A_i\| + \|A_i\|)^2$$

$$< \frac{\eta(2-3\eta)}{4(1-\eta)^2} + \frac{(1-2\eta)^2}{4(1-\eta)^2} = \frac{1}{4}\,.$$

Therefore, from Lemma 1 it follows that the Eq. (4) has maximal solution $\tilde{X}_L$ and

$$\|\tilde{X}_L^{-1}\| < 2\|\tilde{Q}^{-1}\|, \quad \frac{1}{2}\|\tilde{Q}\| \le \|\tilde{X}_L\| \le \|\tilde{Q}\|.$$

We consider equation (9) and the notations φ and ψ in the proof of Theorem 3. For φ and ψ we have

$$\|\varphi\| \ge \|\Delta X_L\|\Big(1 - \|X_L^{-1}\|\|\tilde{X}_L^{-1}\|\sum_{i=1}^{m}\|A_i\|^2\Big)$$

$$\ge \|\Delta X_L\|\left[1 - \frac{(1-2\eta)^2}{1-\eta}\right] = \frac{c_3}{1-\eta}\|\Delta X_L\|\,,$$

$$\|\psi\| \le \|\Delta Q\| + \sum_{i=1}^{m}\|\Delta A_i\|\|\tilde{X}_L^{-1}\|\big(\|\tilde{A}_i\| + \|A_i\|\big)$$

$$\le \|\Delta Q\| + \frac{2\|Q^{-1}\|}{1-\eta}\sum_{i=1}^{m}\|\Delta A_i\|\big(2\|A_i\| + \|\Delta A_i\|\big) = \frac{c_3}{1-\eta}E_3\,,$$

hence by (9) we have $\|\Delta X_L\| \le \frac{1-\eta}{c_3}\|\varphi\| = \frac{1-\eta}{c_3}\|\psi\| \le E_3$. □

Remark 2. *If* $\eta = \frac{1}{2} - \|Q^{-1}\|\Big(\sum_{i=1}^{m}\|A_i\|^2\Big)^{\frac{1}{2}} > 0$ *and*

$$\|\Delta A_j\| < \frac{\eta(2-3\eta)}{4(1-2\eta)^2 + 2\sqrt{4(1-2\eta)^4 + m\|Q^{-1}\|^2\eta(2-3\eta)}},$$

$j = 1, 2, \ldots, m$, *then the condition (iii) in Theorem 5 is satisfied.*

The disadvantage of the estimates in theorems 2 and 3 is that they depend on the coefficients of the original equation and those of the perturbed equation.

4 Numerical experiments

In this section we illustrate the theoretical results by three numerical examples. We consider Eq. (1) in case of $m = 3$ and $n = 5$.

For comparing the results with Duan's result in first example we have $Q = I$ and $\tilde{Q} = I$.

Example 1. *We consider Eq. (1) with matrix coefficients*

$$Q := I, \quad A_i := VA_{0i}V, \quad i = 1, 2, 3,$$

where

$$\begin{aligned} A_{01} &= 10^{-2}\text{diag}(19\ 15\ 15\ 12\ 22), \\ A_{02} &= 10^{-2}\text{diag}(26\ 21\ 20\ 27\ 29), \\ A_{03} &= 10^{-2}\text{diag}(19\ 21\ 29\ 25\ 22), \\ V &= I - 2vv^T/5 \quad \text{with} \quad v = (1\ 1\ 1\ 1\ 1)^T, \end{aligned}$$

and perturbed equation (4) with perturbations in the coefficients $\Delta A_i = V\Delta A_{0i}V$*,* $i = 1, 2, 3$ *with*

$$\begin{aligned} \Delta A_{01} &= 10^{-j}\text{diag}(1\ 1\ 2\ 1\ 2), \\ \Delta A_{02} &= 10^{-j}\text{diag}(1\ 1\ 1\ 2\ 1), \\ \Delta A_{03} &= 10^{-j}\text{diag}(1\ 2\ 1\ 2\ 1). \end{aligned}$$

The maximal solution X_L of Eq. (1) with commuting matrices Q and A_i, $i = 1, 2, \ldots, m$ can be computed by the formula

$$X_L = \frac{1}{2}\left[Q + \left(Q^2 - 4\sum_{i=1}^{m} A_i^* A_i\right)^{\frac{1}{2}}\right]. \tag{15}$$

For computing X_L and $\tilde{X}_L$ in Example 1 we use formula (15).

In Table 1 the perturbation estimates for Example 1 with $j = 4, 5, 6, 7$ are listed. For the estimate E_2 is denoted asterisk, because the condition (i) in Theorem 4 is violated. We see that the estimate E_Y followed by E_1 is sharpest, but E_Y depends to the maximal solution X_L.

