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ON THE EQUATIONS OF GEOMETRICALLY NONLINEAR ELASTIC PLATES WITH ROTATIONAL DEGREES OF FREEDOM*

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Abstract

We consider the general model of 6-parametric elastic plates, in which the rotation tensor field is an independent kinematic field. In this context we show the existence of global minimizers to the minimization problem of the total potential energy.

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1 Introduction

The general non-linear theory of 6-parametric elastic shells (3 parameters for the translation and 3 parameters for the rotational degrees of freedom) has

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been established and presented in the books of Libai and Simmonds [1] and Chróścielewski, Makowski and Pietraszkiewicz [2]. This approach to shell theory is of great importance due to its generality and its efficiency for the treatment of complex shell problems.

In this short note we present an existence results for the equations of geometrically nonlinear elastic plates, in the framework of the 6-parametric shell theory. Using the direct methods of the calculus of variations, we establish the existence of global minimizers for the corresponding minimization problem of the total potential energy. First, we consider the case of isotropic and homogeneous plates. Then, we extend the existence theorem to the more general situation of composite elastic plates.

2 Geometrically nonlinear elastic plates

Consider an elastic plate which occupies in the reference (undeformed) configuration the region $\Omega = \{(x,y,z)\,|\,(x,y)\in\omega,\ z\in\big[-\frac{h}{2},\frac{h}{2}\big]\}$ of the three-dimensional Euclidean space. Here $h>0$ is the thickness of the plate and $\omega\subset\mathbb{R}^2$ is a bounded, open domain with Lipschitz boundary $\partial\omega$. Relative to an inertial frame $(O,\mathbf{e}_i)$, with $\mathbf{e}_i$ orthonormal vectors $(i=1,2,3)$, the position vector $\mathbf{r}$ of any point of Ω can be written as

$$\mathbf{r}(x,y,z) = x\,\mathbf{e}_1 + y\,\mathbf{e}_2 + z\,\mathbf{e}_3\,, \qquad (x,y)\in\omega, \quad z\in\big[-\frac{h}{2},\frac{h}{2}\big]. \tag{1}$$

In the deformed configuration, we denote by $\mathbf{m}:\omega\subset\mathbb{R}^2\to\mathbb{R}^3$ the surface deformation mapping, so that $\mathbf{m}=\mathbf{m}(x,y)$ represents the position vector of the points of the base surface of the plate (shell). Let the vector field $\mathbf{u}=\mathbf{u}(x,y)$ designate the translations (displacements) and the proper orthogonal tensor field $\mathbf{R}=\mathbf{R}(x,y)$ denote the rotations of the shell cross-sections. Then the deformed configuration of the plate is given by

$$\mathbf{m}(x,y) = x\,\mathbf{e}_1 + y\,\mathbf{e}_2 + \mathbf{u}(x,y), \qquad \mathbf{d}_i = \mathbf{R}\,\mathbf{e}_i\,, \quad i=1,2,3. \tag{2}$$

The vectors $\mathbf{d}_i$ introduced in (2) are three orthonormal vectors (usually called *directors*) attached to any point of the deformed base surface $\mathcal{S}=\mathbf{m}(\omega)$. Thus, the rotation tensor field $\mathbf{R}(x,y)\in SO(3)$ can be written as

$$\mathbf{R} = \mathbf{d}_i\otimes\mathbf{e}_i\,. \tag{3}$$

We employ the usual tensor notation and the Einstein's convention of summation over repeated indices. The Latin indices $i, j, ...$ take the values $\{1, 2, 3\}$ and the Greek indices $\alpha, \beta, ...$ range over the set $\{1, 2\}$. The partial derivative with respect to x will be denoted by $(\cdot)_{,x} = \frac{\partial}{\partial x}(\cdot)$.

