Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 7, No. 2/2015

Higher Order Boundary Value Problem for Impulsive Differential Inclusions*

Johnny Henderson† Abdelghani Ouahab‡ Samia Youcefi§

Abstract

In this paper, we present some existence results for the higher order impulsive differential inclusion:

$$\begin{cases} x^{(n)}(t) \in F(t, x(t), x'(t), \ldots, x^{(n-1)}(t)), \ a.e.\ t \in J = [0, \infty), \ t \neq t_k, \\ \hfill k = 1, \ldots, \\ \Delta x^{(i)}|_{t=t_k} = I_{ik}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)), \ i = 0, 1, \ldots, n-1, \\ \hfill k = 1, \ldots, \\ x^{(i)}(0) = x_{0i}, \ (i = 0, 1, \ldots, n-2), \ x^{(n-1)}(\infty) = \beta x^{(n-1)}(0), \end{cases}$$

where $F : \mathbb{R}_+ \times E \times E \times \cdots \times E \to \mathcal{P}(E)$ is a multifunction, $x_{0i} \in E, i = 0, 1, \ldots, n-1$, $0 = t_0 < t_1 < \cdots < t_m < \cdots$, $\lim_{k\to\infty} t_k = \infty$, $I_{ki} \in C(E \times \cdots \times E, E)$ $(i = 1, \ldots, n-1, \ k = 1, \ldots,)$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-), x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively, $x^{(n-1)}(\infty) = \lim_{t\to\infty} x^{(n-1)}(t)$, and $(E, |\cdot|)$ is real separable Banach space. We present some existence results when the right-hand side multi-valued nonlinearity can be either convex or nonconvex.

MSC: 34K45, 34A60, 47D62, 35R12

---

*Accepted for publication in revised form on August 10-th, 2015

†Johnny_Henderson@baylor.edu, Department of Mathematics, Baylor University, Waco, Texas 76798 – 7328 USA

‡agh_ouahab@yahoo.fr, Laboratory of Mathematics, University of Sidi Bel Abbes, P.O. Box 89, 22000, Sidi Bel Abbes, Algeria

§youcefi.samia@yahoo.com, Laboratory of Mathematics, University of Sidi Bel Abbes, P.O. Box 89, 22000, Sidi Bel Abbes, Algeria

keywords: Impulsive differential inclusions, multi-valued maps, decomposable set, fixed point.

1 Introduction

Differential equations with impulses were considered for the first time in the 1960's by Milman and Myshkis [20]. Their work was followed by a period of active research, mostly in Eastern Europe during 1960-1970, culminating with the monograph by Halanay and Wexler [15].

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations act instantaneously or in the form of "impulses". As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biotechnology, industrial robotics, pharmcokinetics, optimal control, and so forth. Again, associated with this development, a theory of impulsive differential equations has been given extensive attention. Works recognized as landmark contributions include [14, 21]. The existence theory of impulsive differential equations in Banach space was studied by Guo [11, 12, 13]. There are also many different studies in biology and medicine for which impulsive differential equations are good models (see for instance, [2] and the references therein).

In recent years, many examples of differential equations with impulses with fixed moments have flourished in several contexts. In the periodic treatment of some diseases, impulses correspond to administration of a drug treatment or a missing product. In environmental sciences, seasonal changes of the water level of artificial reservoirs are often considered as impulses.

More precisely, we will consider nth order impulsive differential inclusions of the form,

$$x^{(n)}(t) \in F(t, x(t), x'(t), \ldots, x^{(n-1)}(t)), \quad \text{a.e. } t \in J = [0,\infty)\backslash\{t_1, \ldots\} \quad (1.1)$$

$$\Delta x^{(i)}|_{t=t_k} = I_{ik}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)),\ i = 0, 1, \ldots, n-1,\ k = 1, \ldots, \quad (1.2)$$

$$x^{(i)}(0) = x_{0i},\ (i = 0, 1, \ldots, n-2,),\ x^{(n-1)}(\infty) = \beta x^{(n-1)}(0), \quad (1.3)$$

where $F : \mathbb{R}_+ \times E \times E \times \cdots \times E \to \mathcal{P}(E)$ is a multifunction, $x_{0i} \in E, i = 0, 1, \ldots, n-1,\quad 0 = t_0 < t_1 < \cdots < t_m < \cdots, \lim\limits_{k\to\infty} t_k = \infty,\ I_{ik} \in C(E \times$

$\cdots \times E, E)$ $(i = 1, \ldots, n-1,\ k = 1, \ldots,)$, $\Delta x^{(i)}|_{t=t_k} = x^{(i)}(t_k^+) - x^{(i)}(t_k^-)$, where $x^{(i)}(t_k^+) = \lim_{h\to 0^+} x^{(i)}(t_k + h)$ and $x^{(i)}(t_k^-) = \lim_{h\to 0^+} x^{(i)}(t_k - h)$ represent the right and left limits of $x^{(i)}(t)$ at $t = t_k$, respectively, $x^{(n-1)}(\infty) = \lim_{t\to\infty} x^{(n-1)}(t)$, and $(E, |\cdot|)$ is real separable Banach space.

Our goal in this work is to give some existence results when the right-hand side multi-valued nonlinearity can be either convex or nonconvex. Some auxiliary results from multi-valued analysis are gathered together in Section 2. In the Section 3, we give an existence result based on nonlinear alternative of Leray-Schauder type for condensing maps (in the convex case). In Section 4, some existence results are obtained based on the nonlinear alternative of Leray-Schauder type and on the Covitz and Nadler fixed point theorem for contractive multi-valued maps (in the nonconvex case).

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

Let (X, d) be a metric space and Y be a subset of X. We denote:

- $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$ and
- $\mathcal{P}_p(X) = \{Y \in \mathcal{P}(X) : Y$ has the property "p"$\}$, where p could be: *cl*=closed, *b*=bounded, *cp*=copmact, *cv*=convex, etc.

Thus

- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
- $\mathcal{P}_{b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
- $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$, where X is a Banach space
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$,
- $\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$, etc.

In what follows, by E we shall denote a separable Banach space over the field of real numbers $\mathbb{R}$, and by $\bar{J}$ a closed bounded interval in $\mathbb{R}$. We let

$$C(\bar{J}, E) = \{x : \bar{J} \to E \mid x \text{ is continuous}\}.$$

We consider the Tchebyshev norm:

$$\|\cdot\|_\infty : C(\bar{J}, E) \to [0, \infty)$$

defined as follows:

$$\|x\|_\infty = \max\{|x(t)| \ : \ t \in \bar{J}\},$$

where $|\cdot|$ stands for the norm in E. Then $(C(\bar{J}, E), \|\cdot\|_\infty)$ is a Banach space.

The following are classical concepts:

A function $x : \mathbb{R}_+ \to E$ is called *measurable* provided for every open $U \subset E$ the set:

$$x^{-1}(U) = \{t \in \mathbb{R}_+ \mid x(t) \in U\}$$

is Lebesgue measurable.

We shall say that a measurable function $x : \mathbb{R}_+ \to E$ is *Bochner integrable* provided the function $|x| : \mathbb{R}_+ \to [0, \infty)$ is Lebesgue integrable function.