Example 2. *We consider Eq. (1) with matrix coefficients*

$$Q := VQ_0V, \quad A_i := VA_{0i}V, \quad i = 1, 2, 3,$$

Table 1: Results for Example 1.

j	4	5	6	7
$\|\Delta X_L\|$	$3.6681e-04$	$3.6649e-05$	$3.6646e-06$	$3.6646e-07$
S_{err}	$4.8404e-03$	$4.7954e-04$	$4.7909e-05$	$4.7905e-06$
E'_Y	$4.8217e-03$	$4.7935e-04$	$4.7907e-05$	$4.7905e-06$
E_Y	$3.0784e-03$	$3.0604e-04$	$3.0586e-05$	$3.0585e-06$
E_1	$4.7922e-03$	$4.7906e-04$	$4.7904e-05$	$4.7904e-06$
E_2	$*$	$*$	$*$	$*$
E_3	$6.4669e-03$	$6.4647e-04$	$6.4645e-05$	$6.4644e-06$

where

$$\begin{aligned}
&Q_0 = 10^{-3}\mathrm{diag}(1\ 5\ 6\ 7\ 8),\\
&A_{01} = 10^{-6}\mathrm{diag}(19\ 15\ 15\ 12\ 22),\\
&A_{02} = 10^{-6}\mathrm{diag}(26\ 21\ 20\ 27\ 29),\\
&A_{03} = 10^{-6}\mathrm{diag}(19\ 21\ 29\ 25\ 22),\\
&V = I - 2vv^T/5 \quad \textit{with} \quad v = (1\ 1\ 1\ 1\ 1)^T,
\end{aligned}$$

and perturbed equation (4) with perturbations in the coefficients $\Delta A_i = V\Delta A_{0i}V$, $i = 1,2,3$ *with*

$$\begin{aligned}
&\Delta Q_0 = 10^{-(j+1)}\mathrm{diag}(2\ 2\ 1\ 1\ 1),\\
&\Delta A_{01} = 10^{-j}\mathrm{diag}(1\ 1\ 2\ 1\ 2),\\
&\Delta A_{02} = 10^{-j}\mathrm{diag}(3\ 1\ 1\ 2\ 1),\\
&\Delta A_{03} = 10^{-j}\mathrm{diag}(1\ 2\ 1\ 2\ 1).
\end{aligned}$$

For computing X_L and $\tilde{X}_L$ in Example 2 we use formula (15).

In Table 2 the perturbation estimates for Example 2 with $j = 4,5,6,7$ are listed. For the estimates E'_Y, E_Y and E_2 in case of $j = 4$ are denoted three asterisks, because the conditions (iii) in theorems 2 and 4, respectively, are violated. Otherwise, we see again that the estimate E_Y followed by E_1 is sharpest, but E_Y depends to the maximal solution X_L.

Example 3. *We consider Eq. (1) with matrix coefficients*

$$A_1 = \alpha\begin{pmatrix} 1 & 0 & 0 & 0 & 1\\ -1 & 1 & 0 & 0 & 1\\ -1 & -1 & 1 & 0 & 1\\ -1 & -1 & -1 & 1 & 1\\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}, A_2 = \alpha\begin{pmatrix} 1 & -1 & -1 & -1 & -1\\ 1 & 1 & -1 & -1 & -1\\ 1 & 0 & 1 & -1 & -1\\ 1 & 0 & 1 & 1 & -1\\ 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

Table 2: Results for Example 2.

j	4	5	6	7
$\|\Delta X_L\|$	$1.8379e-04$	$5.4513e-06$	$4.4407e-07$	$4.3405e-08$
E'_Y	$***$	$1.3331e-05$	$9.9950e-07$	$9.6787e-08$
E_Y	$***$	$6.9808e-06$	$5.3050e-07$	$5.1464e-08$
E_1	$4.4804e-04$	$1.3096e-05$	$9.9881e-07$	$9.6780e-08$
E_2	$***$	$9.1330e-05$	$9.1330e-06$	$9.1330e-07$
E_3	$7.9299e-04$	$2.2409e-05$	$1.6720e-06$	$1.6151e-07$

$$A_1 = \alpha \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}, \quad Q := X_L + \sum_{i=1}^{3} A_i^* X_L^{-1} A_i,$$

where $\alpha = 1/2000$ *and* $X_L = \frac{1}{191}\mathrm{diag}(1\ 2\ 3\ 2\ 1)$*, and perturbed equation (4) with perturbations in the coefficients*

$$\Delta A_i = \frac{C_i}{10^j \|C_i\|}, \quad i = 1,2,3, \quad \textit{and} \quad \Delta X_L = \frac{C_4^T + C_4}{10^{0.71j}\|C_4^T + C_4\|},$$

where C_i*,* $i = 1,2,3,4$ *are random matrices generated by* Matlab *function* **rand**.

In Table 3 the perturbation estimates for Example 3 with $j = 5,6,7,8$ are listed. In case of $j = 5$ for all estimates are denoted two asterisks, because the conditions (ii) in the respectively theorems are violated. Also, in case of $j = 6$ for estimates E'_Y, E_Y are denoted two asterisks.

Table 3: Results for Example 3.

j	5	6	7	8
$\|\Delta X_L\|$	$2.8184e-04$	$5.4954e-05$	$1.0715e-05$	$2.0893e-06$
E'_Y	$**$	$**$	$2.7153e-04$	$5.2024e-05$
E_Y	$**$	$**$	$2.6657e-04$	$5.1528e-05$
E_1	$**$	$1.3513e-03$	$2.6806e-04$	$5.1941e-05$
E_2	$**$	$2.8882e-03$	$5.3835e-04$	$1.0303e-04$
E_3	$**$	$1.9309e-03$	$3.6002e-04$	$6.8910e-05$

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