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ON THE ANISOTROPIC NORM OF DISCRETE TIME STOCHASTIC SYSTEMS WITH STATE DEPENDENT NOISE*

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Abstract

The purpose of this paper is to determine conditions for the boundedness of the anisotropic norm of discrete–time linear stochastic systems with state dependent noise. It is proved that these conditions can be expressed in terms of the feasibility of a specific system of matrix inequalities.

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1 Introduction

Since the early formulation and developments due to E. Hopf and N. Wiener in the 1940's, the filtering problems received much attention. The famous

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results of Kalman and Bucy derived two decades later ([9], [10]) and their successful implementation in many applications including aerospace, signal processing, geophysics, etc., strongly stimulated the reasearch in this area. A comprehensive survey of linear filtering and estimation can be found in [8]. An important issue concerning the filtering performance is the robustness with respect to the modelling uncertainty of the system which state is estimated. It is known that the filter performance deteriorates due to the modelling errors. Many papers have been devoted to the robust filtering and in the presence of parametric uncertainty (see *e.g.* [4], [6], [11] and the references therein). There are applications where the system parameters are subject to random perturbations requiring stochastic models with *state–dependent noise* (or *multiplicative noise*). Such stochastic systems have been intensively studied over the last few decades (see [22] for early references) by considering their H_2 and H_∞ norms ([6], [20]). Recalling that H_2 optimization may not be suitable when the considered signals are strongly colored (e.g. periodic signals), and that H_∞-optimization may poorly perform when these signals are weakly colored (e.g. white noise), compromises between these two approaches were seeked, mostly by considering multi objective optimization (see e.g. [1] and [14]).

In the recent years, a considerable effort has been made to characterize the so called anisotropic norm of linear deterministic systems [5], [12], [18], [19]. The anisotropic norm offers and intermediate topology between the H_2 and H_∞ norms, and as such it provides a single-objective optimization approach alternative, to the multi objective approach of e.g. [1] and [14].

In [19] it is proved a Bounded Real Lemma type result for the anisotropic norm of stable deterministic systems. It is shown that the boundedness norm condition implies to solve a nonconvex optimization problem with reciprocical variables.

The aim of the present paper is to investigate a procedure to determine the anisotropic norm for *stochastic systems with state–dependent noise* and to derive conditions for the boundedness of this norm in this case. Such characterization will allow future developments in the control and estimation algorithms to this class of stochastic systems which seems to have some important applications (see e.g. [6] and [16]).

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathbf{R}^n$ denotes the n dimensional Euclidean space, $\mathbf{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbf{R}^{n\times n}$

means that P is symmetric and positive definite. The trace of a matrix Z is denoted by $Tr\{Z\}$, $col\{a,\ b\}$ denotes a column vector obtained with the concatenation of the vectors a and b. We also denote by $\mathcal{N}(C)$ the basis for the right null space of C.

2 Problem Statement

Consider the following discrete-time stochastic system we denote by F with state- multiplicative noise:

$$x_{k+1} = (A + H\xi_k)x_k + Bw_k \text{ and } z_k = Cx_k + Dw_k \tag{1}$$

where $x_k \in \mathbf{R}^n$ denotes the state vector at moment k, $w_k \in \mathbf{R}^m$ stands for the input, $z_k \in \mathbf{R}^p$ represents the output and $\xi_k \in \mathbf{R}$ is a random discrete-time white noise sequence, with zero mean and unit covariance.

We consider the class of w_k produced by the following *generating filter* with m inputs and m outputs denoted by G:

$$h_{k+1} = (\alpha + \eta\xi_k)h_k + \beta v_k \text{ and } w_k = \gamma h_k + \delta v_k, \tag{2}$$

where v_k is a white noise sequence, independent of ξ_k and also with zero mean and unit covariance. Throughout the paper both stochastic systems (1) and (2) are assumed *exponentially stable in mean square.* Recall that a stochastic system of form (1) is called exponentially stable in mean square if there exist $c_1 > 0$ and $c_2 \in (0, 1)$ such that $E[|x_k|^2] \leq c_2 c_1^k |x_0|^2$ for all $k \geq 0$ and for any initial condition $x_0 \in \mathbf{R}^n$ at $k = 0$, where E denotes the expectation and $|\cdot|$ stands for the Euclidian norm. Consider the estimate $\hat{w}_k$ of w_k based on past measurements, namely,

$$\hat{w}_k = E\{w_k | w_j, j < k\} \tag{3}$$

and denote the estimation error by

$$\tilde{w}_k = w_k - \hat{w}_k. \tag{4}$$

The mean anisotropy of G is then obtained by the Szego-Kolmogorov formula:

$$\bar{A}(G) = -\frac{1}{2} ln\ det \left(\frac{mE(\tilde{w}_k \tilde{w}_k^T}{Tr\{w_k w_k^T\}} \right). \tag{5}$$

