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# Ideals generated by linear forms and symmetric algebras*

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**Abstract**

We consider ideals generated by linear forms in the variables $X_1 \dots, X_n$ in the polynomial ring $R[X_1, \dots, X_n]$, being $R$ a commutative, Noetherian ring with identity. We investigate when a sequence $a_1, a_2, \dots, a_m$ of linear forms is an $s-$sequence, in order to compute algebraic invariants of the symmetric algebra of the ideal $I = (a_1, a_2, \dots, a_m)$.

**MSC**: 13C10, 13C15, 13D02

**keywords:** Symmetric algebra, linear forms, regularity.

## 1 Introduction

Let $M$ be a finitely generated module on a commutative ring $R$ with identity. Let $A = (a_{ij})$ be a $n \times m$ matrix, with entries in $R$, $I_k(A)$ the ideal generated by the $k \times k$ minors of $A$, $1 \leq k \leq \min(m,n)$, and let $\varphi : R^m \longrightarrow R^n$ be a module homomorphism. We denote by $I_k(\varphi)$ the ideal $I_k(A)$, where $A = (a_{ij})$ is the $n \times m$ matrix associated to $\varphi$, for an appropriate choice of the bases.
Let

$$R^m \xrightarrow{\varphi} R^n \to M \to 0 \tag{1}$$

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be a free presentation of the module $M$. If we consider the symmetric algebras of the modules in (1), the presentation ideal $J$ of $\mathrm{Sym}_R(M)$ is generated by the linear forms in the variables $Y_j$, $1 \leq j \leq n$:

$$a_i = \sum_{j=1}^{n} a_{ji} Y_j, \qquad 1 \leq i \leq m$$

The theory of $s-$sequences has been recently introduced by Herzog, Restuccia, Tang ([4]) and it permits to compute the invariants of $\mathrm{Sym}_R(M)$ starting from the main algebraic invariants of quotients of $R$, via the initial ideal $\mathrm{in}_<(J)$, with respect to a suitable term order, introduced in $R[Y_1, \ldots, Y_n]$, where $n$ is the number of elements in a minimal system of generators of $M$. In this paper we are interested to the case the ideal $J$ is generated by an $s-$sequence. The problem is part of a wider context, precisely:
*Given an ideal* $I = (a_1, \ldots, a_m) \subset R[X_1, \ldots, X_n]$ *generated by linear forms in the variables* $X_1, \ldots, X_n$*, we want to study when* $I$ *is generated by an* $s-$*sequence and to compute the standard invariants of* $\mathrm{Sym}_R(I)$ *in terms of the corresponding invariants of special quotients of the ring* $R$*.*
Standard invariants of $\mathrm{Sym}_R(I)$ are Krull dimension, multiplicity, depth and regularity, denoted respectively by $\dim(\mathrm{Sym}_R(I))$, $\mathrm{e}(\mathrm{Sym}_R(I))$, $\mathrm{depth}(\mathrm{Sym}_R(I))$ with respect to the maximal graded ideal, $\mathrm{reg}(\mathrm{Sym}_R(I))$. The first three invariants are classical. For the last invariant, we recall that $\mathrm{reg}(\mathrm{Sym}_R(I))$ is the Castelnuovo-Mumford regularity of the graded module $I$. Its importance is briefly explained in Einsenbud-Goto theorem which is an interesting description of regularity in terms of graded Betti numbers of $M$ ([3]). In general the problem is hard, but if $I$ is generated by an $s-$sequence, our approach gives some interesting results. In Section 1, we recall some results obtained about ideals generated by linear forms as relation ideals of special symmetric algebras. At the end of the section, we consider some basic properties about $s-$sequences and, additionally, we recall how to compute the invariants ([7], [8]). In Section 2, for the classes of ideals studied in the previous section, we find sufficient and necessary conditions so that they are generated by $s-$sequences. In this direction a main result is:

**Theorem** *Let* $J = (a_1, \ldots, a_m) \subset S = R[Y_1, \ldots, Y_n]$*, be an ideal generated by* $m$ *linear forms,* $a_i = \sum_{j=1}^{n} a_{ij} Y_j$, $a_{ij} \in R, 1 \leq i \leq m, 1 \leq j \leq n$*. If* $\mathrm{depth}(I_k(\varphi)) \geq m - k + 1$, $1 \leq k \leq m$*, then*

- $(i)$ $J$ *is generated by an* $s-$*sequence of* $m$ *elements;*
- $(ii)$ $\mathrm{Sym}_S(J) \cong S[Z_1, \ldots, Z_m]/K$*, where* $K$ *is an ideal generated by linear forms in the variables* $Z_j$, $1 \leq j \leq m$ *and* $in_< K = (J_1 Z_1, J_2 Z_2, \ldots,$

*$J_mZ_m$), where $J_1, \dots, J_m$ are the annihilator ideals of the sequence $a_1, \dots, a_m$;*

*(iii)* $J_{i-1} : (a_i) = J_{i-1}$ *with* $J_{i-1} = (a_1, \dots, a_{i-1})$.

## 2 Notations and basic results

This section deals with ideals generated by linear forms of a polynomial ring $R[Y_1, \dots, Y_n]$ over a commutative, noetherian ring $R$ with identity. We are interested in ideals that are kernels of epimorphisms of symmetric algebras, in particular they are ideals of relations of symmetric algebras of finitely generated modules on $R$. We give a list of results that will be useful in the following.

**Definition 2.1.** Let $\underline{a} = \{a_1, \dots, a_n\}$ be a sequence of elements of $R$. The sequence $\underline{a}$ is called a $d-$sequence if $\underline{a}$ is a minimal generating system for the ideal $(a_1, \dots, a_n)$ and $(a_1, \dots, a_i) : a_{i+1}a_k = (a_1, \dots, a_i) : a_k$ for all $i = 0, \dots, n-1$, $k \geq i+1$.

**Definition 2.2.** Let $I$ be an ideal of the ring $R$. $I$ is an almost complete intersection if the number of its generators is $\operatorname{depth}(I) + 1$.

Let $A = (a_{ij})$ be a $m \times n$ matrix, $I_k(A)$ the ideal generated by all $k \times k$ minors of the matrix $A$, $1 \leq k \leq \min(m, n)$. By definition, we have

$$I_0(A) = R \quad \text{and} \quad I_k(A) = 0 \quad \text{for } k > \min(m, n)$$

Let $R^m \overset{\varphi=(a_{ij})}{\longrightarrow} R^n$ an homomorphism between free modules. We denote by $I_k(\varphi)$ the ideal $I_k(A)$, where $A$ is the matrix associated to $\varphi$, for a convenient choice of the bases. Let

$$R^m \overset{\varphi=(a_{ij})}{\longrightarrow} R^n \longrightarrow M \longrightarrow 0$$

be a free presentation of the module $M$.
The following results are known. The kernel of the canonical epimorphism

$$S = \operatorname{Sym}_R(R^n) \to \operatorname{Sym}_R(M) \to 0$$

is a complete intersection if and only if

$$\operatorname{depth}(I_k(\varphi)) \geq m - k + 1, \quad 1 \leq k \leq m$$

*Proof.* See [1], Proposition 3. □

The kernel of the canonical epimorphism

$$S = \mathrm{Sym}_R(R^n) \to \mathrm{Sym}_R(M) \to 0$$

is an almost complete intersection with $\mathrm{depth}(a_1, \ldots, a_{m-1}) = m - 1$ if and only if

$$\mathrm{depth}(I_k(\varphi')) \geq (m-1) - k + 1, \quad 1 \leq k \leq m-1$$

where

$$\varphi = \varphi' + \varphi'', \quad \varphi' : R^{m-1} \to R^m$$

such that

$$\varphi'(f_1) = a_1, \ldots, \varphi'(f_{m-1}) = a_{m-1},$$

where $f_1, \ldots, f_{m-1}$ form a standard basis for $R^{m-1}$. Moreover, the kernel can be generated by a $d-$sequence if and only if

$$Z_1 \cap IK = B_1$$

where $Z_1$ and $B_1$ are respectively the 1-cycle and the 1-boundary of the Koszul complex $K$ over $a_1, \ldots, a_m$ and $I$ is an ideal of $R$.