We let:

$$L^1(\mathbb{R}_+, E) = \{x : \mathbb{R}_+ \to E \mid x \ \text{ is Bochner integrable}\}.$$

Let us add that two functions $x_1, x_2 : J \to E$ such that the set $\{x_1(t) \neq x_2(t) \mid t \in \mathbb{R}_+\}$ has Lebesgue measure equal to zero are considered as equal.

Then, we are able to define on L^1,

$$\|x\|_{L^1} = \int_0^\infty |x(t)|\, dt.$$

It is well-known that:

$$(L^1(\mathbb{R}_+, E), \|\cdot\|_{L^1})$$

is a Banach space.

Definition 1. *Let (Ω, Σ, μ) be a finite measure space. A subset $\mathcal{C}$ in $L^1(\Omega, \Sigma, \mu)$ is called uniformly integrable if, for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that, for each measurable subset $\mathcal{R} \subset \Sigma$ whose $\mu(\mathcal{R}) < \delta(\epsilon)$, we have*

$$\int_{\mathcal{R}} |f(\omega)| d\mu(\omega) < \epsilon.$$

Remark 1. *Let $\mathcal{C} \subset L^1(\Omega, \Sigma, \mu)$, then:*

(i) if $\mu(\Omega) < \infty$ and $\mathcal{C}$ is bounded in $L^p(\Omega, \Sigma, \mu)$ where $p > 1$, then $\mathcal{C}$ is uniformly integrable.

(ii) if there exist $p \in L^1(\Omega, \mu, \mathbb{R}_+)$ such that

$$|f(\omega)| \leq p(\omega), \text{ for each } f \in \mathcal{C} \text{ and a.e. } \omega \in \Omega,$$

then $\mathcal{C}$ is uniformly integrable.

Let $K \subset X$. We define $\mathcal{K}$ by

$$\mathcal{K} = \{f \in L^1(\Omega, \Sigma, \mu) : \ f(\omega) \in K \text{ a.e. } \omega \in \Omega\}.$$

Theorem 1. *[8] Let (Ω, Σ, μ) be a finite measure space and X a Banach space, and let $\mathcal{K}$ be a bounded uniformly integrable subset of $L^1(\Omega, \Sigma, \mu)$. Suppose that given $\epsilon > 0$ there exists a measurable set Ω_ϵ and a weakly compact set $K_\epsilon \subset X$ such that $\mu(\Omega \backslash \Omega_\epsilon) < \epsilon$ and for each $f \in \mathcal{K}$, $f(\omega) \in K_\epsilon$ for almost all $\omega \in \Omega_\epsilon$. Then $\mathcal{K}$ is a relatively weakly compact subset of $L^1(\Omega, \Sigma, \mu)$.*

Next we present a new result due to Vrabie [23].

Theorem 2. *Let $(\Omega, \sum, \mu)$ be a $\sigma-$finite measure space, let $\{\Omega_k : k \in \mathbb{N}\}$ be a subfamily of $\sum$ such that*

$$\begin{cases} \mu(\Omega_k) < \infty & \text{for } k = 0, 1, \ldots, \\ \Omega_k \subset \Omega_{k+1} & \text{for } k = 0, 1, \ldots, \\ \cup_{k=0}^{\infty} \Omega_k = \Omega, \end{cases}$$

and let X be a Banach space. Let $K \subset L^1(\Omega, \mu, X)$ be bounded and uniformly integrable in $L^1(\Omega_k, \mu, X)$, for $k = 0, 1, \ldots$, and

$$\lim_{k \to \infty} \int_{\Omega \backslash \Omega_k} |f(\omega)| d\mu(\omega) = 0$$

uniformly for $f \in K$. If for each $\gamma > 0$ and each $k \in \mathbb{N}$, there exist a weakly compact subset $C_{\gamma,k} \subset X$ and a measurable subset $\Omega_{\gamma,k} \subset \Omega_k$ with $\mu(\Omega \backslash \Omega_{\gamma,k}) \leq \gamma$ and $f(\Omega_{\gamma,k}) \subset C_{\gamma,k}$ for all $f \in K$, then K is weakly relatively compact in $L^1(\Omega, \sum, \mu)$.

2.1 Multi-valued analysis

Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued map $G : X \to \mathcal{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in X for each bounded set B of X, i.e., $\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty)$. The map G is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, subset of X and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$. Also, G is said to be *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multi-valued map G is completely continuous with nonempty compact values, then G is *u.s.c.* if and only if G has a closed graph (i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). Finally, we say that G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$.

A multi-valued map $G : \mathbb{R}_+ \to \mathcal{P}_{cl}(X)$ is said to be *measurable* if for each $x \in E$, the function $Y : \mathbb{R}_+ \to X$ defined by

$$Y(t) = \operatorname{dist}(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\},$$

is Lebesgue measurable.

Definition 2. *A measure of noncompactness β is called*

(a) Monotone if $\Omega_0, \Omega_1 \in \mathcal{P}(X)$ $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$.

(b) Nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in X, \Omega \in \mathcal{P}(X)$.

(c) Invariant with respect to the union with compact sets if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset X$ and $\Omega \in \mathcal{P}(X)$.

(d) Real if $\mathcal{A}=\overline{\mathbb{R}}_+ = [0, \infty]$ and $\beta(\Omega) < \infty$ for every bounded Ω.

(e) Semi-additive if $\beta(\Omega_0 \cup \Omega_1) = \max(\beta(\Omega_0), \beta(\Omega_1))$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(X)$.

(f) Lower-additive if β is real and $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(X)$.

(g) Regular if the condition $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

Definition 3. *A sequence $\{v_n\}_{n\in\mathbb{N}} \subset L^1([a, b], X)$ is said to be semi-compact if*

(a) it is integrably bounded, i.e. if there exists $\psi \in L^1([a, b], \mathbb{R}^+)$ such that

$$\|v_n(t)\| \leq \psi(t), \quad \text{for a.e. } \ t \in [a, b] \ \text{ and every } \ n \in \mathbb{N},$$

(b) the image sequence $\{v_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in X for a.e. $t \in [a, b]$.

Lemma 1. *[18] Every semi-compact sequence in $L^1([a,b],X)$ is weakly compact in $L^1([a,b],X)$.*

Lemma 2. *[18] If $F : X \to \mathcal{P}(Y)$ is u.s.c., then $\mathcal{G}r(F)$ is a closed subset of $X \times Y$ Conversely, if F is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.*

Next we state the nonlinear alternative of Leray-Schauder type for condensing maps.

Lemma 3. *[18] Let $V \subset X$ be a bounded open neighborhood of zero and $N : \overline{V} \to \mathcal{P}_{cp,cv}(X)$ a $\beta-$condensing u.s.c. multi-map, where β is a nonsingular measure of noncompactness defied on subsets of X. If N satisfies the boundary condition*

$$x \notin N(x)$$

for all $x \in \partial V$ and $0 < \lambda < 1$, then the set $Fix(N) = \{x \in V, x \in N(x)\}$ is nonempty.