We denote the class of admissible filters G with $\bar{A}(G) < a$ by $\mathcal{G}_a$. We note that the anisotropy $\bar{A}(G)$ of w_k is a measure of its whiteness. Namely, if w_k is white, then it can not be estimated (i.e. its optimal estimate is just zero) and $\tilde{w}_k = w_k$ which leads to $\bar{A}(G) = 0$. On the other hand, if w_k can be perfectly estimated, then $\bar{A}(G)$ tends to infinity.

The *a–anisotropic norm* of the system F is defined as

$$||F||_a := sup_{G \in \mathcal{G}_a} \frac{||FG||_2}{||G||_2}, \tag{6}$$

where $||G||_2$ denotes the H_2–type norm of the system (2), namely $||G||_2 := \lim_{k \to \infty} E[|w_k|^2]^{\frac{1}{2}}$, the sequence w_k, $k = 0, 1, ...$ being determined with null initial conditions in (2). The computation of this norm may allow us to analyze the disturbance attenuation properties for a given F or to design feedback controllers which give rise to closed-loop systems F.

3 Generating Filter Mean Anisotropy

We first aim at computing $\bar{A}(G)$ in terms of $\alpha, \beta, \gamma, \delta, \eta$. To this end we define $\hat{h}_k = E\{h_k | w_j, j < k\}$ and we have $\hat{w}_k = \gamma \hat{h}_k$ and $\tilde{w}_k = \gamma \tilde{h}_k + \delta v_k$ where $\tilde{h}_k := h_k - \hat{h}_k$ denotes the state estimation error. Therefore,

$$E\{\tilde{w}_k \tilde{w}_k^T\} = \gamma X \gamma^T + \delta \delta^T \tag{7}$$

where $X := E\{\tilde{h}_k \tilde{h}_k\}$. Also, $E\{w_k w_k^T\} = \gamma Q \gamma^T + \delta \delta^T$ where Q is the solution of the Lyapunov equation

$$Q = \alpha Q \alpha^T + \eta Q \eta^T + \beta \beta^T. \tag{8}$$

To complete the explicit computation of $\bar{A}(G)$ it remains to derive X. We have the following result.

Lemma 1. *The optimal filter gain L for which $\alpha - L\gamma$ is stable and X is minimized, is given by:*

$$L^* = (\alpha X \gamma^T + \beta \delta^T) \left(\delta \delta^T + \gamma X \gamma^T \right)^{-1} \tag{9}$$

where X is the stabilizing solution of the Riccati equation

$$X = \alpha X \alpha^T - (\alpha X \gamma^T + \beta \delta^T) \left(\delta \delta^T + \gamma X \gamma^T \right)^{-1} (\gamma X \alpha^T + \delta \beta^T) + \eta Q \eta^T + \beta \beta^T \tag{10}$$

and Q is the solution of the Lyapunov equation (8).

Proof: Consider the estimator

$$\hat{h}_{k+1} = \alpha \hat{h}_k + L(w_k - \gamma \hat{h}_k) \tag{11}$$

From the latter and (2) one obtains

$$\begin{bmatrix} \tilde{h}_{k+1} \\ h_{k+1} \end{bmatrix} = \begin{bmatrix} \alpha - L\gamma & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} \tilde{h}_k \\ h_k \end{bmatrix} + \begin{bmatrix} 0 & \eta \\ 0 & \eta \end{bmatrix} \begin{bmatrix} \tilde{h}_k \\ h_k \end{bmatrix} \xi_k + \begin{bmatrix} \beta - L\delta \\ \beta \end{bmatrix} v_k.$$