*Proof.* See [8], Theorem 6. □

Let $M$ be a finitely generated module on $R$, with generators $f_1, f_2, \ldots, f_n$. We denote by $(a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$ the relation matrix, by $\mathrm{Sym}_i(M)$ the $i$-th symmetric power of $M$, and by $\mathrm{Sym}_R(M) = \bigoplus_{i \geq 0} \mathrm{Sym}_i(M)$ the symmetric algebra of $M$. Note that

$$\mathrm{Sym}_R(M) = R[Y_1, \ldots, Y_n]/J,$$

where

$$J = (g_1, \ldots, g_m), \text{ and } g_i = \sum_{j=1}^{n} a_{ij} Y_j.$$

We consider $S = R[Y_1, \ldots, Y_n]$ a graded ring by assigning to each variable $Y_i$ the degree 1 and to the elements of $R$ the degree 0. Then $J$ is a graded ideal and the natural epimorphism $S \to \mathrm{Sym}_R(M)$ is a homomorphism of graded $R-$algebras.
Let $<$ be a monomial order on monomials in $Y_1, \ldots, Y_n$ with $Y_1 < Y_2 < \ldots < Y_n$. We call admissible such an order. For any polynomial $f \in$

$R[Y_1,\dots,Y_n]$, $f = \sum_\alpha a_\alpha Y^\alpha$, we put $\mathrm{in}_<(f) = a_\alpha Y^\alpha$ where $Y^\alpha$ is the largest monomial in $f$ with respect to $<$ with $a_\alpha \neq 0$, and we set

$$\mathrm{in}_<(J) = (\mathrm{in}_<(f) : f \in J).$$

For $i = 1,\dots,n$ we set $M_i = \sum_{j=1}^{i} Rf_j$, and let $I_i$ be the colon ideal $M_{i-1} : \langle f_i \rangle$. In other words, $I_i$ is the annihilator of the cyclic module $M_i/M_{i-1}$ and so $M_i/M_{i-1} \cong R/I_i$. For convenience we also set $M_0 = 0$.

**Definition 2.3.** The colon ideals $I_i$, $1 \leq i \leq n$, are called annihilator ideals of the sequence $f_1,\dots,f_n$.

Notice that $(I_1Y_1, I_2Y_2,\dots,I_nY_n) \subseteq \mathrm{in}_<(J)$, and the ideals coincide in degree 1.

**Definition 2.4.** The generators $f_1,\dots,f_n$ of $M$ are called an $s-$sequence (with respect to an admissible order $<$), if

$$\mathrm{in}_<(J) = (I_1Y_1, I_2Y_2,\dots,I_nY_n)$$

If in addition $I_1 \subset I_2 \subset \dots \subset I_n$, then $f_1,\dots,f_n$ is called a strong $s-$sequence.

The invariants of the symmetric algebra of a module which is generated by an $s-$sequence can be computed by the corresponding invariants of quotients of $R$. We have

**Proposition 2.1.** *Let $M$ be generated by an $s-$sequence $f_1,\dots,f_n$, with annihilator ideals $I_1,\dots,I_n$. Then*

*1.* $d := \dim(\mathrm{Sym}_R(M)) = \max\limits_{\substack{0\leq r\leq n,\\ 1\leq i_1<\dots<i_r\leq n}} \{\dim(R/(I_{i_1}+\dots+I_{i_r})) + r\};$

*2.* $\mathrm{e}(\mathrm{Sym}_R(M)) = \sum\limits_{\substack{0\leq r\leq n, 1\leq i_1<\dots<i_r\leq n\\ \dim(R/(I_{i_1}+\dots+I_{i_r}))=d-r}} \mathrm{e}(R/(I_{i_1}+\dots+I_{i_r})).$