Lemma 4. *[18] Let W be a closed bounded convex subset of a Banach space X and $\mathcal{F} : \mathcal{W} \to \mathcal{P}_{cp}(W)$ be a closed*
$\beta-$condensing multi-map where β is a monotone MNC on X. Then $Fix(F)$ is nonempty and compact.

For more details on multi-valued maps we refer to the books Hu and Papageorgiou [17] and Kamenskii *et al* [18].

3 Convex case

Before stating the results of this section we consider the following spaces.

$$\begin{aligned} PC \quad &= \Big\{x : \mathbb{R}_+ \to E \mid x(t_k^-), x(t_k^+) \text{ exist with } \ x(t_k) = x(t_k^-), \\ &\quad x_k \in C(J_k, E), \ k = 1, \dots \Big\}, \end{aligned}$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}], \ k = 0, \dots.$

$$DPC(\mathbb{R}_+, E) = \{x \in PC : \sup_{t \in J} e^{-t}|x(t)| < \infty\},$$

It is clear that $DPC(\mathbb{R}_+, E)$ is a Banach space with norm

$$\|x\|_B = \sup_{t \in \mathbb{R}_+} e^{-t}|x(t)|.$$

$$
\begin{aligned}
PC^{(n-1)} &= \Big\{x \in PC(\mathbb{R}_+, E) \mid x^{(i)} \text{ exist and it is continuous at } \ t \neq t_k, \\
& \quad i = 1, \dots, n-1, \text{and } \ x^{(i)}(t_k^-), x^{(i)}(t_k^+) \text{ exist with} \\
& \quad x^{(i)}(t_k^-) = x^{(i)}(t_k), k = 1, \dots\Big\},
\end{aligned}
$$

$DPC^{n-1}(\mathbb{R}_+, E) = \{y \in PC^{(n-1)} : \ \sup_{t\in J} e^{-t}|x^{(i)}(t)| < \infty \quad i = 1, \dots, n-1\}$
is a Banach space with the norm

$$\|x\|_D = \max(\|x\|_B, \|x'\|_B, \dots, \|x^{(n-1)}\|_B).$$

Set

$$
\begin{aligned}
AC(J, E) \quad = \quad & \{y : [a, b] \to E \text{ absolutely continuous,} \\
& y(t) = y(a) + \int_a^t y'(s)ds, \text{ and } y' \in L^1([a, b], E)\}.
\end{aligned}
$$

in general, on interval $[a, b]$, there need not exist $y'(t)$, for almost all $t \in [a, b]$ with $y' \in L^1([a, b], E)$ and

$$y(t) = y(a) + \int_a^t y'(s)ds.$$

It is so if E satisfies the Radon-Nikodym property, in particular, if E is reflexive. Moreover, we have the following.

Lemma 5. *[1] Suppose $y : [a, b] \to E$ is absolutely continuous, y' exists a.e., and*

$$|y'(t)| \leq l(t) \text{ a.e. for some } l \in L^1([a, b], E).$$

Then $y' \in L^1([a, b], E)$

$$\int_\tau^t y'(s)ds = y(t) - (\tau), \quad t, \tau \in [a, b].$$

Let us start by defining what we mean by a solution of problem (1.1)-(1.3).

Definition 4. *We say that the function $x \in PC^{(n-1)}$ is a solution of the system (1.1)-(1.3) if $x_{0i} = x^{(i)}(0), i = 0 \ , \dots, n-1$ and there exists $v(\cdot) \in L^1([0, \infty), E)$, such that $v(t) \in F(t, x(t), x'(t), \dots, x^{(n-1)}(t))$ a.e $[0, \infty)$, and such that $x^{(n)}(t) = v(t)$, and the impulsive systems $\Delta x^{(i)}|_{t=t_k} = I_{ki}(x(t_k))$, $i = 0, 1, \dots, n-1, \ k = 1, 2, \dots,$ are satisfied.*

A fundamental notation for a solution of problem (1.1)-(1.3) is given by the following auxiliary result.

Lemma 6. *[13]. Let $f \in L^1(\mathbb{R}_+, E)$ and $\beta \in \mathbb{R}\backslash\{1\}$. Then x is the unique solution of the impulsive boundary value problem,*

$$x^{(n)}(t) = f(t), \quad t \in J := [0, \infty),\ t \neq t_k, \quad k = 1, \ldots, \tag{3.1}$$

$$\Delta x^{(i)}|_{t=t_k} = I_{ki}(x(t_k^-), x'(t_k), \ldots, x^{(n-1)}(t_k)),\ i = 0, \ldots, n-1,\ k = 1, \ldots, \tag{3.2}$$

$$x^{(i)}(0) = x_{0i},\ i = 0, \ldots, n-1,\ x^{(n-1)}(\infty) = \beta x^{(n-1)}(0), \tag{3.3}$$

if and only if x is a solution of impulsive integral differential equation

$$x(t) = \begin{cases} \sum_{j=0}^{n-2} \frac{t^j}{j!} x_{0i} + \frac{t^{n-1}}{(\beta-1)(n-1)!} \int_0^\infty f(s)ds \\ + \frac{t^{n-1}}{(\beta-1)(n-1)!} \sum_{k=1}^{\infty} I_{n-1k}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \\ + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s)ds \\ \sum_{0<t_k<t} \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{j!} I_{jk}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \quad \text{if } t \in [0, \infty). \end{cases} \tag{3.4}$$

Let $F : J \times E \times \ldots \times E \to \mathcal{P}_{cp,cv}(E)$ be a Carathéodory multimap which satisfies the following assumptions:

$(H1)$ There exist functions $a, b_j \in L^1(J, \mathbb{R}_+), j = 0, \ldots, n-1$, such that

$$\|F(t, z_0, z_1, \ldots, z_{n-1})\|_{\mathcal{P}} \leq a(t) + \sum_{j=0}^{n-1} b_j(t)|z_j|$$

for a.e. $t \in J, z_j \in E, (j = 0, \ldots, n-1)$,

$$a^* = \int_0^\infty a(t)dt < \infty, \quad b_j^* = \int_0^\infty b_j(t)e^t dt < \infty, \quad j = 0, \ldots, n-1,$$

and F has a measurable selection.

$(H2)$ There exist nonnegative constants c_{ikj}, d_{ik} $(i, j = 0, \ldots, n-1; k = 1, 2, \ldots)$ such that

$$|I_{ik}(z_0, z_1, \ldots, z_{(n-1)})| \leq d_{ik} + \sum_{j=0}^{n-1} c_{ikj}|z_j|,$$

$\forall z_j \in E$, $(i, j = 0, \ldots, n-1; k = 1, 2, \ldots)$,

$$d^* = \sum_{k=1}^{\infty} d_k^*, \quad c^* = \sum_{k=1}^{\infty} e^{t_k}(\sum_{j=0}^{n-1} c_{kj}^*) < \infty,$$

where

$$d_k^* = \max\{d_{ik}, i = 0, \ldots, n-1\}, \; c_{kj}^* = \max\{c_{ikj}, i = 0, \ldots, n-1\}.$$

$(H3)$ There exists $p \in L^1(J, \mathbb{R}^+)$ such that, for every bounded subset D in $DPC^{n-1}(J, E)$,

$$\chi(F(t, D^{(i)}(t))) \leq p(t)\chi_D(D), \forall t \in J; (i = 0, \ldots, n-1),$$

with

$$p^* = \int_0^{\infty} p(t)e^t dt < \infty,$$

where $D^{(i)}(t) = \{x^{(i)}(t), x \in D\}$, and χ is the Hausdorff MNC.

