

SYNCHRONIZATION IN NETWORKS OF COUPLED HARMONIC OSCILLATORS WITH STOCHASTIC PERTURBATION AND TIME DELAYS*

Yilun Shang[†]

Abstract

In this paper, we investigate the leader-follower synchronization of coupled second-order linear harmonic oscillators with the presence of random noises and time delays. The interaction topology is modeled by a weighted directed graph and the weights are perturbed by white noise. On the basis of stability theory of stochastic differential delay equations, algebraic graph theory and matrix theory, we show that the coupled harmonic oscillators can be synchronized almost surely with random perturbation and time delays. Numerical examples are presented to illustrate our theoretical results.

MSC: 93E15, 05C82, 34C15

keywords: synchronization; time delay; harmonic oscillator; consensus; random noise.

*Accepted for publication on January 12, 2012.

[†]shylmath@hotmail.com. Institute for Cyber Security, University of Texas at San Antonio, San Antonio, Texas 78249, USA

1 Introduction

Synchronization, as an emergent collective phenomenon of a population of units with oscillatory behaviors, is one of the most intriguing in nature and plays a significant role in a variety of disciplines such as biology, sociology, physics, chemistry and technology [2, 20, 26, 28]. One celebrated model for synchronization is the Kuramoto model [12], which is described by a system of structured ordinary differential equations and often used to model synchronization of oscillators in different fields of physics, engineering and biology. The original Kuramoto formulation assumes full connectivity of the network, that is, the interaction topology is a complete graph. Recent works generalize the Kuramoto model to nearest neighbor interaction and the underlying topologies may be general networks, see e.g. [1, 13, 21]. Wireless sensor network is also a field where synchronization is an important problem to deal with. Many distributed applications on wireless networks require accurate clock synchronization, see e.g. [4, 27]. Another classical model for synchronization is the harmonic oscillator network [3, 23, 29], which is the very subject of the present paper. Recently, Ren [23] investigates synchronization of coupled second-order linear harmonic oscillators with local interaction. Due to the linear structure, the ultimate trajectories to which each oscillator converges over directed fixed networks are shown explicitly and milder convergence conditions than those in the case of Kuramoto model [13] are derived.

Since noise is ubiquitous in nature, technology, and society [26], the motion of oscillator is inevitably subject to disturbance in the environments. In biological and communication networks, time delay is also unavoidable due to finite communication speed [31, 32]. Although random noise and time delay have been considered extensively in exploring synchronization and consensus problems by means of theoretical and numerical methods, they have seldom been analytically treated in synchronization of coupled harmonic oscillators. Motivating this idea, the objective of this paper is to deal with leader-following synchronization conditions for coupled harmonic oscillators over general directed topologies with the presence of noise perturbation and communication time delays. The main tools used here are borrowed from algebraic graph theory, matrix theory and stochastic differential delay equation theory.

The synchronization of harmonic oscillator networks treated here are related to the second-order consensus dynamics, see e.g. [14, 17, 24, 30, 31, 32].

In the literature regarding consensus problems, agents are usually considered to be governed by first-order dynamics (see [19] and references therein). The second-order consensus problems are more challenging and especially meaningful for the implementation of coordination and control in networked systems. A continuous-time average consensus algorithm for double-integrator dynamics over undirected network topologies is proposed in [30]. Ref. [24] extends the results of [30] to the case of directed interaction. In [31], the authors address a second-order consensus problem with time delays and directed fixed topology. Ref. [32] derives a necessary and sufficient condition for the second-order consensus with the communication delay, that is, the underlying topology contains a directed spanning tree. Ref. [14] analyzes the discrete-time consensus problem with nonuniform time delay and switching topologies. With a selected Lyapunov-Razumikhin function, the authors in [17] present sufficient consensus conditions for a locally passive multi-agent system over a packet-switched communication network with the presence of packet time-delay. In contrast to the above works, where the consensus equilibrium for the velocities of agents is a constant, the positions and velocities are synchronized to achieve oscillating motion by utilizing harmonic oscillator schemes (c.f. Remark 4 below).

On the other hand, the leader-following consensus problem of a group of second-order dynamics agents is one of the main research topics in agent-based problems, as is the setup considered in this paper (see also Remark 1 below). An algorithm for distributed estimation of the active leader's unmeasurable state variables is introduced in [9]. By a Lyapunov-based approach, it is shown that the followers will track the leader when the undirected inter-agent topology is a connected graph. Ref. [10] further extends the result to directed switching topologies. The varying-velocity leader and time-varying delays are considered in [22]. In [8], a distributed observers design is proposed to achieve the leader-following in an undirected switching network topology. However, random noise issues are typically not addressed in the above works.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries and present the coupled harmonic oscillator network model. In Section 3, we analyze the synchronization stability of this model and give sufficient conditions for almost surely convergence. Numerical examples are given in Section 4 to validate our theoretical results. Finally, the conclusion is drawn in Section 5.

