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THE NASH EQUILIBRIUM IN OPEN LOOP LINEAR QUADRATIC GAMES FOR POSITIVE SYSTEMS*

Neli Baeva †

Abstract

We consider two-player linear quadratic differential games for positive linear systems with an open loop information structure. The Newton method to obtain the stabilizing solution of a corresponding Riccati equation is presented in the literature. Here, we propose a new iterative method, where the Sylvester iteration to a decoupled Riccati equation is applied. Moreover, the convergence properties of this modification are investigated and the sufficient condition to apply the modification is derived. The performances of the proposed algorithm are illustrated on some numerical examples.

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keywords: open loop Nash equilibrium, generalized Riccati equation, stabilizing solution, nonnegative solution.

1 Introduction

The problem to compute the stabilizing nonnegative solution to the set of Riccati equation is an important problem with many practical applications. Our investigation is motivated from the paper of Jank and Kremer [10] and the paper of Azevedo-Perdicoulis and Jank [1], where the problem of finding

*Accepted for publication on January 12-th 2017
†nelilnb@abv.bg Technical University of Varna, 1 Studentska str., Varna 9010, Bulgaria.

a open loop deterministic Nash equilibrium for a two-player infinite-horizon linear-quadratic differential game is studied. This equilibrium is defined as a pair of linear state feedback strategies stabilizing the closed-loop system.

It is well known the Engwerdas book [9] on linear-quadratic game theory, which is an excellent introduction to the modern modern linear-quadratic game models and their practical implementations. There is a special chapter on noncooperative open loop information games. It is showed that a Nash equilibrium exists if and only if some Riccati differential equation has a solution. Moreover, some numerical procedures are provided to compute the unique Nash equilibrium points. In this paper, we consider the case of non zero sum games which is fundamentally investigated in [14].

However, the considered game is studied on positive systems and players's strategies are presented via the stabilizing solution of the associated nonsymmetric Riccati equation. Applications of the positive systems occur naturally in ecological and economic systems [6],a special class of positive switched systems [8], many biological models as the epidemiological model considered in [7].

The concept of a Nash equilibrium in games considering different information structures has been introduced [4, 5]. Following their findings we refer that the deterministic feedback Nash equilibria are characterized by the solutions of different type of Riccati equations with a stability property.

Let us consider the nonsymmetric matrix Riccati equation

$$0 \ = \mathcal{R}(X_1, X_2)\,, \tag{1}$$

where

$$\begin{aligned} \mathcal{R}(X_1, X_2) = \begin{pmatrix} \mathcal{R}_1(X_1, X_2) \\ \mathcal{R}_2(X_1, X_2) \end{pmatrix} := -\begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ -\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} A - \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} + \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \end{aligned} \tag{2}$$

where the notations are : $-A$ is an $n \times n$ M-matrix, Q_j is an $n \times n$ symmetric nonnegative matrix, $S_j = B_j\, R_{jj}^{-1}\, B_j^T$ is a nonpositive matrix, $j = 1, 2$. In addition, the matrices $B_j, R_{jj}, j = 1, 2$ come from the state of each player. B_j is an $n \times m_j$ nonnegative matrix and R_{jj} is an $m_j \times m_j$ negative definite matrix with ${R_{jj}}^{-1}$ is a nonnegative matrix $j = 1, 2$, and moreover X_1, X_2 are $n \times n$ unknown matrices.

The Newton method to calculate the left-right nonnegative stabilizing solution of a nonsymmetric algebraic matrix Riccati equation in a two-player linear-quadratic differential game is proposed in [10, 1]. In Theorem 5 of

[10] the convergence properties of the Newton method for a two-player differential game with an open loop information structure are derived.