According with the results derived for instance in [6, 7] concerning the computation of the H_2 norm of stochastic systems with state–dependent noise, the H_2 norm of the above system with the output $\tilde{h}_k$ equals $\left[Tr\left(\mathcal{C}\mathcal{P}\mathcal{C}^T\right)\right]^{\frac{1}{2}}$ where $\mathcal{C} = \begin{bmatrix} I & 0 \end{bmatrix}$ and the stochastic controllability Gramian

$$\mathcal{P} = \begin{bmatrix} X & Z \\ Z^T & Q \end{bmatrix}$$

is the solution of the Lyapunov equation

$$\mathcal{P} = \mathcal{A}\mathcal{P}\mathcal{A}^T + \mathcal{D}\mathcal{P}\mathcal{D}^T + \mathcal{B}\mathcal{B}^T, \tag{12}$$

where the following notations have been introduced

$$\mathcal{A} := \begin{bmatrix} \alpha - L\gamma & 0 \\ 0 & \alpha \end{bmatrix}, \mathcal{D} := \begin{bmatrix} 0 & \eta \\ 0 & \eta \end{bmatrix}, \mathcal{B} := \begin{bmatrix} \beta - L\delta \\ \beta \end{bmatrix}.$$

Then direct algebraic computations show that the blocks (1,1) and (2,2) of equation (12) give

$$X = (\alpha - L\gamma)\, X\, (\alpha - L\gamma)^T + \eta Q \eta^T + (\beta - L\delta)(\beta - L\delta)^T, \tag{13}$$

and (8), respectively. The above equation (13) can be readily written as:

$$X = \alpha X \alpha^T - (\alpha X \gamma^T + \beta \delta^T)\left(\delta\delta^T + \gamma X \gamma^T\right)^{-1}(\gamma X \alpha^T + \delta \beta^T) + \eta Q \eta^T + \beta\beta^T + (L - L^*)\left(\delta\delta^T + \gamma X \gamma^T\right)(L - L^*)^T$$

where L^* is given by (9). Noting that Q satisfies (8), the theorem follows by the monotonicity property of discrete-time Riccati equations (see for instance [2], [17], [21]).

The stability of $\alpha - L\gamma$ directly follows from the fact that X is the stabilizing solution of (10).

We can, therefore, now present the formula for the mean anisotropy of the generating filter (2):

$$\bar{A}(G) = -\frac{1}{2} ln \; det \left(\frac{m(\gamma X \gamma^T + \delta\delta^T)}{Tr(\gamma Q \gamma^T + \delta\delta^T)} \right) \tag{14}$$

where X and Q respectively satisfy (10) and (8). Thus the condition $\bar{\Lambda}(G) < a$ becomes

$$-\frac{1}{2} ln \; det \left(\frac{m(\gamma X \gamma^T + \delta\delta^T)}{Tr(\gamma Q \gamma^T + \delta\delta^T)} \right) < a$$

which gives

$$det \left(\gamma X \gamma^T + \delta\delta^T \right) > e^{-\frac{2a}{m}} \left(Tr \left(\gamma Q \gamma^T + \delta\delta^T \right) \right)^m .$$

One can show that the above condition is fulfilled if there exists $q > 0$ such that

$$\gamma X \gamma^T + \delta\delta^T > qI_m > e^{-\frac{2a}{m}} \left(\gamma Q \gamma^T + \delta\delta^T \right). \tag{15}$$

Recall that in the above developments X denotes the stabilizing solution of the Riccati equation (10). Considering instead of this equation the inequality

$$\begin{gathered} X < \alpha X \alpha^T - (\alpha X \gamma^T + \beta\delta^T) \left(\delta\delta^T + \gamma X \gamma^T \right)^{-1} (\gamma X \alpha^T + \delta\beta^T) \\ + \eta Q \eta^T + \beta\beta^T \end{gathered} \tag{16}$$

with $X > 0$, from the monotonicity properties of the stabilizing solution of the Riccati equation with respect to the free term, it follows that if $\tilde{X} > 0$ verifies (16) then $\tilde{X} < X$ where X is the solution of (10). Therefore the left side inequality in (15) is fulfilled by the solution X of the Riccati equation if it holds for a solution $\tilde{X} > 0$ of (16).

Defining

$$\mathcal{G} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \tag{17}$$

it follows based on Schur complements arguments that (16) is equivalent with the inequality

$$\begin{bmatrix} -X + \eta Q \eta^T & 0 \\ 0 & 0 \end{bmatrix} + \mathcal{G} \begin{bmatrix} X & 0 \\ 0 & I_m \end{bmatrix} \mathcal{G}^T > 0. \tag{18}$$

With the notation (17) the left side inequality in (15) may be written in the equivalent form

$$\begin{bmatrix} 0 & I_m \end{bmatrix} \mathcal{G} \begin{bmatrix} X & 0 \\ 0 & I_m \end{bmatrix} \mathcal{G}^T \begin{bmatrix} 0 \\ I_m \end{bmatrix} > qI_m. \tag{19}$$

Remark 1. (i) *S*ince $X > 0$ it follows that condition (18) is fulfilled if $\eta Q \eta^T > X$;

(ii) *If* $\delta\delta^T > qI_m$, the left side inequality in (15) is automatically fulfilled for any $X > 0$.