*and, if $f_1,\dots,f_n$ is a strong $s-$sequence, then*

*1'.* $d = \max\limits_{0\leq r\leq n} \{\dim(R/I_r) + r\};$

*2'.* $\mathrm{e}(\mathrm{Sym}_R(M)) = \sum\limits_{\substack{r\\ \dim(R/I_r)=d-r}} \mathrm{e}(R/I_r).$

*If $R = K[X_1,\dots,X_m]$ and we assume that $M$ is generated by a strong s-sequence of elements of the same degree, with annihilator ideals $I_1 \subset \dots \subset I_n$, we have*

3. $\operatorname{reg}(\operatorname{Sym}_R(M)) \leq \max\{\operatorname{reg}(I_i) : i = 1, \dots, n\}$;

4. $\operatorname{depth}(\operatorname{Sym}_R(M)) \geq \min\{\operatorname{depth}(R/I_i) + i : i = 0, 1, \dots, n\}$.

*Proof.* See [4], Proposition 2.4., 2.6. □

## 3 Relation ideals generated by $d$-sequences

The aim of this section is to study ideals generated by linear forms that are relation ideals of symmetric algebras and to describe their invariants via the $s-$sequence theory. Let $J$ be the ideal of relations of the symmetric algebra $\operatorname{Sym}_R(M)$ of a module $M$.

Let $J = (a_1, \dots, a_m) \subset S = R[Y_1, \dots, Y_n]$ be an ideal generated by $m$ linear forms $a_i = \sum_{j=1}^{n} a_{ji} Y_j$ on the variables $Y_j$. If $\operatorname{depth}(I_k(\varphi)) \geq m-k+1$, $1 \leq k \leq m$, then

(*i*) $J$ is generated by an $s-$sequence of $m$ elements;

(*ii*) $\operatorname{Sym}_S(J) \cong S[Z_1, \dots, Z_m]/K$, where $K$ is an ideal generated by linear forms on the variables $Z_j$ and $\operatorname{in}_<(K) = (J_1 Z_1, \dots, J_m Z_m)$, where $J_1, \dots, J_m$ are the annihilator ideals of the $s-$sequence generating the ideal $J$;

(*iii*) $J_{i-1} : (a_i) = J_{i-1}$, $J_i = (a_1, \dots, a_{i-1}), i = 2, \dots, m$ and $J$ is generated by a strong $s$-sequence.

*Proof.* (*i*) By Theorem 2, $J$ is generated by a regular sequence, then it is generated by an $s-$sequence with respect to the reverse lexicographic order on the monomials in the variables $Y_j$ with $Y_n > \dots > Y_1$.
(*ii*) Since $J$ is generated by a strong $s-$sequence, the ideal $K$ has a Gröbner basis that is linear in the variables $Z_1, \dots, Z_m$, then $\operatorname{in}_<(K) = (J_1 Z_1, J_2 Z_2, \dots, J_m Z_m)$.
(*iii*) Since $a_1, \dots, a_m$ is a regular sequence, by definition we have

$$J_1 = 0 : (a_1) = (0),$$

$$J_i = (a_1, a_2, \dots, a_{i-1}) : (a_i) = (a_1, a_2, \dots, a_{i-1}) \quad i = 2, \dots, m$$

then the assertion holds. Now, it results $J_{i-1} \subsetneq J_i$, $i = 2, \dots, m$, and the $s-$sequence is strong. □

The assertion (*ii*) of the theorem 3 gives information about the initial ideal of the relation ideal of the first syzygy module of the ideal $J$.