In this paper we use the following notations: $\mathbf{R}^{n\times s}$ stands for $n \times s$ real matrices. The inequality $X \geq 0$ ($X > 0$) means that all elements of the matrix (or vector) X are real nonnegative (positive) and we call the matrix X nonnegative (positive). For the matrices $A = (a_{ij}), B = (b_{ij})$, we write $A \geq B (A > B)$ if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ hold for all indexes i and j. An $n \times n$ matrix A is called a Z-matrix if it has nonpositive off-diagonal entries. Any Z-matrix A can be presented as $A = \alpha I - N$ with N being a nonnegative matrix, and it is called a nonsingular M-matrix if $\alpha > \rho(N)$, where $\rho(N)$ is the spectral radius of N. A matrix A is called asymptotically stable (or Hurwitz) if the eigenvalues of A have a negative real part. A symmetric matrix A is called positive definite (semidefinite) matrix if all eigenvalues are positive (nonnegative).

We apply some properties of the matrix equation $AXB = C$, i.e. it is equivalent to the linear system $(B^T \otimes A)\, vec(X) = vec(C)$, where the sign $\otimes$ denotes the Kronecker matrix product and the *vec* operator arranges the columns of a matrix into a column vector. An usual Gaussian elimination technique for solving this system requires $O(n^6)$ operations.

The Newton method is given by the following set of recursive equations [10, 1] :

$$\begin{aligned} &-X_{i+1}(A - SX_i) - (D - X_i S)\, X_{i+1} = Q + X_i\, S\, X_i \\ &i = 0, 1, 2, \dots , \end{aligned} \tag{3}$$

where

$$D = \begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix}, Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, S = (S_1 \ \ S_2)\ .$$

The following theorem proves that the Newton matrix sequence monotonically increasing.

Theorem 1. *(Theorem 5, [10]) Suppose additionally for the nonsymmetric Riccati equation positive system (1) that the matrix* $-A$ *is an M-matrix and* $Q_i \geq 0, i = 1, 2$ *and* $S \leq 0$*. Assume further that there exists a* $P \geq 0$*, such that* $\mathcal{R}(P) > 0$*, then the Newton sequence* $\{X_i\}_{i\in N}$ *initialized with* $X_0 = 0$ *is well defined and converges monotonically to a solution* $X \geq 0$*.* X *is the smallest solution in the set of all nonnegative solutions.*

In fact, Theorem 4 [10] statements that the Newton sequence $\{X_i\}_{i\in N}$ converges to a solution $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \geq 0$ and the component matrices

X_1 and X_2 defines the Nash equilibrium strategy (u_1^*, u_2^*) as follows: $u_i^* = -R_{ii}^{-1} B_i^T X_i x^*$, $i = 1, 2$, with x^* being the solution of the closed loop equation

$$\dot{x} = (A - S_1X_1 - S_2X_2)\, x\,, \quad x(0) = x_0\,.$$

In this paper we will improve the introduced Newton iteration. We consider a new iterative method for computing the nonnegative minimal solution to (1), where two sequences of Sylvester algebraic equations are constructed. Numerical examples have introduced so as to demonstrate the effectiveness of the proposed algorithms. The new method is faster than the Newton method because it solves the Sylvester matrix equations at each iterative step in comparison to the system of linear equations with high dimensional structure. In this investigation, we are motivated by the interesting papers [11, 12] where the Lyapunov type iterations are applied to general Riccati equations and fundamental properties are derived.

2 Convergence properties of a new method

We present the matrix functions $\mathcal{R}_1(X_1, X_2), \mathcal{R}_2(X_1, X_2)$ in the following type:

$$\begin{aligned}
\mathcal{R}_1(X_1, X_2) &= -(A - S_1X_1)^T X_1 - X_1(A - S_1X_1 - S_2X_2) - Q_1 - X_1S_1X_1 \\
\mathcal{R}_2(X_1, X_2) &= -(A - S_2X_2)^T X_2 - X_2(A - S_1X_1 - S_2X_2) - Q_2 - X_2S_2X_2\,.
\end{aligned}$$