Using again Schur complement arguments it follows that the right side inequality in (15) is equivalent with

$$\begin{bmatrix} 0 & I_m \end{bmatrix} \mathcal{G} \begin{bmatrix} X & 0 \\ 0 & I_m \end{bmatrix} \mathcal{G}^T \begin{bmatrix} 0 \\ I_m \end{bmatrix} < qe^{\frac{2a}{m}} I_m. \tag{20}$$

4 Anisotropic Norm Computation

We note that the anisotropy $\bar{A}(G)$ of w_k is a measure of its whiteness. Namely, if w_k is white, then it can not be estimated (i.e. its optimal estimate is just zero) and $\tilde{w}_k = w_k$ which leads to $\bar{A}(G) = 0$. In the case of $\eta = 0$ (i.e. the case without mutiplicative noise) this corresponds to $G = \lambda I$ for some $\lambda > 0$, where G notation is abused to be the transfer function matrix of the generating system. If on the other hand the transfer matrix function corresponding to G is rank deficient (namely w has frequency bands with zero power spectrum) on some finite interval of frequencies, then $\bar{A}(G)$ tends to infinity. These facts may provide intuitive explanation to the result of [12] where it is shown that $||F||_a$ of (6) coincides with the H_2 norm at $a \to 0+$ whereas it coincides with the H_∞-norm for $a \to +\infty$. We note that [12] also provides asymptotic expansions of $||F||_a$ in the vicinity of those two extremes.

Appending (2) to (1), and defining the augmented state-vector $\bar{x}_k = col\{x_k, h_k\}$ we readily obtain:

$$\bar{x}_{k+1} = (\bar{A} + \bar{H}\xi k)\bar{x}_k + \bar{B}w_k \text{ and } z_k = \bar{C}\bar{x}_k + \bar{D}v_k \tag{21}$$

where

$$\bar{A} = \begin{bmatrix} A & B\gamma \\ 0 & \alpha \end{bmatrix}, \bar{B} = \begin{bmatrix} B\delta \\ \beta \end{bmatrix}, \bar{H} = \begin{bmatrix} H & 0 \\ \eta & 0 \end{bmatrix} \tag{22}$$

and

$$\bar{C} = \begin{bmatrix} C & D\gamma \end{bmatrix}, \bar{D} = D\delta \tag{23}$$

We now note that

$$||G||_2^2 = Tr\{\gamma Q \gamma^T + \delta\delta^T\} \tag{24}$$

where Q satisfies (8) and that

$$||FG||_2^2 = Tr\{\bar{C}P\bar{C}^T + \bar{D}\bar{D}^T\} \tag{25}$$

where

$$P = \bar{A}P\bar{A}^T + \bar{B}\bar{B}^T + \bar{H}P\bar{H}^T \tag{26}$$

Applying Schur's complements arguments, the following linear matrix inequalities, therefore, characterize $||F||_a < \theta$:

$$\begin{bmatrix} -P & \bar{B} & \bar{H}P & \bar{A}P \\ \bar{B}^T & -I & 0 & 0 \\ P\bar{H}^T & 0 & -P & 0 \\ P\bar{A}^T & 0 & 0 & -P \end{bmatrix} < 0 \tag{27}$$

and

$$\begin{bmatrix} -Q & \beta & \eta Q & \alpha Q \\ \beta^T & -I & 0 & 0 \\ Q\eta^T & 0 & -Q & 0 \\ Q\alpha^T & 0 & 0 & -Q \end{bmatrix} < 0 \tag{28}$$

where

$$Tr\{\bar{C}P\bar{C}^T + \bar{D}\bar{D}^T\} - Tr\{\gamma Q\gamma^T + \delta\delta^T\}\theta^2 < 0. \tag{29}$$