$(H4)$ There exists $l_{ik} > 0$ such that, for every bounded subset D in $DPC^{n-1}(J, E)$,

$$\chi(I_k(D^{(i)}(t))) \leq l_{ik}\chi_D(D), (i = 0, \ldots, n-1; k = 1, 2, \ldots),$$

$$l^* = \sum_{k=1}^{\infty} l_k^*, \quad l^{**} = \sum_{k=1}^{\infty}\sum_{j=0}^{n-1} l_{jk} < \infty,$$

where

$$l_k^* = \max\{l_{ik}, i = 0, \ldots, n-1\},$$

and

$$\chi_D(D) = \max\{\sup_{t \in J}(\chi(D^{(i)}(t))), i = 0, \ldots, n-1\}.$$

(H5) There exists a nonnegative constant q such that

$$q := \frac{1}{\beta - 1}(p^* + l^*) + l^{**} + \|p\|_{L1} < 1.$$

Lemma 7. *[11] Let $D \subset DPC^{(n-1)}$ be bounded set such that $D^{(i)}$ is equicontinous and $\lim_{t\to+\infty} e^{-t}|u^{(i)}(t)| = 0$ uniformly for every $u \in D$. Then*

$$\alpha_D(D) = \max\{\sup_{t\in J} e^{-t}\alpha(D^{(i)}(t)) : i = 0, 1, \ldots, n-1\}$$

is a measure of noncompactness in $DPC^{(n-1)}$, where α is the Kurataowski measure of noncompactness on bounded sets in E.

Theorem 3. *[3] Let E be a Banach space. The Kuratowski and Hausdorff MNCs are related by the inequalities*

$$\chi(B) \leq \alpha(B) \leq 2\chi(B), \quad \text{for every } B \in \mathcal{P}_b(E).$$

Theorem 4. *Assume that hypotheses $(H1)-(H5)$ hold. Then the BVP (1.1)–(1.3) has at least one solution.*

Proof. Let $N : DPC^{(n-1)}(J,E) \to \mathcal{P}(DPC^{(n-1)}(J,E))$ be defined by

$$N(x) = \left\{ h \in DPC^{(n-1)} : h(t) = \left\{ \begin{array}{l} \displaystyle\sum_{j=0}^{n-2} \frac{t^j}{j!} x_{0i} \\ +\frac{t^{n-1}}{(\beta-1)(n-1)!}\displaystyle\int_0^\infty v(s)ds \\ +\frac{t^{n-1}}{(\beta-1)(n-1)!}\displaystyle\sum_{k=1}^{\infty} I_{n-1k}(x(t_k), x'(t_k), \\ \ldots, x^{(n-1)}(t_k))) \\ +\frac{1}{(n-1)!}\displaystyle\int_0^t (t-s)^{n-1} v(s)ds \\ +\displaystyle\sum_{0<t_k<t}\sum_{j=0}^{n-1} \frac{(t-t_k)^j}{j!} I_{jk}(x(t_k), x'(t_k), \\ \ldots, x^{n-1}(t_k)) \\ \quad \text{if } t \in [0,\infty), \end{array} \right. \right\}$$

where

$$v \in S_{F,x} = \{v \in L^1(J,E) : v(t) \in F(t, x(t), x'(t), \ldots, x^{(n-1)}(t)), \text{ a.e } t \in \mathbb{R}_+\}.$$

$(H1)$ implies that the set $S_{F,x}$ is nonempty. Since for each $x \in DPC^{n-1}$ the nonlinearity F takes convex values, the selection set $S_{F,x}$ is convex and therefore N has convex values. Under assumptions $(H1), (H2)$, N sends bounded sets into bounded and equicontinuous sets.

Step 1. For bounded $D \subset DPC^{n-1}$, we show that for all $h \in N(D)$,

$$e^{-t}|h^{(i)}(t)| \to 0 \text{ as } t \to \infty,$$

independent of $y \in D$. Let $h \in N(y)$. Then there exists $v \in S_{F,y}$ such that

$$h^{(i)}(t) = \begin{cases} \sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} x_{0i} + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \int_0^\infty v(s)ds \\ + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \sum_{k=1}^{\infty} I_{n-1k}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \\ + \frac{1}{(n-i-1)!} \int_0^t (t-s)^{n-i-1} v(s)ds \\ + \sum_{0<t_k<t} \sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(x(t_k), x'(t_k), \ldots, x^{n-1)})), \ t \in [0,\infty). \end{cases}$$

Thus

$$\begin{aligned} e^{-t}|h^{(i)}(t)| \ \leq \ & e^{-t}\left(\sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!}\right) \max\{|x_{0i}| : \ i = 0, \ldots, n-1\} \\ & + \frac{e^{-t}t^{n-i-1}}{(\beta-1)(n-i-1)!}\left(a^* + \sum_{j=0}^{n} b_j^* R\right) \\ & + \frac{e^{-t}t^{n-i-1}}{(\beta-1)(n-i-1)!} \sum_{k=1}^{\infty} (d_{n-1k} + e^{t_k} \sum_{i=0}^{n-1} c_{ki} R) \\ & + \frac{e^{-t}t^{n-i-1}}{(n-i-1)!}\left(a^* + \sum_{j=0}^{n} b_j^* R\right) \\ & + te^{-t} \sum_{0<t_k<t} \sum_{j=i}^{n-1} (d_{ik} + e^{t_k} \sum_{i=0}^{n-1} c_{ki} R) \\ & \to 0 \text{ as } t \to \infty. \end{aligned}$$

Step 2. To see that N is a $\beta-$condensing operator for a suitable MNC β, let $mod_C(D)$ the modulus of quasi-equicontinuity of the set of functions D defined by

$$mod_C(D) = \max\{\lim_{\delta \to 0} \sup_{x \in D} \max |x^{(i)}(\tau_1) - x^{(i)}(\tau_2)|, i = 0, \ldots, n-1\}.$$

Then $mod_C(D)$ defines an MNC in DPC^{n-1} which satisfies all of the properties in Definition 2 except regularity. Given the Hausdorff MNC χ, let γ be the real MNC defined on bounded subsets on DPC^{n-1} by

$$\gamma(D) = \sup_{t\in J} e^{-t}\chi_D(D(t)).$$

Let $D \in DPC^{n-1}$ be bounded and define the following MNC on bounded subsets of DPC^{n-1} by

$$\beta(D) = \max_{D\in\Delta(DPC^{n-1})}(\gamma(D), mod_C(D)),$$

where $\Delta(DPC^{n-1})$ is the collection of all denumerable bounded subsets of D. Then the MNC β is monotone, regular, and nonsingular. To show that N is $\beta-$condensing, let $D \in DPC^{n-1}$ be bounded set and

$$\beta(D) \leq \beta(N^i(D)). \tag{3.5}$$

We will show that D is relatively compact. Let $\{x_m, m \in \mathbb{N}\} \subset D$ and let

$$N = L_1 + L_2 \circ \Gamma_1 \circ S_F + \Gamma \circ S_F,$$

where $L_1 : DPC^{n-1} \to DPC^{n-1}$ is defined by

$$\begin{aligned}(L_1x)(t) \quad = \quad & \sum_{j=0}^{n-2} \frac{t^{j-i}}{(j-i)!}x_{0i} + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!}\times \\ & \sum_{k=1}^{\infty} I_{n-1k}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \\ & + \sum_{0<t_k<t}\sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)).\end{aligned}$$