It is easy to derive the identities for any symmetric matrices Y_1, Y_2:

$$\begin{aligned}
&\mathcal{R}_1(X_1, X_2) = -(A - S_1Y_1)^T X_1 - X_1(A - S_1Y_1 - S_2Y_2) \\
&-Q_1 - X_1S_1Y_1 - (Y_1 - X_1)S_1X_1 - X_1S_2(Y_2 - X_2)\,, \\
&\mathcal{R}_2(X_1, X_2) = -(A - S_2Y_2)^T X_2 - X_2(A - S_1Y_1 - S_2Y_2) \\
&-Q_2 - X_2S_2Y_2 - (Y_2 - X_2)S_2X_2 - X_2S_1(Y_1 - X_1)\,.
\end{aligned} \tag{4}$$

According to above presentation of $\mathcal{R}_1(X_1, X_2), \mathcal{R}_2(X_1, X_2)$ we derive the new iteration

$$-(A - S_1X_1^{(k)})^T X_1^{(k+1)} - X_1^{(k+1)}(A - S_1X_1^{(k)} - S_2X_2^{(k)}) = Q_1 + X_1^{(k)}S_1X_1^{(k)} \tag{5}$$

$$-(A - S_2X_2^{(k)})^T X_2^{(k+1)} - X_2^{(k+1)}(A - S_1X_1^{(k)} - S_2X_2^{(k)}) = Q_2 + X_2^{(k)}S_2X_2^{(k)}\,. \tag{6}$$

In our investigation we exploit the fact that the following statements are equivalent for a Z-matrix (-A):

(a) $-A$ is a nonsingular M-matrix;

(b) $I_n \otimes (-A^T) + (-A^T) \otimes I_n$ is a nonsingular M-matrix;

(c) A is asymptotically stable.

We derive a theorem where the convergence properties of the Sylvester iteration described by (5)-(6) are established:

Theorem 2. *Assume there exist symmetric nonnegative matrices* $\hat{X}_1, \hat{X}_2$ *and* $X_1^{(0)} = 0, X_2^{(0)} = 0$ *such that* $\mathcal{R}_i(\hat{X}_1, \hat{X}_2) \geq 0, i = 1, 2,$ *and* $-A$ *is a nonsingular M-matrix. Then, the matrix sequences* $\{X_1^{(k)}, X_2^{(k)}\}_{k=0}^{\infty}$ *defined by (5)-(6) satisfy:*

(i) $\hat{X}_i \geq X_i^{(k+1)} \geq X_i^{(k)}$ *and* $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \leq 0$ *for* $i = 1, 2\,,\ k = 0, 1, \ldots$*;*

(ii) *The matrices* $I_n \otimes [-(A - S_1 X_1^{(k)})^T] + [-(A - S_1 X_1^{(k)} - S_2 X_2^{(k)})^T] \otimes I_n$ *and* $[-(A - S_2 X_2^{(k)})^T] + [-(A - S_1 X_1^{(k)} - S_2 X_2^{(k)})^T] \otimes I_n$ *are M-matrices for* $k = 0, 1, \ldots$*;*

(iii) *The matrix sequences* $\{X_1^{(k)}, X_2^{(k)}\}_{k=0}^{\infty}$ *converge to the nonnegative minimal solution* $\tilde{X}_1, \tilde{X}_2$ *to the set of Riccati equations (1) with* $\tilde{X}_i \leq \hat{X}_i$ *.*

Proof: Using iteration (5) we construct the matrix sequences

$$X_1^{(0)} = X_2^{(0)} = 0, X_1^{(1)}, X_2^{(1)}, X_1^{(2)}, X_2^{(2)}, \ldots, X_1^{(r)}, X_2^{(r)}, \ldots$$