We next partition P as follows:

$$P = \begin{bmatrix} R & M \\ M^T & S \end{bmatrix}. \tag{30}$$

Using this notation and (17), the inequality (27) can be rewritten as

$$\mathcal{Z} + \mathcal{P}^T \mathcal{G} \mathcal{Q} + \mathcal{Q}^T \mathcal{G}^T \mathcal{P} < 0 \tag{31}$$

where

$$\mathcal{Z} = \begin{bmatrix} -R & -M & 0 & HR & HM & AR & AM \\ -M^T & -S & 0 & \eta R & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 \\ RH^T & R\eta^T & 0 & -R & -M & 0 & 0 \\ M^T H^T & 0 & 0 & -M^T & -S & 0 & 0 \\ RA^T & 0 & 0 & 0 & 0 & -R & -M \\ M^T A^T & 0 & 0 & 0 & 0 & -M^T & -S \end{bmatrix}. \tag{32}$$

and where

$$\mathcal{P} = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \\ B^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

and

$$\mathcal{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & S \\ 0 & I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{34}$$

Then according to the so–called Projection Lemma (see e.g. [3], p. 22) there exists $\mathcal{G}$ for which the condition (31) if and only if

$$W_{\mathcal{P}}^T \mathcal{Z} W_{\mathcal{P}} < 0 \tag{35}$$

and

$$W_{\mathcal{Q}}^T \mathcal{Z} W_{\mathcal{Q}} < 0 \tag{36}$$

where $W_{\mathcal{P}}$ and $W_{\mathcal{Q}}$ are bases of the null spaces of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Denoting $\mathcal{N}_{\mathcal{B}^T} = \mathcal{N}(B^T)$ we readily obtain:

$$W_{\mathcal{P}} = \begin{bmatrix} \mathcal{N}_{\mathcal{B}^T} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \quad \text{and} \quad W_{\mathcal{Q}} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, we rewrite (28), as:

$$\bar{\mathcal{Z}} + \bar{\mathcal{P}}^T \mathcal{G} \bar{\mathcal{Q}} + \bar{\mathcal{Q}}^T \mathcal{G}^T \bar{\mathcal{P}} < 0 \tag{37}$$

where

$$\bar{\mathcal{Z}} = \begin{bmatrix} -Q & 0 & \eta Q & 0 \\ 0 & -I & 0 & 0 \\ Q\eta^T & 0 & -Q & 0 \\ 0 & 0 & 0 & -Q \end{bmatrix}$$

and

$$\bar{\mathcal{P}}^T = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \bar{\mathcal{Q}} = \begin{bmatrix} 0 & 0 & 0 & Q \\ 0 & I & 0 & 0 \end{bmatrix}.$$

Remark 2. *The solutions $\mathcal{G}$ of (31) (and similarly of (37)) may be expressed using the following parameterization (see the proof in [15], p. 30) $\mathcal{G} = \Phi_1 + \Phi_2 \mathcal{L} \Phi_3$ with the parameter $\mathcal{L}$ such that $\mathcal{L}^T \mathcal{L} < I$, where Φ_1, Φ_2 and Φ_3 depends on $\mathcal{Z}$, $\mathcal{P}$ and $\mathcal{Q}$.*

Further, based on the notations introduced above, the condition (29) becomes

$$Tr\left\{\begin{bmatrix} C & \begin{bmatrix} 0 & D \end{bmatrix} \mathcal{G} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} R & M \\ M^T & S \end{bmatrix} \begin{bmatrix} C^T \\ \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{G}^T \begin{bmatrix} 0 \\ D^T \end{bmatrix} \end{bmatrix} \right. \\ \left. + \begin{bmatrix} 0 & D \end{bmatrix} \mathcal{G} \begin{bmatrix} 0 \\ D^T \end{bmatrix} - \theta^2 \begin{bmatrix} 0 & I_m \end{bmatrix} \mathcal{G} \begin{bmatrix} Q & 0 \\ 0 & I_m \end{bmatrix} \mathcal{G}^T \begin{bmatrix} 0 \\ I_m \end{bmatrix} \right\} < 0. \tag{38}$$

The above developments are concluded in the following result.

Theorem 1. *The a–anisotropic norm of the stochastic system with state–dependent noise (1) is less than $\theta > 0$ if the system of matrix inequalities (18)–(20), (31), (37), (38) are feasible with respect to the scalar $q > 0$ and to the matrices $\mathcal{G}$, η, $Q > 0$, $X > 0$, $P > 0$ where $\mathcal{G}$ and P are defined by (17) and (30), respectively.*

5 Final remarks

The boundedness conditions for the anisotropic norm given by Theorem 1 require to solve a sign–indefinite quadratic optimization problem. The following research will be devoted to the development of numerical algorithms based on semidefinite programming to solve this optimization problem.

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