The local equilibrium equations for 6-parametric plates are [1, 2]:

$$\mathrm{Div}_s\,\mathbf{N} + \mathbf{f} = \mathbf{0}, \qquad \mathrm{Div}_s\,\mathbf{M} + \mathrm{axl}(\mathbf{N}\mathbf{F}^T - \mathbf{F}\mathbf{N}^T) + \mathbf{c} = \mathbf{0}. \tag{4}$$

Here, $\mathbf{f}$ and $\mathbf{c}$ are the external surface resultant force and couple vector fields, $\mathbf{N}$ and $\mathbf{M}$ are the internal surface stress resultant and stress couple resultant tensors (of the first Piola–Kirchhoff stress tensor type), Div_s is the surface divergence operator, while $\mathbf{F} = \mathrm{Grad}_s\,\mathbf{m} = \mathbf{m}_{,x} \otimes \mathbf{e}_1 + \mathbf{m}_{,y} \otimes \mathbf{e}_2$ is the surface gradient of deformation. The superscript $(\cdot)^T$ denotes the transpose and $\mathrm{axl}(\cdot)$ is the axial vector of any skew-symmetric tensor.

To formulate the boundary conditions, we take a disjoint partition of the boundary curve $\partial\omega = \partial\omega_d \cup \partial\omega_f$, $\partial\omega_d \cap \partial\omega_f = \emptyset$, with $\mathrm{length}(\partial\omega_d) > 0$. We consider the following boundary conditions [2, 3]

$$\mathbf{u} - \mathbf{u}^* = \mathbf{0}, \qquad \mathbf{R} - \mathbf{R}^* = \mathbf{0} \quad \text{along } \partial\omega_d\,, \tag{5}$$

$$\mathbf{N}\nu - \mathbf{n}^* = \mathbf{0}, \qquad \mathbf{M}\nu - \mathbf{m}^* = \mathbf{0} \quad \text{along } \partial\omega_f\,, \tag{6}$$

where $\mathbf{n}^*$ and $\mathbf{m}^*$ are the external boundary resultant force and couple vectors applied along $\partial\omega_f$, and ν is the external unit normal vector to $\partial\omega$.

In the general resultant theory of shells, the strain measures are the strain tensor $\mathbf{E}$ and the bending tensor $\mathbf{K}$, given by [2, 4]

$$\mathbf{E} = \mathbf{R}^T \big[\, (\mathbf{m}_{,x} - \mathbf{d}_1) \otimes \mathbf{e}_1 + (\mathbf{m}_{,y} - \mathbf{d}_2) \otimes \mathbf{e}_2 \,\big], \tag{7}$$

$$\mathbf{K} = \mathbf{R}^T \big[\, \mathrm{axl}(\mathbf{R}_{,x}\mathbf{R}^T) \otimes \mathbf{e}_1 + \mathrm{axl}(\mathbf{R}_{,y}\mathbf{R}^T) \otimes \mathbf{e}_2 \,\big]. \tag{8}$$

One can prove that the following relation holds for any rotation tensor $\mathbf{Q} \in SO(3)$ and any second order skew-symmetric tensor $\mathbf{A} \in \mathbf{so}(3)$

$$\mathrm{axl}(\mathbf{Q}\mathbf{A}\mathbf{Q}^T) = \mathbf{Q}\,\mathrm{axl}(\mathbf{A}). \tag{9}$$

If we write this relation for $\mathbf{Q} = \mathbf{R}$ and $\mathbf{A} = \mathbf{R}^T\mathbf{R}_{,x}$ we obtain

$$\mathbf{R}^T \mathrm{axl}(\mathbf{R}_{,x}\mathbf{R}^T) = \mathrm{axl}(\mathbf{R}^T\mathbf{R}_{,x})\,. \tag{10}$$

By (8) and (10), the bending tensor $\mathbf{K}$ can be expressed in the simpler form

$$\mathbf{K} = \mathrm{axl}(\mathbf{R}^T\mathbf{R}_{,x}) \otimes \mathbf{e}_1 + \mathrm{axl}(\mathbf{R}^T\mathbf{R}_{,y}) \otimes \mathbf{e}_2\,. \tag{11}$$