We apply identity (4) to $\mathcal{R}_1(X_1^{(s)}, X_2^{(s)})$ for $Y_j = X_j^{(s-1)}, j = 1, 2$:

$$\begin{aligned} & R_1(X_1^{(s)}, X_2^{(s)}) \\ = \;& -(A - S_1 X_1^{(s-1)})^T X_1^{(s)} - (X_1^{(s-1)} - X_1^{(s)}) S_1 X_1^{(s)} \\ & -Q_1 - X_1^{(s)} S_1 X_1^{(s-1)} - X_1^{(s)} (A - S_1 X_1^{(s-1)} - S_2 X_2^{(s-1)}) \\ & -X_1^{(s)} S_2 (X_2^{(s-1)} - X_2^{(s)})\,. \end{aligned}$$

According to the first iteration equation (5) we obtain:

$$\begin{aligned} & R_1(X_1^{(s)}, X_2^{(s)}) \\ = \;& +X_1^{(s-1)} S_1 X_1^{(s-1)} - (X_1^{(s-1)} - X_1^{(s)}) S_1 X_1^{(s)} \\ & -X_1^{(s)} S_1 X_1^{(s-1)} - X_1^{(s)} S_2 (X_2^{(s-1)} - X_2^{(s)}) \\ = \;& +(X_1^{(s-1)} - X_1^{(s)}) S_1 (X_1^{(s-1)} - X_1^{(s)}) - X_1^{(s)} S_2 (X_2^{(s-1)} - X_2^{(s)})\,. \end{aligned} \tag{7}$$

We will prove by induction the following statements for $r = 0, \ldots$:

(A) $\mathcal{R}_i(X_1^{(r)}, X_2^{(r)}) \le 0, i = 1, 2$ and matrices $I_n \otimes [-(A - S_1 X_1^{(r)})^T] + [-(A - S_1 X_1^{(r)} - S_2 X_2^{(r)})^T] \otimes I_n$ and $[-(A - S_2 X_2^{(r)})^T] + [-(A - S_1 X_1^{(r)} - S_2 X_2^{(r)})^T] \otimes I_n$ are M-matrices;

(B) $X_i^{(r+1)} \ge X_i^{(r)}, i = 1, 2$;

(C) $\hat{X}_i \ge X_i^{(r+1)}, i = 1, 2$.

Assume that $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \le 0$ and the matrices $I_n \otimes [-(A - S_1 X_1^{(k)})^T] + [-(A - S_1 X_1^{(k)} - S_2 X_2^{(k)})^T] \otimes I_n$ and $I_n \otimes [-(A - S_2 X_2^{(k)})^T] + [-(A - S_1 X_1^{(k)} - S_2 X_2^{(k)})] \otimes I_n$ are M-matrices and $\hat{X}_i \ge X_i^{(k+1)} \ge X_i^{(k)}, i = 1, 2$. We will prove the statements (A)-(B)-(C) for $r = k+1$. We compute $X_1^{(k+1)}, X_2^{(k+1)}$ via (5)-(6).

Applying (7) for $R_1(X_1^{(k+1)}, X_2^{(k+1)})$ we conclude that $R_1(X_1^{(k+1)}, X_2^{(k+1)}) \le 0$ because $S_j \le 0, j = 1, 2$ and $X_j^{(k+1)} \ge X_j^{(k)} \ge 0, j = 1, 2$. Analogously, we conclude $R_2(X_1^{(k+1)}, X_2^{(k+1)}) \le 0$.

Next, we will prove that $I_n \otimes [-(A - S_1 X_1^{(k+1)})^T] + [-(A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)})] \otimes I_n$ and $I_n \otimes [-(A - S_2 X_2^{(k+1)})^T] + [-(A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)})] \otimes I_n$ are M-matrices. We consider difference $R_1(X_1^{(k+1)}, X_2^{(k+1)}) - \mathcal{R}_1(\hat{X}_1, \hat{X}_2)$, where $\mathcal{R}_1(\hat{X}_1, \hat{X}_2)$ is presented via (7) with $Y_j = X_j^{(k+1)}, j = 1, 2$. Thus

$$\begin{aligned} & R_1(X_1^{(k+1)}, X_2^{(k+1)}) - \mathcal{R}_1(\hat{X}_1, \hat{X}_2) \\ = \; & -(A - S_1 X_1^{(k+1)})^T (X_1^{(k+1)} - \hat{X}_1) \\ & -(X_1^{(k+1)} - \hat{X}_1)(A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)}) \\ & -(X_1^{(k+1)} - \hat{X}_1) S_1 (X_1^{(k+1)} - \hat{X}_1) + \hat{X}_1 S_2 (X_2^{(k+1)} - \hat{X}_2) \,. \end{aligned}$$