$L_2 : \mathbb{R}_+ \to B(E)$ is defined by

$$L_2(x) = \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!}x.$$

$S_F : DPC^{n-1}(J,E) \to L^1(J,E)$ is defined by

$$S_F(x) = \{v \in L^1(J,E) : v \in F(t, x(t), x'(t), \ldots, x^{n-1}(t)), \text{a.e } t \in J\}.$$

$\Gamma_1 : L^1(J,E) \to DPC^{n-1}(J,E)$ is defined by

$$\Gamma_1(g)(t) = \int_0^{\infty} g(s)ds, t \in [0,\infty),$$

and

$$\Gamma(g)(t) = \int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!} g(s)ds.$$

Then

$$|\Gamma g_1(t) - \Gamma g_2(t)| \leq \int_0^t e^{t-s}|g_1(s) - g_2(s)|ds.$$

Moreover, each element $h_m \in N(x_m)$ can be represented as

$$h_m^{(i)} = L_1(x_m) + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)}\Gamma_1(g_m) + \Gamma(g_m), \quad (3.6)$$

with some $g_m \in S_F(x_m)$ and (3.5) yields

$$\beta(\{h_m, m \in \mathbb{N}\}) \geq \beta(\{x_m, m \in \mathbb{N}\}). \quad (3.7)$$

From hypothesis $(H3)$, for a.e. $t \in J$, we have

$$\chi(\{g_m(t), m \in \mathbb{N}\}) \leq e^t p(t)\gamma(\{x_m\}_{m=1}^{\infty}), \quad (3.8)$$

and then,

$$e^{-t}\chi(\{g_m(t), m \in \mathbb{N}\}) \leq p(t)\gamma(\{x_m\}_{m=1}^{\infty}).$$

We have

$$\chi(\{\Gamma(g_m)(t)\}_{m=1}^{\infty}) \leq e^t\gamma(\{x_m\}_{m=1}^{\infty}) \int_0^t p(s)ds,$$

then

$$e^{-t}\chi(\{\Gamma(g_m)(t)\}_{m=1}^{\infty}) \leq \gamma(\{x_m\}_{m=1}^{\infty}) \int_0^t p(s)ds,$$

$$\chi(\{\tfrac{t^{n-i-1}}{(\beta-1)(n-i-1)}\Gamma_1(g_m)(t)\}_{m=1}^{\infty}) \leq \frac{e^t}{\beta-1}\gamma(\{x_m\}_{m=0}^{\infty})p^*.$$

And so,

$$e^{-t}\chi(\{\frac{t^{n-i-1}}{(\beta-1)(n-i-1)}\Gamma_1(g_m)(t)\}_{m=1}^{\infty}) \leq \frac{p^*}{\beta-1}\gamma(\{x_m\}_{m=1}^{\infty}),$$

and

$$\chi(L_1\{x_m(t)\}_{m=1}^{\infty}) \leq e^t(\frac{l^*}{\beta-1} + l^{**})\gamma(\{x_m\}_{m=1}^{\infty}),$$

so that,

$$e^{-t}\chi(L_1\{x_m(t)\}_{m=1}^{\infty}) \leq (\frac{l^*}{\beta-1} + l^{**})\gamma(\{x_m\}_{m=1}^{\infty}).$$

(3.6) and the lower additivity of γ yield

$$\gamma(\{h_m\}_{m=1}^{\infty}) \leq \left[\frac{1}{\beta-1}(p^*+l^*)+l^{**}+\|p\|_{L1}\right]\gamma(\{x_m\}_{m=1}^{\infty}). \tag{3.9}$$

Therefore

$$\gamma(\{x_m\}_{m=1}^{\infty}) \leq \gamma(\{h_m\}_{m=1}^{\infty}) \leq q\gamma(\{x_m\}_{m=1}^{\infty}). \tag{3.10}$$

Since $0 < q < 1$, we infer that

$$\gamma(\{x_m\}_{m=1}^{\infty}) = 0. \tag{3.11}$$

Next, we show that $mod_C(B) = 0$ i.e, the set B is equicontinuous. This is equivalent to showing that every $\{h_m^i\} \subset N^i(B)$ satisfies this property. Given a sequence $\{h_m\}$, there exist sequences $\{x_m\} \subset B$ and $\{g_m\} \subset S_{F,x_m}$ such that

$$h_m^i = L_1(x_m) + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)}\Gamma_1(g_m) + \Gamma(g_m).$$

From (3.11), we infer that

$$\chi_D(\{x_m(t)\} = 0, \text{ for a.e.} t \in [0,\infty).$$

Hypothesis $(H1)$ in turn implies that

$$\chi(\{g_m(t)\} = 0, \text{ for a.e.} t \in [0,\infty).$$

From $(H1)$, the sequence $\{g_m\}$ is integrable bounded, hence semi-compact in $L^1(\Omega_k, E),\ k \in \mathbb{N},\ \Omega_k = [0,k]$. Given $\gamma \in (0,1)$ and K_γ a measurable set of $\mathbb{R}_+$ such that $\lambda(K_\gamma) \leq \gamma$, then $\lambda(\Omega_k \setminus \Omega_{\gamma,k}) \leq \gamma$, where $\Omega_{\gamma,k} = \Omega_k \setminus K_\gamma$,

$$g_n(\Omega_{\gamma,k}) \subseteq C_{\gamma,k} := \overline{\{g_m(t) : t \in \Omega_k \setminus K_\gamma,\ m \in \mathbb{N}\}},\ n \in \mathbb{N},$$

and

$$\lim_{k\to\infty}\int_{\Omega\setminus\Omega_k}|g_m(t)|d\lambda(t) \leq \lim_{k\to\infty}\int_k^{\infty} p(t)d\lambda(t) = 0,\ \Omega = [0,\infty).$$

Hence $\{g_m :\ m \in \mathbb{N}\}$ is weakly compact in $L^1([0,\infty), E)$. Using Mazur's lemma, we deduce that, up to a subsequence, $\{h_m\}$ is relatively compact. Therefore $\beta(\{h_m\}_{m=1}^{\infty}) = 0$ which implies that $\beta(\{x_m\}_{m=1}^{\infty}) = 0$. We have proved that B is relatively compact and so the map N is $\beta-$condensing.

Step 3. By essentially the same method used in [14, Theorem 10.2], it can be proved that N has a closed graph and is a locally compact operator.