In the case of plates, the strain tensor $\mathbf{E}$ and the bending tensor $\mathbf{K}$ can be written in component form relative to the tensor basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ as

$$\mathbf{E} = E_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha = (\mathbf{m}_{,x} \cdot \mathbf{d}_i - \delta_{i1})\mathbf{e}_i \otimes \mathbf{e}_1 + (\mathbf{m}_{,y} \cdot \mathbf{d}_i - \delta_{i2})\mathbf{e}_i \otimes \mathbf{e}_2\,, \qquad (12)$$

$$\begin{aligned}\mathbf{K} = K_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha = (\mathbf{d}_{2,x}\cdot\mathbf{d}_3)\mathbf{e}_1\otimes \mathbf{e}_1 + (\mathbf{d}_{3,x}\cdot\mathbf{d}_1)\mathbf{e}_2\otimes \mathbf{e}_1 + (\mathbf{d}_{1,x}\cdot\mathbf{d}_2)\mathbf{e}_3\otimes \mathbf{e}_1 \\ +(\mathbf{d}_{2,y}\cdot\mathbf{d}_3)\mathbf{e}_1 \otimes \mathbf{e}_2 + (\mathbf{d}_{3,y}\cdot\mathbf{d}_1)\mathbf{e}_2 \otimes \mathbf{e}_2 + (\mathbf{d}_{1,y}\cdot\mathbf{d}_2)\mathbf{e}_3 \otimes \mathbf{e}_2\,,\end{aligned} \qquad (13)$$

where δ_{ij} is the Kronecker symbol.

Let $W = W(\mathbf{E},\mathbf{K})$ be the strain energy density of the elastic plate. According to the hyperelasticity assumption, the constitutive equations are

$$\mathbf{N} = \mathbf{R}\,\frac{\partial W}{\partial \mathbf{E}}\;, \qquad \mathbf{M} = \mathbf{R}\,\frac{\partial W}{\partial \mathbf{K}}\;. \qquad (14)$$

The strain energy density for physically linear isotropic plates is [5]

$$\begin{aligned} W(\mathbf{E},\mathbf{K}) &= W_{\text{mb}}(\mathbf{E}) + W_{\text{bend}}(\mathbf{K}),\\ 2W_{\text{mb}}(\mathbf{E}) &= \alpha_1 \text{tr}^2\mathbf{E}_{\|} + \alpha_2 \text{tr}\mathbf{E}_{\|}^2 + \alpha_3 \text{tr}(\mathbf{E}_{\|}^T\mathbf{E}_{\|}) + \alpha_4 \mathbf{e}_3\mathbf{E}\mathbf{E}^T\mathbf{e}_3,\\ 2W_{\text{bend}}(\mathbf{K}) &= \beta_1 \text{tr}^2\mathbf{K}_{\|} + \beta_2 \text{tr}\mathbf{K}_{\|}^2 + \beta_3 \text{tr}(\mathbf{K}_{\|}^T\mathbf{K}_{\|}) + \beta_4 \mathbf{e}_3\mathbf{K}\mathbf{K}^T\mathbf{e}_3, \end{aligned} \qquad (15)$$

where the coefficients α_k , β_k are constant material parameters, and we use the notations $\mathbf{E}_{\|} = \mathbf{E} - (\mathbf{e}_3 \otimes \mathbf{e}_3)\mathbf{E}$ and $\mathbf{K}_{\|} = \mathbf{K} - (\mathbf{e}_3 \otimes \mathbf{e}_3)\mathbf{K}$.

3 Existence of minimizers

Let us define the admissible set $\mathcal{A}$ by

$$\mathcal{A} = \big\{(\mathbf{m},\mathbf{R}) \in \mathbf{H}^1(\omega,\mathbb{R}^3)\times \mathbf{H}^1(\omega, SO(3)) \;\big|\; \mathbf{m}_{\big|\partial\omega_d} = \mathbf{m}^*,\; \mathbf{R}_{\big|\partial\omega_d} = \mathbf{R}^*\big\}. \qquad (16)$$

The boundary conditions in (16) are to be understood in the sense of traces. We assume the existence of a function $\Lambda(\mathbf{u},\mathbf{R})$ representing the potential of the external surface loads $\mathbf{f},\mathbf{c}$, and boundary loads $\mathbf{n}^*,\mathbf{m}^*$ [4].

Consider the two-field minimization problem associated to the deformation of elastic plates: find the pair $(\hat{\mathbf{m}},\hat{\mathbf{R}}) \in \mathcal{A}$ which realizes the minimum of the functional

$$I(\mathbf{m},\mathbf{R}) = \int_\omega W(\mathbf{E},\mathbf{K})\,\text{d}\omega - \Lambda(\mathbf{u},\mathbf{R}) \qquad \text{for} \qquad (\mathbf{m},\mathbf{R}) \in \mathcal{A}. \qquad (17)$$

Here the strain tensor $\mathbf{E}$ and the bending tensor $\mathbf{K}$ are expressed in terms of $(\mathbf{m}, \mathbf{R})$ by the relations $(2)_2$, (7) and (8).