2 Problem formulation

By convention, \mathbb{R} represents the real number set; I_n is an $n \times n$ identity matrix. For any vector x , x^T denotes its transpose and $\|x\|$ its Euclidean norm. For a matrix A , denote by $\|A\|$ the operator norm of A , i.e. $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$. $\text{Re}(z)$ denotes the real part of $z \in \mathbb{C}$.

Throughout the paper we will use the following concepts on graph theory (see e.g. [6]) to capture the topology of the network interactions.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph with the set of vertices $\mathcal{V} = \{1, 2, \dots, n\}$ and the set of arcs $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The vertex i in \mathcal{G} represents the i th oscillator, and a directed edge $(i, j) \in \mathcal{E}$ means that oscillator j can directly receive information from oscillator i . The set of neighbors of vertex i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called the weighted adjacency matrix of \mathcal{G} with nonnegative elements and $a_{ij} > 0$ if and only if $j \in \mathcal{N}_i$. The in-degree of vertex i is defined as $d_i = \sum_{j=1}^n a_{ij}$. The Laplacian of \mathcal{G} is defined as $L = D - \mathcal{A}$, where $D = \text{diag}(d_1, d_2, \dots, d_n)$. A directed graph \mathcal{G} is called strongly connected if there is a directed path from i to j between any two distinct vertices $i, j \in \mathcal{V}$. If there exists a directed path from vertex i to vertex j , then i is said to be reachable for j . If a vertex i is reachable for every other vertex in \mathcal{G} , then we say i is globally reachable in \mathcal{G} . In this case, we also say that \mathcal{G} has a directed spanning tree with root i .

Consider n coupled harmonic oscillators connected by dampers and each attached to fixed supports by identical springs with spring constant k . The resultant dynamical system can be described as

$$\ddot{x}_i + kx_i + \sum_{j \in \mathcal{N}_i} a_{ij}(\dot{x}_i - \dot{x}_j) = 0, \quad i = 1, \dots, n \quad (1)$$

where $x_i \in \mathbb{R}$ denotes the position of the i th oscillator, k serves as a positive gain, and a_{ij} characterizes interaction between oscillators i and j as mentioned before.

Here we study a leader-follower version of the above system, and moreover, communication time delay and stochastic noises during the propagation of information from oscillator to oscillator are introduced. In particular, we consider the dynamical system of the form:

$$\begin{aligned}
& \ddot{x}_i(t) + kx_i(t) + \sum_{j \in \mathcal{N}_i} a_{ij} (\dot{x}_i(t - \tau) - \dot{x}_j(t - \tau)) + b_i (\dot{x}_i(t - \tau) - \dot{x}_0(t - \tau)) \\
& + \left[\sum_{j \in \mathcal{N}_i} \sigma_{ij} (\dot{x}_i(t - \tau) - \dot{x}_j(t - \tau)) + \rho_i (\dot{x}_i(t - \tau) - \dot{x}_0(t - \tau)) \right] \dot{w}_i(t) = 0, \\
& i = 1, \dots, n, \quad (2)
\end{aligned}$$

$$\ddot{x}_0(t) + kx_0(t) = 0, \quad (3)$$

where τ is the time delay and x_0 is the position of the virtual leader, labeled as oscillator 0, which follows Equation (3) describing an undamped harmonic oscillator. We thus concern another directed graph $\bar{\mathcal{G}} \supset \mathcal{G}$ associated with the system consisting of n oscillators and one leader. Let $B = \text{diag}(b_1, \dots, b_n)$ be a diagonal matrix with nonnegative diagonal elements and $b_i > 0$ if and only if $0 \in \mathcal{N}_i$. Let $W(t) := (w_1(t), \dots, w_n(t))^T$ be an n -dimensional standard Brownian motion. Hence, $\dot{w}_i(t)$ is one-dimensional white noise. To highlight the presence of noise, it is natural to assume that $\sigma_{ij} > 0$ if $j \in \mathcal{N}_i$, and $\sigma_{ij} = 0$ otherwise; $\rho_i > 0$ if $0 \in \mathcal{N}_i$, and $\rho_i = 0$ otherwise. Also let $A_\sigma = (\sigma_{ij}) \in \mathbb{R}^{n \times n}$ and $B_\sigma = \text{diag}(\rho_1, \dots, \rho_n)$ be two matrices representing the intensity of noise. Moreover, let $\sigma_i = \sum_{j=1}^n \sigma_{ij}$, $D_\sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$, and $L_\sigma = D_\sigma - A_\sigma$.