Let $J = (a_1, \dots, a_m) \subset R[Y_1, \dots, Y_n] = S$ be an ideal generated by linear forms in the variables $Y_i$ that form a regular sequence. Then:

(*i*) $\dim(\mathrm{Sym}_S(J)) = \dim R + n + 1$; If $R = K[X_1, \ldots, X_t]$ and we suppose that $\deg a_i = a$ for all $i$, then

(*ii*) $\mathrm{e}(\mathrm{Sym}_S(J)) = \sum_{i=1}^{m} a^{i-1}$.

(*iii*) If $R$ is the polynomial ring $\mathrm{reg}(\mathrm{Sym}_S(J)) \leq (m-1)(a-1)+1$;

(*iv*) $\mathrm{depth}(\mathrm{Sym}_S(J)) = \mathrm{depth}(S) + 1$, if $R$ is Cohen-Macaulay.

*Proof.* Since $J$ is generated by a strong $s-$sequence, we can compute the standard invariants, using Proposition 2.1, then

(i) $\dim(\mathrm{Sym}_S(J)) = \max_{0 \leq r \leq m}\{\dim(S/J_r) + r, r = 0, \ldots, m\} =$ $\dim(S/(a_1, \ldots, a_{m-1})) + m = \dim(S) - m + 1 + m = \dim(S) + 1$.

(ii) $\mathrm{e}(\mathrm{Sym}_R(J)) = \sum_{0 \leq r \leq m} \mathrm{e}(R/J_r)$,
being $J_r$ generated by a regular sequence. It follows that $\mathrm{e}(\mathrm{Sym}_S(J)) = \sum_{i=1}^{m} a^{i-1}$, where $a$ is the degree of the generators of $J_i$.

(iii) $\mathrm{depth}(\mathrm{Sym}_S(J)) \geq \min_{0 \leq r \leq m}\{\mathrm{depth}(S/J_r) + r, r = 0, \ldots, m\} =$
$= \min\{\mathrm{depth}(S) - m + 1 + m\} = \mathrm{depth}(S) + 1$.
If $R$ is Cohen-Macaulay, $\dim(S) = \mathrm{depth}(S)$, then

$\mathrm{depth}(S) + 1 \leq \mathrm{depth}(\mathrm{Sym}_S(J)) \leq \dim(\mathrm{Sym}_S(J)) = \dim(S) + 1 = \mathrm{depth}(S) + 1$.

(iv) $\mathrm{reg}(\mathrm{Sym}_S(J)) \leq (m-1)(a-1)+1$, being $a$ the degree of any generator of $J$ ([7], Proposition 1).

□

(*i*) follows from $\mathrm{Sym}_R(J) = \mathcal{R}(J)$, since $J$ is generated by a regular sequence and $\dim(\mathcal{R}(J)) = \dim(R[Y_1, \ldots, Y_n]) + 1$ ([2], [9]).

Let $J = (a_1, \ldots, a_{m-1}, a_m) \subset S = R[Y_1, \ldots, Y_n]$ be an ideal generated by $m$ linear forms. Suppose that:

$$\mathrm{depth}(a_1, \ldots, a_{m-1}) = m - 1, \quad Z_1 \cap JK = B_1, \quad J = (a_1, \ldots, a_{m-1})$$

Then we have

(i) $J$ is generated by a $d-$sequence;

(ii) $\mathrm{Sym}_S(J) \cong S[Z_1, \ldots, Z_m]/K$, where $K$ is an ideal generated by linear forms in the variables $Z_j$, $\mathrm{in}_< K = (J_1 Z_1, \ldots, J_m Z_m)$, $J_1, \ldots, J_m$ are the annihilator ideals of the sequence $a_1, \ldots, a_m$;

(iii) The annihilator ideals of $J$ are such that

$$J_{i-1} : (a_i) = J_{i-1}, \quad i = 1, \ldots, m-1$$

and $J_{m-1} : (a_m) = J_m$, the last annihilator ideal.

*Proof.* (*i*) By theorem 2, the elements $a_1, \ldots, a_m$ form a $d-$sequence and then a strong $s-$sequence with respect to the reverse lexicographic order on the monomials in the variables $Z_i$ and with $Z_m > Z_{m-1} > \ldots > Z_1$.