Step 4. A priori bounds on solutions.
Let $x \in DPC^{n-1}$ be such that $x \in N(x)$. Then there exists $v \in S_{F,x}$ such that

$$x(t) = \begin{cases} \displaystyle\sum_{j=0}^{n-2} \frac{t^j}{j!} x_{0i} \\ +\frac{t^{n-1}}{(\beta-1)(n-1)!} \displaystyle\int_0^\infty v(s)ds \\ +\frac{t^{n-1}}{(\beta-1)(n-1)!} \displaystyle\sum_{k=1}^{\infty} I_{n-1k}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k))) \\ +\frac{1}{(n-1)!} \displaystyle\int_0^t (t-s)^{n-1} v(s)ds \\ + \displaystyle\sum_{0<t_k<t} \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{j!} I_{jk}(x(t_k), x'(t_k), \dots, x^{n-1}(t_k)) \\ \quad \text{if } t \in [0, \infty). \end{cases}$$

We have

$$x^{(i)}(t) = \begin{cases} \displaystyle\sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} x_{0i} + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \int_0^\infty v(s)ds \\ +\frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \displaystyle\sum_{k=1}^{\infty} I_{n-1k}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)) \\ +\frac{1}{(n-i-1)!} \displaystyle\int_0^t (t-s)^{n-i-1} v(s)ds \\ + \displaystyle\sum_{0<t_k<t} \sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)), \\ \quad \text{if } t \in [0, \infty). \end{cases}$$

Then

$$\begin{aligned} e^{-t}|x^i(t)| \quad &\leq \quad \sum_{j=i}^{n-2} |x_{0i}| + \left(\frac{1}{|\beta-1|} + 1\right)(a^* + d^*) \\ &\quad + \left(\frac{1}{|\beta-1|} + 1\right)(c^* + \sum_{j=0}^{n-1} b_j^*)\|x\|_D. \end{aligned}$$

Hence

$$\|x\|_D \leq \frac{\sum_{j=i}^{n-2} |x_{0i}| + (\frac{1}{|\beta-1|} + 1)(a^* + d^*)}{1 - (\frac{1}{|\beta-1|} + 1)(c^* + \sum_{j=0}^{n-1} b_j^*)} := M_i, \quad (i = 0, \dots, n-1).$$

Finally

$$\|x\|_D \leq \max(M_i, i = 0, \ldots, n-1) := M.$$

From Lemma 3, we deduce that N has at least one fixed point denoted by x. Moreover since $Fix(N)$ is bounded, by Lemma 4, $Fix(N)$ is compact. $\square$

4 Nonconvex case

In this section we present a second existence result for problem (1.1)–(1.3) when the multi-valued nonlinearity is not necessarily convex. In the proof, we will make use of the nonlinear alternative of Leray-Schauder type for condensing map, combined with a selection theorem due to Bressan and Colombo [6], for lower semicontinious multi-valued maps with decomposable values. Also, another result is presented as an application of the fixed point theorem for contractive multi-valued operators. Let $\mathcal{A}$ be a subset of $J \times B$. $\mathcal{A}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $\mathcal{A}$ belongs to the σ-algebra generated by all sets of the form $N \times D$ where N is Lebesgue measurable in J and D is Borel measurable in B. A subset $\mathcal{A}$ of $L^1(J, E)$ is decomposable if, for all $u, v \in \mathcal{A}$ and $N \subset L^1(J, E)$ measurable, the function $u\widetilde{\chi}_N + v\widetilde{\chi}_{J \setminus N} \in \mathcal{A}$, where $\widetilde{\chi}$ stands for the characteristic function of the set A. Let X be a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is *lower semi-continuous (l.s.c.)* if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in E.

Definition 5. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J, E))$ be a multivalued operator. We say that N has property (BC) if*

1) N is lower semi-continuous (l.s.c.);

2) N has nonempty closed and decomposable values.

Let $F : J \times E \to \mathcal{P}(E)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C(J, E) \to \mathcal{P}(L^1(J, E))$$

by letting

$$\mathcal{F}(y) = \{v \in L^1(J, E) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated to F.

Definition 6. *Let $F : J \times E \to \mathcal{P}(E)$ be a multivalued function with nonempty compact values. We say that F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continous and has nonempty closed and decomposable values.*

Next we state a selection theorem due to Bressan and Colombo. let Y be a Banach space. Then every *l.s.c.* multi-valued operator decomposable values has a continuous selection.

Theorem 5. *[6] Let Y be separable metric space and let $N : Y \to \mathcal{P}(L^1(J, E))$ be a multivalued operator which has property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $f : Y \to L^1(J, E)$ such that $f(x) \in N(x)$ for every $x \in Y$.*

Lemma 8. *[10] Let $F : J \times Y \to \mathcal{P}_{cp}(Y)$ be an integrably bounded multimap satisfying*

($\mathcal{H}_{lsc}$) *$F : J \times Y \to \mathcal{P}(Y)$ is a nonempty compact valued multi-map such that*

(a) the mapping $(t, y) \mapsto F(t, y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;

(b) the mapping $y \mapsto F(t, y)$ is l.s.c. for a.e. $t \in J$.

Then F is of lower semi-continuous type.

Theorem 6. *Suppose that hypotheses $(H1) - (H5)$ and the conditions*

(A1) $F : J \times E \times E \ldots \times E \longrightarrow \mathcal{P}_{cl}(E)$ is a multi-valued map such that:
a) $(t, x_0, x_1, \ldots, x_n) \mapsto F(t, x_0, x_1, \ldots, x_n)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
b) $(x, u) \mapsto F(t, x_0, x_1, \ldots, x_n)$ is lower semi-continuous for a.e. $t \in J$;

(A2) $F(t, x_0, x_1, \ldots, x_n) \subset G(t)$ for a.e $t \in J$ and for all $(x_0, x_1, \ldots, x_n) \in E \times E \ldots \times E$ and with $G : J \to \mathcal{P}_{w,cp,c}(E)$ integral bounded;

are satisfied. Then the impulsive boundary value problem (1.1)–(1.3) has at least one solution.

Proof. First, let $\mathcal{F} : DPC^{(n-1)} \to \mathcal{P}(DPC^{(n-1)})$ be defined by

$$\mathcal{F}(x) = \{v \in L^1([0, \infty), E) : v(t) \in F(t, x(t), x'(t), \ldots, x^{(n-1)}(t))), \text{ a.e. } t \in [0, \infty)\}.$$

Now, we establish the properties of $\mathcal{F}(\cdot)$. Analogous results can be found in Halidias and Papageorgiou [16]. We prove that $\mathcal{F}(\cdot)$ has nonempty, closed, decomposable values and is l.s.c.

For the nonempty part, from hypothesis $(A2)$ we have $F(\cdot, x(\cdot), x'(\cdot), \ldots, x^{(n-1)}(\cdot))$ is a measurable multifunction. Then there exists a sequence of measurable selections $\{f_m(t) : m \geq 1\}$ of F such that

$$F(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) = \overline{\{f_m(t) : m \geq 1\}}.$$

From $(A2)$, we have $f_m(\cdot) \in G(\cdot)$. Using the fact that G has weakly compact values, we pass to a subsequence if necessary to get $f_{m_k}(\cdot)$ converges weakly to $f(\cdot)$ in E. Since $\{f_{m_k} : k \geq 1\} \subseteq \{f_m : m \geq 1\}$, then $f \in \overline{\{f_m : m \geq 1\}}$. By Mazur's Lemma there exists $v_m(t) \in \overline{conv}\{f_{m_k}(t) : m \geq 1\}$ such that $v_m(\cdot)$ converges strongly to $f(\cdot)$ in E. So $f(t) \in F(t, x(t), x'(t), \ldots, x^{(n-1)}(t))$, for a.e. $t \in [0, \infty)$. Therefore, for every $x \in DPC^{n-1}$, $\mathcal{F}(x) \neq \emptyset$. The closedness and decomposability of the values of $\mathcal{F}(\cdot)$ are easy to check.