We rewrite the above equation

$$L_1^{(k+1)} \, vec(X_1^{(k+1)} - \hat{X}_1) = vec\,(V_1^{(k)}),$$

where

$$L_1^{(k+1)} = I_n \otimes [-(A - S_1 X_1^{(k+1)})^T] + (-(A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)})^T) \otimes I_n \,,$$

and

$$\begin{aligned} V_1^{(k)} \; = \; & R_1(X_1^{(k+1)}, X_2^{(k+1)}) - \mathcal{R}_1(\hat{X}_1, \hat{X}_2) \\ & +(X_1^{(k+1)} - \hat{X}_1) S_1 (X_1^{(k+1)} - \hat{X}_1) - \hat{X}_1 S_2 (X_2^{(k+1)} - \hat{X}_2) \,. \end{aligned}$$

Since $\mathcal{R}_1(\hat{X}_1, \hat{X}_2) \ge 0, R_1(X_1^{(k+1)}, X_2^{(k+1)}) \le 0$ and $S_j \le 0, j = 1, 2$ and hence, together with $\hat{X}_1 \ge X_1^{(k)} \ge 0$ we infer $V_1^{(k)} \le 0$.

Therefore, the matrix $L_1^{(k+1)}$ is an M-matrix. In similar way we prove that the matrix $L_2^{(k+1)} = I_n \otimes [-(A - S_2 X_2^{(k+1)})^T] + (-(A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)})^T) \otimes I_n$ is an M-matrix.

Further on, we can apply the recursive equations (5)-(6) to find the matrix $X_1^{(k+2)}, X_2^{(k+2)}$. We will prove $X_j^{(k+2)} \geq X_j^{(k+1)}, j = 1, 2$.

We consider the difference between iteration equation (5) and $R_1(X_1^{(k+1)}, X_2^{(k+1)})$. We derive

$$\begin{aligned} 0 - R_1(X_1^{(k+1)}, X_2^{(k+1)}) =& -(A - S_1 X_1^{(k+1)})^T (X_1^{(k+2)} - X_1^{(k+1)}) \\ &-(X_1^{(k+2)} - X_1^{(k+1)}) (A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)})\,, \end{aligned}$$

and therefore:

$$L_1^{(k+1)}\, vec(X_1^{(k+2)} - X_1^{(k+1)}) = -vec\, (R_1(X_1^{(k+1)}, X_2^{(k+1)}))\,.$$

Since $R_1(X_1^{(k+1)}, X_2^{(k+1)})$ is a nonpositive matrix and $L_1^{(k+1)}$ is an M-matrix, we obtain $X_1^{(k+1)} - X_1^{(k+2)} \leq 0$. Moreover, the inequality $X_2^{(k+1)} - X_2^{(k+2)} \leq 0$ is proved analogously.

Further on, we will prove $\hat{X}_i \geq X_i^{(k+2)}, i = 1, 2$. Consider the matrix

$$-(A - S_1 X_1^{(k+1)})^T (\hat{X}_1 - X_1^{(k+2)}) - (\hat{X}_1 - X_1^{(k+2)})(A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)})\,.$$