(*ii*) $\mathrm{Sym}_S(J) = S[Z_1, \ldots, Z_m]/K$ and $\mathrm{in}_<(K) = (J_2Y_2, J_3Y_3, \ldots, J_{m-1}Y_{m-1}, J_mY_m)$ and $J_i = (a_1, \ldots, a_{i-1})$ for $i = 2, \ldots, m-1$.

□

Let $J = (a_1, \ldots, a_{m-1}, a_m) \subset S = R[Y_1, \ldots, Y_m]$ be an ideal generated by linear forms that are an almost complete intersection $d-$sequence. Put $J_m = (a_1, \ldots, a_{m-1}) : (a_m)$. Then

(i) $\dim(\mathrm{Sym}_S(J)) = \max\{\dim S + 1, dim(S/J_m) + m\}$;

(ii) $\mathrm{depth}(\mathrm{Sym}_S(J)) \geq \min\{\mathrm{depth}(S/J_m), \mathrm{depth}(R) + n + 1\}$, with the equality if $S$ is Cohen-Macaulay; If $R = K[X_1, \ldots, X_t]$ and $\mathrm{deg} a_i = a$ for all $i$, then

(iii) $\mathrm{e}(\mathrm{Sym}_S(J)) = \sum_{i=1}^{m-1} a^{i-1} + \mathrm{e}(S/J_m)$;

(iv)
$$\mathrm{reg}(\mathrm{Sym}_S(J)) \leq \max\{(m-2)(a-1) + 1, \mathrm{reg}(S/J_m)\}.$$

*Proof.* (i) The ideal $J$ is generated by a strong $s-$sequence, because it is generated by a $d-$sequence and

$$(0) = J_1 \subset J_2 \subset \ldots \subset J_{m-1} \subset J_m.$$

(ii) The assertion follows by [4].

(iii) $\dim(\mathrm{Sym}_S(J)) = \dim(S[Z_1, \ldots, Z_m]/J) = \dim(S[Z_1, \ldots, Z_m]/\mathrm{in}_<(J)) =$

$$\begin{aligned}
&= \max\{\dim(S/J_r) + r, r = 0, \ldots, m\} \\
&= \max\{\dim(S/J_{m-1}) + m - 1, \dim(S/J_m + m)\} = \\
&= \max\{\dim(S) - m + 2 + m - 1, \dim(S/J_m) + m\} \\
&= \max\{dim(S) + 1, \dim(S/J_m) + m\}.
\end{aligned}$$

(iv) Since $\operatorname{reg}(\operatorname{Sym}_S(J)) \leq \max\{(m-2)(a-1)+1, \operatorname{reg}(S/J_m)\}$, the assertion follows from [7], Proposition 1.

□

Let $f = \sum_{i=1}^{n} a_i Y_i$ be a linear form, $f \in R[Y_1, \ldots, Y_n]$. Suppose that $(0 : f) = (0 : f^2)$, then we have:

(i) $I = (f)$ is generated by an $s-$sequence;

(ii) $\operatorname{Sym}_S(I) = S[Z]/J$, and $\operatorname{in}_<(J) = (I_1 Z)$, $I_1 = (0 : f)$ the annihilator ideal of the sequence $\{f\}$;

*Proof.* By the condition $(0 : f) = (0 : f^2)$, the sequence $\{f\}$ is a $d-$sequence, then $\{f\}$ is an $s-$sequence ([4], Corollary 3.3.), using the reverse lexicographic order on the monomials in the unique variable $Y$. In this case $I_0 = (0)$, $I_1 = (0 : f) = (0 : f^2)$ is the unique annihilator ideal of $I$. □

Let $f = \sum_{i=1}^{n} a_i Y_i$ be a linear form, $f \in R[Y_1, \ldots, Y_n] = S$, $(0 : f) = (0 : f^2)$ and let $I = (f)$. We have:

(i) $\dim(\operatorname{Sym}_S(I)) = \dim(R) + n + 1$;