To prove that $\mathcal{F}(\cdot)$ is $l.s.c.$, $(H1)$ and $(A1)$ imply by Lemma 8 that F is of lower semi-continuous type. Using the Theorem 5 of Bressan and Colombo [6], we get that there is a continuous selection

$$f : DPC^{(n-1)} \to L^1([0, \infty), E)$$

such that $f(x) \in \mathcal{F}(x)$ for every $x \in DPC^{n-1}$. We consider the following problem:

$$x^{(n)}(t) = f(x)(t), \quad \text{a.e. } t \in J \backslash \{t_1, \ldots, t_m\} \tag{4.1}$$

$$\Delta x^{(i)}|_{t=t_k} = I_{ik}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)), \;\; i = 0, 1, \ldots, n-1, \;\; k = 1, \ldots, \tag{4.2}$$

$$x^{(i)}(t) = x_{0i}, \;\; (i = 0, 1, \ldots, n-2), \;\; x^{(n-1)}(\infty) = \beta x^{(n-1)}(0), \tag{4.3}$$

Transform the problem (4.1)-(4.3) into a fixed point problem. Consider the operator $P^i : DPC^{(n-1)} \to DPC^{(n-1)}$ defined by

$$P^i(x) = \begin{cases} \displaystyle\sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} x_{0i} + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \int_0^\infty f(x(s))ds \\ \displaystyle + \tfrac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \sum_{k=1}^{\infty} I_{n-1k}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \\ \displaystyle + \tfrac{1}{(n-i-1)!} \int_0^t (t-s)^{n-i-1} f(x(s))ds \\ \displaystyle + \sum_{0<t_k<t} \sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)), \\ \text{if } t \in [0, \infty). \end{cases} \tag{4.4}$$

We shall show that the single-valued operator P^i is completely continuous. From Step 1 through Step 3 of the proof Theorem 4, we can check that P^i maps bounded sets into bounded sets in DPC^{n-1} and P^i is condensing.

Then P^i is a completely continuous. There exists $b^* > 0$ such that, for every solution x of the problem (4.1)–(4.3), we have

$$\|x\|_D \leq b^*.$$

Let

$$U = \{x \in DPC^{(n-1)} : \|x\|_D < b^* + 1\}.$$

From the choice of U there is no $x \in \partial U$ such that $x = \lambda P^i(x)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray Schauder type, we deduce that P^i has a fixed point x in U is a solution of the problem (4.1)–(4.2). Then there exists x which is a solution to problem (1.1)–(1.3) on $[0, \infty)$. □

In this next part we present a second result for the problem (1.1)–(1.3) with a non-convex valued right-hand side. Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider the Hausdorff-Pompeiu metric [5, 22]

$H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{a\in\mathcal{A}} d(a, \mathcal{B}), \sup_{b\in\mathcal{B}} d(\mathcal{A}, b)\right\},$$

where $d(\mathcal{A}, b) = \inf_{a\in\mathcal{A}} d(a, b),\ d(a, \mathcal{B}) = \inf_{b\in\mathcal{B}} d(a, b)$.
Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized (complete) metric space (see [19]).

Definition 7. *A multivalued operator $G : X \to \mathcal{P}_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(G(x), G(y)) \leq \gamma d(x, y), \quad \textit{for each } x,\ y \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Our considerations are based on the following fixed point theorem for contractive multivalued operators given by Covitz and Nadler

Lemma 9. *[19] Let (X, d) be a complete metric space. If $G : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Let us introduce the following hypotheses:

(A3) $F : J \times B \times E \longrightarrow \mathcal{P}_{cp}(E)$; $(t, x_0, x_1, \ldots, x_n) \longmapsto F(t, x_0, x_1, \ldots, x_n)$ is measurable for each $(x_0, x_1, \ldots, x_n) \in E \times E \times \cdots \times E$.

(A4) There exists a function $l \in L^1(J, \mathbb{R}^+)$ such that, for *a.e.* $t \in J$

and all $(x_0, x_1, \ldots, x_n), (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n) \in E \times E \times \cdots \times E$,

$$H_d(F(t, x_0, x_1, \ldots, x_n), F(t, \bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n) \leq l(t) \sum_{j=0}^{n} |x_j - \bar{x}_j|$$

and

$$H_d(0, F(t, 0, 0, \ldots, 0)) \leq l(t) \quad \text{for a.e. } \ t \in J,$$

with

$$\int_0^\infty l(s)e^s ds < \infty.$$

(A5) There exist constants c_{ik} such that

$$|I_{ik}(x_0, x_1, \ldots, x_n) - I_{ik}(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n)| \leq \sum_{k=1}^{\infty} c_{ik} e^{-t_k} |x_i - \bar{x}_i|$$

with

$$\sum_{k=1}^{\infty} c_{ik} < \infty \text{ and} \sum_{k=1}^{\infty} \sum_{j=i}^{n-1} c_{ik} e^{-t_k} < \infty, i = 0, 1, \ldots, n-1.$$

Theorem 7. *Let Assumptions (A3)–(A6) be satisfied. If, in addition,*

$$\int_0^\infty l(s)e^s ds + \|l\|_{L^1} + \sum_{k=1}^{\infty} c_{jk} + \sum_{0<t_k<t} \sum_{j=i}^{n-1} c_{jk} e^{-t_k} < 1,$$

then the BVP (1.1)-(1.3) has at least one solution.

Proof. In order to transform problem (1.1)-(1.3) into a fixed point problem, let the multi-valued operator $N : DPC^{n-1} \to \mathcal{P}(DPC^{n-1})$ be as defined in Theorem 4. We shall show that N satisfies all the assumptions of Lemma 9. The proof will be given in one step.

Step 1. $N(x) \in \mathcal{P}_{cl}(DPC^{n-1})$ for each $x \in DPC^{n-1}$.