The external loading potential $\Lambda(\mathbf{u}, \mathbf{R})$ is decomposed additively

$$\Lambda(\mathbf{u}, \mathbf{R}) = \Lambda_\omega(\mathbf{u}, \mathbf{R}) + \Lambda_{\partial\omega_f}(\mathbf{u}, \mathbf{R}), \tag{18}$$

where $\Lambda_\omega(\mathbf{u}, \mathbf{R})$ is the potential of the external surface loads $\mathbf{f}, \mathbf{c}$, while $\Lambda_{\partial\omega_f}(\mathbf{u}, \mathbf{R})$ is the potential of the external boundary loads $\mathbf{n}^*, \mathbf{m}^*$

$$\Lambda_\omega(\mathbf{u}, \mathbf{R}) = \int_\omega \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}\omega + \Pi_\omega(\mathbf{R}), \quad \Lambda_{\partial\omega_f}(\mathbf{u}, \mathbf{R}) = \int_{\partial\omega_f} \mathbf{n}^* \cdot \mathbf{u} \, \mathrm{d}s + \Pi_{\partial\omega_f}(\mathbf{R}). \tag{19}$$

The load potentials $\Pi_\omega : \mathbf{L}^2(\omega, SO(3)) \to \mathbb{R}$ and $\Pi_{\partial\omega_f} : \mathbf{L}^2(\omega, SO(3)) \to \mathbb{R}$ are assumed to be continuous and bounded operators. Let us present next the main existence result corresponding to isotropic elastic plates.

Theorem 1 *Assume that the external loads and the boundary data satisfy the regularity conditions*

$$\mathbf{f} \in \mathbf{L}^2(\omega, \mathbb{R}^3), \;\; \mathbf{n}^* \in \mathbf{L}^2(\partial\omega_f, \mathbb{R}^3), \;\; \mathbf{m}^* \in \mathbf{H}^1(\omega, \mathbb{R}^3), \;\; \mathbf{R}^* \in \mathbf{H}^1(\omega, SO(3)). \tag{20}$$

Consider the minimization problem (16), (17) for isotropic plates, i.e. when the strain energy density W is given by the relations (15). If the constitutive coefficients satisfy the conditions

$$\begin{aligned} &2\alpha_1 + \alpha_2 + \alpha_3 > 0, \quad \alpha_2 + \alpha_3 > 0, \quad \alpha_3 - \alpha_2 > 0, \quad \alpha_4 > 0, \\ &2\beta_1 + \beta_2 + \beta_3 > 0, \quad \beta_2 + \beta_3 > 0, \quad \beta_3 - \beta_2 > 0, \quad \beta_4 > 0, \end{aligned} \tag{21}$$

then the problem (16), (17) admits at least one minimizing solution pair $(\hat{\mathbf{m}}, \hat{\mathbf{R}}) \in \mathcal{A}$.

For the proof, we apply the direct methods of the calculus of variations and we follow the same steps as in the proof of Theorem 4.1 from [6].

4 Composite plates

The modeling of composite shells in the nonlinear 6-parametric general theory of shells has been presented in [7]. In this case, the strain energy density can be written using the matrix notation in the following way [7]

$$W(\mathbf{E}, \mathbf{K}) = \frac{1}{2} \, \mathbf{v}^T \mathbf{C} \, \mathbf{v} \,, \tag{22}$$

where $\mathbf{C}$ is a 12×12 matrix containing the constitutive coefficients, and $\mathbf{v}$ is a 12×1 column vector of the forms

$$\mathbf{C}_{12\times12} = \begin{bmatrix} \mathbf{A}_{4\times4} & \mathbf{0}_{4\times2} & \mathbf{B}_{4\times4} & \mathbf{0}_{4\times2} \\ \mathbf{0}_{2\times4} & \mathbf{S}_{2\times2} & \mathbf{0}_{2\times4} & \mathbf{0}_{2\times2} \\ \mathbf{B}_{4\times4} & \mathbf{0}_{4\times2} & \mathbf{D}_{4\times4} & \mathbf{0}_{4\times2} \\ \mathbf{0}_{2\times4} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times4} & \mathbf{G}_{2\times2} \end{bmatrix}, \qquad \mathbf{v}_{12\times1} = \begin{bmatrix} \mathbf{e}_{4\times1} \\ \varepsilon_{2\times1} \\ \mathbf{k}_{4\times1} \\ \kappa_{2\times1} \end{bmatrix}. \quad (23)$$