We execute some algebraic manipulations on the above matrix and after applying the equalities

$$A^T \hat{X}_1 + \hat{X}_1 A = -\mathcal{R}_1(\hat{X}_1, \hat{X}_2) + \hat{X}_1 S_1 \hat{X}_1 + \hat{X}_1 S_2 \hat{X}_2 - Q_1$$

and

$$\begin{aligned} &-(A - S_1 X_1^{(k+1)})^T X_1^{(k+2)} - X_1^{(k+2)} (A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)}) \\ &= +Q_1 + X_1^{(k+1)} S_1 X_1^{(k+1)} \end{aligned}$$

we obtain

$$-\mathcal{R}_1(\hat{X}_1, \hat{X}_2) + \hat{X}_1 S_1 (\hat{X}_1 - X_1^{(k+1)}) + \hat{X}_1 S_2 (\hat{X}_2 - X_2^{(k+1)})\,.$$

Thus

$$\begin{aligned} &-(A - S_1 X_1^{(k+1)})^T (\hat{X}_1 - X_1^{(k+2)}) \\ &-(\hat{X}_1 - X_1^{(k+2)})(A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)}) \\ &= -\mathcal{R}_1(\hat{X}_1, \hat{X}_2) + \hat{X}_1 S_1 (\hat{X}_1 - X_1^{(k+1)}) + \hat{X}_1 S_2 (\hat{X}_2 - X_2^{(k+1)})\,. \end{aligned} \tag{8}$$

Consider the matrix equation (8). The matrix $I_n \otimes [-(A - S_1 X_1^{(k+1)})^T] + [-(A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)})] \otimes I_n$ is an M-matrix. The right-hand side of the above equation is nonpositive. Thus $X_1^{(k+2)} - \hat{X}_1 \leq 0$ and $X_2^{(k+2)} - \hat{X}_2 \leq 0$.

Hence, the induction process has been completed.

Thus the matrix sequence $\{X_1^{(k)}, X_2^{(k)}\}_{k=0}^{\infty}$ is monotonically increasing and bounded above by $\hat{X}_1, \hat{X}_2$ (in the elementwise ordering). We denote $\lim_{k\to\infty}(X_1^{(k)}, X_2^{(k)}) = (\tilde{X}_1, \tilde{X}_2)$. By taking the limits in (5)-(6) it follows that $(\tilde{X}_1, \tilde{X}_2)$ is a solution of $\mathcal{R}_i(X_1, X_2) = 0, i = 1, 2$ with the property $\tilde{X}_i \leq \hat{X}_i$ and matrices $I_n \otimes [-(A - S_1\tilde{X}_1)^T] + [-(A - S_1\tilde{X}_1 - S_2\tilde{X}_2)^T] \otimes I_n$ and $[-(A - S_2\tilde{X}_2)^T] + [-(A - S_1\tilde{X}_1 - S_2\tilde{X}_2)^T] \otimes I_n$ are M-matrices.

The proof is complete. □

In addition, we refer the Corollary 1 [10] where is proved that if $-A + S_1\hat{X}_1 + S_1\hat{X}_2$ is an M-matrix and $-\begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix} + \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix} (S_1 \quad S_2)$ is an M-matrix then $\begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}$ a left- right stabilizing solution to (1).

3 Numerical examples

We carry out some numerical experiments for computing the stabilizing solution to a nonsymmetric Riccati equation (1). The Newton method (3) and the Silvester method (5)-(6) are applied and compared on some examples. The numerical experiments are constructed following the approach applied in [13].

We consider a two-player game and two numerical examples. The matrix coefficients A, B_i, Q_i and R_{ii} for $i = 1, 2$ are defined using the Matlab description.

Example 1. The matrix coefficients are:

A=abs(randn(n))/100; s=max(abs(eig(A)))+4.5;

for i=1:n, A(i,i)=-(A(i,i))-s; end

B_1 = abs(randn(n,1))/3;

B_2 = eye(n,n); B_2(n,n)=n/3; B_2(1,1)=n/3;

Q_1=zeros(n,n); Q_1(1,1)=n; Q_1(n,n)=1;

Q_2 =Q_1;

R_{11}=-1;

R_{22} = -eye(n,n); R_{22}(1,1)=-40; R_{22}(n,n)=-40;

Example 1 is executed for different values of n, also 100 runs are completed for each value of n. We take $X_1^{(0)} = X_2^{(0)} = 0$ and thus $\mathcal{R}_i(\mathbf{X}^{(0)}) =$

Table 1. Comparison between two iterations.