Indeed, let $(x_m)_{m\geq 0} \in N(x)$ such that $x_m \longrightarrow \tilde{x}$ in DPC^{n-1}. Then there exists $v_m \in S_{F,x}$ such that for each $t \in [0,\infty)$

$$x_m^{(i)}(t) = \begin{cases} \displaystyle\sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} x_{0i} + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \int_0^\infty v_m(s)ds \\ + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \displaystyle\sum_{k=1}^{\infty} I_{n-1k}(x_m(t_k), x_m'(t_k), \ldots, x_m^{(n-1)}(t_k)) \\ + \frac{1}{(n-i-1)!} \displaystyle\int_0^t (t-s)^{n-i-1} v_m(s)ds \\ + \displaystyle\sum_{0<t_k<t} \sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(x_m(t_k), x_m'(t_k), \ldots, x_m^{(n-1)}(t_k)). \end{cases}$$

Since $v_m(t) \in F(t, x(t), x'(t), \ldots, x^{(n-1)}(t))$,we may pass to a subsequence if necessary to get that v_m converges almost everywhere to some v in E. From $(A4)$, we have

$$|v_m(t)| \leq e^t l(t)(M+1), \quad \|x\|_D \leq M.$$

Also by $(A4)$, we get

$$v(t) \in F(t, \widetilde{x}(t), \widetilde{x}'(t), \ldots, \widetilde{x}^{(n-1)}(t)), \quad \text{a.e.} \quad t \in [0,\infty).$$

Thus

$$\widetilde{x}^{(i)}(t) = \begin{cases} \displaystyle\sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} x_{0i} + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \int_0^\infty v(s))ds \\ + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \displaystyle\sum_{k=1}^{\infty} I_{n-1k}(\widetilde{x}(t_k), \widetilde{x}'(t_k), \ldots, \widetilde{x}^{(n-1)}(t_k)) \\ + \frac{1}{(n-i-1)!} \displaystyle\int_0^t (t-s)^{n-i-1} v(s))ds \\ + \displaystyle\sum_{0<t_k<t} \sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(\widetilde{x}(t_k), \widetilde{x}'(t_k), \ldots, \widetilde{x}^{(n-1)}(t_k)). \end{cases}$$

So $\widetilde{x} \in N(x)$. By the same method used in [14] Theorem 9.61, we can easily prove that

$$\begin{aligned} H_d(N(x), N(x_*)) \quad &\leq \quad \Big[\int_0^\infty l(s) e^s ds + \|l\|_{L^1} + \\ &\Sigma_{k=1}^{\infty} c_{jk} + \Sigma_{0<t_k<t} \Sigma_{j=i}^{n-1} c_{jk} e^{-t_k} \Big] \|x - x_*\|_D. \end{aligned}$$

So N^i is a contraction and thus, by Lemma 9, N has a fixed point x which is a solution of the problem (1.1)–(1.3) on $[0, \infty)$.

□

5 Concluding remarks

In this work, we have established the existence of solutions for Problem (1.1)–(1.3) in both the convex case and the nonconvex case for the nonlinearity. In particular, in each case, the Problem is formulated as a fixed point problem for a multi-valued operator, and then applications have been made from multi-valued analysis, topological fixed point theory, and measure of noncompactness in obtaining solutions.

While in this paper, we have focused on the existence of solutions for impulsive boundary value problems for higher order differential inclusions on the half-line, results concerning boundary value problems for first order impulsive differential equations and inclusions on bounded intervals can be found in [4, 9] and the references therein

Moreover, existence results for nth order impulsive integrodifferential equations on the half-line can be found, to name a few, in [11, 12, 13] and the references therein.

Acknowledgement. The authors would like to thank anonymous referees for their careful reading of the manuscript and pertinent comments; their constructive suggestions substantially improved the quality of the work.

References

[1] J. Andres and R. Bader, *Asymptotic boundary value problems in Banach spaces*, J. Math. Anal. Appl., 274, (2002), 437 – 457.

[2] Z. Agur, L. Cojocaru, G. Mazaur, R. M. Anderson and Y. L. Danon, *Pulse mass measles vaccination across age cohorts*, Proc. Nat. Acad. Sci. USA, 90, (1993), 11698 – 11702.

[3] J. M. Ayerbe Toledano, T. Domínguez Benavides and G. Lopez Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Operator Theory: Advances and Applications 99, Birkhäuser, Basel, 1997.

[4] M. Benchohra, J. J. Nieto and A. Ouahab, *Impulsive differential inclusions involving evolution operators in separable Banach spaces*, Ukrainian Math. J. 64 (2012), 991 – 1018.

[5] V. Berinde and M. Păcurar, *The role of the Pompeiu-Hausdorff metric in fixed point theory*, Creat. Math. Inform. 22, (2013), No. 2, 35 – 42.

[6] A. Bressan and G. Colombo, *Extensions and selections of maps with decomposable values*, Studia Math. 90, (1988), 69 – 86.

[7] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, 580, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[8] J. Diestel, *Remarks on weak compactness in* $L^1(\mu, X)$., Glasgow Math. J. 18, (1977), 87 – 91.

[9] S. Djebali, L. Gòrniewicz, and A. Ouahab, *First order periodic impulsive semi- linear differential inclusions: existence and structure of solution sets*, Math. and Comput. Mod., 52, (2010), 683 – 714.

[10] M. Frigon and A. Granas, *Théorèmes d'existence pour des inclusions différentielles sans convexité*, C. R. Acad. Sci. Paris, Ser. I, 310, (1990), 819 – 822.

[11] D. Guo, *Existence of solutions for nth order impulsive integro-differential equations in a Banach space*, Proceedings of the Third World Congress of Nonlinear Analysts, Part 2 (Catania, 2000), Nonlinear Anal. 47, (2001), No. 2, 741 – 752.

[12] D. Guo, *A class of nth-order impulsive integrodifferential equations in Banach spaces*, Comput. Math. Appl. 44 (2002), No. 10 – 11, 1339 – 1356.

[13] D. Guo, *Multiple positive solutions of a boundary value problem for nth-order impulsive integro-differential equations in Banach spaces*, Nonlinear Anal. 63, (2005), No. 4, 618 – 641.

[14] J. R. Graef, J. Henderson and A. Ouahab, *Impulsive Differential Inclusions A Fixed Point Approach*, De Gruyter Series in Nonlinear Analysis and Applications 20, de Gruyter, Berlin, 2013.

[15] A. Halanay and D. Wexler, *Teoria calitativa a systeme cu impulsuri*, Editura Republicii Socialiste Romania, Bucharest, 1968.

[16] N. Halidias and N. Papageorgiou, *Existence and relaxation results for nonlinear second order multivalued boundary value problems in* $\mathbb{R}^N$, J. Differential Equations 147, (1998), 123 – 154.

[17] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer, Dordrecht, 1997.

[18] M. Kamenskii, V. Obukhovskii and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, de Gruyter, Berlin, 2001.

[19] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, The Netherlands, 1991.

[20] V. D. Milman and A. A. Myshkis, *Random impulses in linear dynamical systems, in "Approximante Methods for Solving Differential Equations*, Publishing house of the Academy of Sciences of Ukainian SSR, Kiev, in Russian, (1963), 64 – 81.

[21] N. A. Perestyuk, V. A. Plotnikov, A. M. Samoilenko and N. V. Skripnik, *Differential Equations with Impulse Effects. Multivalued Right-hand Sides with Discontinuities*, de Gruyter, Berlin, 2011.

[22] D. Pompeiu, *Sur le continuité des fonctions de variables complexes* (These), Gauthier-Villars, Paris, 1905, Ann. Fac. Sci. de Toulouse 7, (1905), 264 – 315.

[23] I. I. Vrabie, *Existence for nonlinear evolution inclusions with nonlocal retarded initial conditions*, Nonlinear Anal. 74, (2011), 7047 – 7060.