Here we have denoted by $\mathbf{e}$, ε, $\mathbf{k}$ and κ the following column vectors of components of the strain and bending tensors for plates

$$\mathbf{e} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{21} \\ E_{12} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_{,x}\cdot\mathbf{d}_1 - 1 \\ \mathbf{m}_{,y}\cdot\mathbf{d}_2 - 1 \\ \mathbf{m}_{,x}\cdot\mathbf{d}_2 \\ \mathbf{m}_{,y}\cdot\mathbf{d}_1 \end{bmatrix}, \ \mathbf{k} = \begin{bmatrix} K_{21} \\ -K_{12} \\ -K_{11} \\ K_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_{3,x}\cdot\mathbf{d}_1 \\ \mathbf{d}_{3,y}\cdot\mathbf{d}_2 \\ \mathbf{d}_{3,x}\cdot\mathbf{d}_2 \\ \mathbf{d}_{3,y}\cdot\mathbf{d}_1 \end{bmatrix},$$

$$\varepsilon = \begin{bmatrix} E_{31} \\ E_{32} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_{,x}\cdot\mathbf{d}_3 \\ \mathbf{m}_{,y}\cdot\mathbf{d}_3 \end{bmatrix}, \qquad \kappa = \begin{bmatrix} K_{31} \\ K_{32} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_{1,x}\cdot\mathbf{d}_2 \\ \mathbf{d}_{1,y}\cdot\mathbf{d}_2 \end{bmatrix}. \quad (24)$$

In view of the above notations, the expression of the strain energy density (22) becomes

$$2\,W(\mathbf{E},\mathbf{K}) = \mathbf{e}^T\mathbf{A}\,\mathbf{e} + \mathbf{e}^T\mathbf{B}\,\mathbf{k} + \mathbf{k}^T\mathbf{B}\,\mathbf{e} + \mathbf{k}^T\mathbf{D}\,\mathbf{k} + \varepsilon^T\mathbf{S}\,\varepsilon + \kappa^T\mathbf{G}\,\kappa. \quad (25)$$

In the above relation we can observe the multiplicative coupling of the strain tensor $\mathbf{E}$ with the bending tensor $\mathbf{K}$ for composite plates. The matrices $\mathbf{A}, \mathbf{B}, \mathbf{D}, \mathbf{S}, \mathbf{G}$ containing the constitutive coefficients for elastic (orthotropic) composite multilayered shells and plates have been determined in [7] in terms of the material/geometrical parameters of the layers.

We can prove the existence of minimizers also for composite plates under the assumption of coercivity and convexity on the strain energy density. More precisely, the following theorem holds.

Theorem 2 ***(Composite, anisotropic plates)*** *Consider the minimization problem (16), (17) associated to the deformation of composite plates, and assume that the external loads and boundary data satisfy the conditions (20). Assume that the strain energy density $W(\mathbf{E},\mathbf{K})$ is a quadratic convex function in $(\mathbf{E},\mathbf{K})$, and moreover W is coercive, i.e.*

$$W(\mathbf{E},\mathbf{K}) \geq c\left(\|\mathbf{E}\|^2 + \|\mathbf{K}\|^2\right), \quad \forall\,\mathbf{E} = E_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha,\ \mathbf{K} = K_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha, \quad (26)$$

for some constant $c > 0$. Then, the minimization problem (16), (17) admits at least one minimizing solution pair $(\hat{\mathbf{y}}, \hat{\mathbf{Q}}) \in \mathcal{A}$.

Finally, we mention that the model of 6-parametric plates has many similarities with the Cosserat plate model proposed and investigated by the second author in [6, 8]. Although this Cosserat model for plates has been obtained independently by a derivation approach, the strain measures of the two models essentially coincide. Moreover, the expressions of the elastic strain energies become identical for isotropic plates, provided one makes a suitable identification of constitutive coefficients in the two approaches.

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