	Newton Iteration (3)			Sylvester Iteration (5)-(6)		
n	*avIt*	CPU	Error	*avIt*	CPU	Error
80	2	5.4s	0.002×10^{-7}	2	5.9s	0.2156×10^{-7}
100	2	9.027s	0.0017×10^{-7}	2	9.069s	0.202×10^{-7}
120	2	14.269s	0.002×10^{-7}	2	13.593s	0.2107×10^{-7}

$-Q_i \leq 0$ (i.e. the matrix is nonpositive). We might note that the conditions of theorems 1 and 2 are fulfilled, i.e. $\mathbf{X}^{(0)} \leq \hat{\mathbf{X}}$, $\mathcal{R}_i(\mathbf{X}^{(0)}) \leq 0$ and $\mathcal{R}_i(\hat{\mathbf{X}}) \geq 0$, $i = 1, 2$. The computed solution $\tilde{\mathbf{X}}$ satisfies the inequality $\tilde{\mathbf{X}} \leq \hat{\mathbf{X}}$.

On the basis of the experiments, performed for $n = 15$, the following summary of results might be outlined. The Newton iteration (3) requires 4.4 average iteration steps (for all 100 runs) while finding the stabilizing nonnegative solution $\tilde{X}_1^N, \tilde{X}_2^N$. In the same time, the modification of Sylvester iteration (5)-(6) requires 5.5 average iteration steps so as to find the stabilizing nonnegative solution $\tilde{X}_1^S, \tilde{X}_2^S$. The CPU time is 0.34s and 0.68s respectively for executing the Newton iteration with 100 runs and the accelerated Newton iteration with 100 runs.

The Newton iteration preserves his advantages in Example 1. It works faster and it executes less average number of iterations. The Sylvester iteration makes the bigger average number of iterations and that's why it is slower than the Newton iteration. Note that the Newton iteration uses block matrix coefficients while the Silvester iteration is applied to the two Silvester equations with non block matrices.

Example 2. The matrix coefficients are:

A=abs(randn(n))*10; s=max(abs(eig(A)))+5;

for i=1:n, A(i,i)=-(A(i,i))-s; end

B_1 = zeros(n,1); B_1(1)=abs(randn)/5; B_1(n)=abs(randn)/5;

B_2=eye(n,n); B_2(n,n)=sqrt(n);

Q_1=0.25*eye(n,n); Q_1(1,n)=n/1; Q_1(n,1)=n/1;

Q_2=0.05*eye(n,n);

for i=1:n-1 Q_2(i,i+1)=0.1; Q_2(i+1,i)=0.1; end

R_{11}=-0.25;

R_{22}=-10*eye(n,n);

Example 2 is executed for different values of $n = 80, 100, 120$, also 150 runs are completed for each value of n. We take $X_1^{(0)} = X_2^{(0)} = 0$ again. Table 1 explains the computational results for different values of n. The

Newton iteration is faster than Silvester iteration for small values of n. However, the Silvester iteration outstrips and it becomes faster than the Newton iteration for $n = 120$. Obviously, the Silvester iteration preserves this advantage for $n > 120$. The Silvester iteration solves two Sylvester matrix equations at each iterative step while the Newton method has to solve a set of linear equations with high dimensional structure. The block matrix structure in the Newton method delayed the implementation of the method for large values of n while the Sylvester iteration demonstrated the advantage in the case of solving two independent linear matrix equations.

4 Conclusion

We have made numerical experiments for computing the stabilizing solution to a nonsymmetric Riccati equation (1) and we have compared the numerical results. The numerical experiments confirm the effectiveness of the proposed Silvester iteration. In addition, the Silvester method can be naturally divided and applied on a two processors computer. Thus, the Silvester iteration is an effective alternative method to the Newton method.

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