(ii) $\operatorname{e}(\operatorname{Sym}_S(I)) = \operatorname{e}(S/(0 : f))$;

(iii) If $R = K[X_1, X_2, \ldots, X_m]$, then

$$\operatorname{depth}(\operatorname{Sym}_S(I)) \geq \operatorname{depth}(S/(0 : f)) + 1;$$

(iv) If $R = K[X_1, X_2, \ldots, X_m]$, then

$$\operatorname{reg}(\operatorname{Sym}_S(I)) \leq \operatorname{reg}(S/(0 : f)) + 1.$$

*Proof.* (i) $\dim(S/I_0) = \dim(S)$ and $\dim(S/(0 : f)) = \dim(S)$, hence $\dim(\operatorname{Sym}_S(I)) = \dim(S) + 1$, by Proposition 2.1.

(ii) Using Proposition 2.1, the sum in (ii) has only one summand $\operatorname{e}(S/I_1)$

(iii) $\operatorname{depth}(S/I_0) = \operatorname{depth}(S)$, and $\operatorname{depth}(S/(0 : f)) \leq \operatorname{depth}(S)$, by Proposition 2.1.

(iv) See [7], Theorem 2.

□

Sequences of linear forms that are $d$-sequences are the simplest examples besides the regular sequences, that generate relation ideals of symmetric algebras and provide a fertile testing ground for general results. In particular, if $R = K[X_1, \ldots, X_n]$ and $I = (X_1^a, \ldots, X_n^a)$, then $\mathrm{Sym}(I) = \mathcal{R}(I) = K[X_1, \ldots,$
$X_n]/J$, where $J$ is the ideal generated by all $2 \times 2$ minors $[i, j]$, $i = 1, \ldots, n$, $j = 1, 2, \ldots$, of the matrix $\begin{pmatrix} X_1^a & \ldots & X_n^a \\ Y_1 & \ldots & Y_n \end{pmatrix}$, $a$ integer, $a \geq 1$, that are linear forms in the variables $Y_1, \ldots, Y_n$. Therefore: Let $J$ be as before and let the minors be ordered lexicographically $[1, 2] > [1, 3] > \ldots > [n-1, n]$. Then $J$ is generated by an $s$-sequence of linear forms in the variables $Y_1, \ldots, Y_n$ with respect to the reverse lexicographic order.

*Proof.* Any $d$-sequence is an $s$-sequence, then the assertion follows. □

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## References


[1] L. Avramov, *Complete Intersections and Symmetric Algebras*, Reports Dept. Math. University of Stockholm, 7, 1980.

[2] W. Bruns, J. Herzog *Cohen-Macaulay Rings, Rev. Ed.*, Cambridge Studies in Advanced Mathemathics 39, Cambridge University Press, 1998.

[3] D. Eisenbud, S. Goto, *Linear Free Resolutions and Minimal Multiplicity*, Journal of Algebra, 88, 1984, pp. 89 – 133.

[4] J. Herzog, G. Restuccia, Z. Tang, *s-Sequences and Symmetric Algebras*, Manuscripta Math., 104, 2001, pp. 479 – 501.

[5] J. Herzog, A. Simis, W.V. Vasconcelos, *Approximation Complexes of Blowing-up rings*, Journal of Algebra, 74(2), 1982, pp. 466 – 493.

[6] C. Huneke, *The Theory of d-Sequences and Powers of Ideals*, Advanced in Math., 37, 1982, pp. 49 – 60.

[7] G. Restuccia, *Symmetric Algebras of Finitely Generated Graded Modules and s-Sequences*, Rend. Sem. Mat., Univ. Pol. Torino, 64(4), 2006, pp. 479 – 496.

[8] G. Restuccia, *Formes Linéaires et Algèbres Symétriques*, Bull. Sc. Math., II série, 110, 1986, pp. 391 − 410.

[9] R. H. Villarreal, *Monomial Algebras*, Marcel Dekker Inc., 2001.