Remark 1. *Consensus problems of self-organized groups with leaders have broad applications in swarms, formation control and robotic systems, etc.; see e.g. [8, 9, 10, 16, 18, 22]. In multi-agent systems, the leaders have influence on the followers' behaviors but usually independent of their followers. One therefore transfers the control of a whole system to that of a single agent, which saves energy and simplifies network control design [5, 11]. Most of the existing relevant literatures assume a constant state leader, while our model serves to be an example of oscillating state leader on this stage.*

Let $r_i = x_i$ and $v_i = \dot{x}_i$ for $i = 0, 1, \dots, n$. By denoting $r = (r_1, \dots, r_n)^T$ and $v = (v_1, \dots, v_n)^T$, we can rewrite the system (2), (3) in a compact form

as:

$$dr(t) = v(t)dt, \quad (4)$$

$$dv(t) = [-kr(t) - (L + B)v(t - \tau) + Bv_0(t - \tau)1]dt + [- (L_\sigma + B_\sigma)v(t - \tau) + B_\sigma v_0(t - \tau)1]dW, \quad (5)$$

$$dr_0(t) = v_0(t)dt, \quad dv_0(t) = -kr_0(t)dt, \quad (6)$$

where 1 denotes an $n \times 1$ column vector of all ones (with some ambiguity; however, the right meaning would be clear in the context).

Remark 2. *Note that v_i depends on the information from its in-neighbors and itself. In the special case that time delay $\tau = 0$ and $A_\sigma = B_\sigma = 0$, algorithms (4)-(6) are equivalent to algorithms (12) and (13) in [23].*

3 Convergence analysis

In this section, the convergence analysis of systems (4)-(6) is given and we show that n coupled harmonic oscillators (followers) are synchronized to the oscillating behavior of the virtual leader with probability one.

Before proceeding, we introduce an exponential stability result for the following n -dimensional stochastic differential delay equation (for more details, see e.g. [7])

$$dx(t) = [Ex(t) + Fx(t - \tau)]dt + g(t, x(t), x(t - \tau))dW(t), \quad (7)$$

where E and F are $n \times n$ matrices, $g : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ which is locally Lipschitz continuous and satisfies the linear growth condition with $g(t, 0, 0) \equiv 0$, $W(t)$ is an m -dimensional standard Brownian motion.

Lemma 1. ([15]) *Assume that there exists a pair of symmetric positive definite $n \times n$ matrices P and Q such that $P(E + F) + (E + F)^T P = -Q$. Assume also that there exist non-negative constants α and β such that*

$$\text{trace}[g^T(t, x, y)g(t, x, y)] \leq \alpha\|x\|^2 + \beta\|y\|^2 \quad (8)$$

for all $(t, x, y) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$. Let $\lambda_{\min}(Q)$ be the smallest eigenvalue of Q . If

$$(\alpha + \beta)\|P\| + 2\|PF\|\sqrt{2\tau(4\tau(\|E\|^2 + \|F\|^2) + \alpha + \beta)} < \lambda_{\min}(Q),$$

then the trivial solution of Equation (7) is almost surely exponentially stable.

We need the following lemma for Laplacian matrix.

Lemma 2.([25]) *Let L be the Laplacian matrix associated with a directed graph \mathcal{G} . Then L has a simple zero eigenvalue and all its other eigenvalues have positive real parts if and only if \mathcal{G} has a directed spanning tree. In addition, $L1 = 0$ and there exists $p \in \mathbb{R}^n$ satisfying $p \geq 0$, $p^T L = 0$ and $p^T 1 = 1$.*

Let

$$\begin{cases} r_0(t) := \cos(\sqrt{k}t)r_0(0) + \frac{1}{k}\sin(\sqrt{k}t)v_0(0), \\ v_0(t) := -\sqrt{k}\sin(\sqrt{k}t)r_0(0) + \cos(\sqrt{k}t)v_0(0). \end{cases}$$

Then it is easy to see that $r_0(t)$ and $v_0(t)$ solve (6). Let $r^* = r - r_01$, $v^* = v - v_01$. Invoking Lemma 2, we can obtain an error dynamics of (4)-(6) as follows

$$d\varepsilon(t) = [E\varepsilon(t) + F\varepsilon(t - \tau)]dt + H\varepsilon(t - \tau)dW(t), \quad (9)$$

where

$$\begin{aligned} \varepsilon &= \begin{pmatrix} r^* \\ v^* \end{pmatrix}, \quad E = \begin{pmatrix} 0 & I_n \\ -kI_n & 0 \end{pmatrix}, \\ F &= \begin{pmatrix} 0 & 0 \\ 0 & -L - B \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 0 & -L_\sigma - B_\sigma \end{pmatrix} \end{aligned}$$

and $W(t)$ is an $2n$ -dimensional standard Brownian motion.

Now we present our main result as follows.

Theorem 1. *Suppose that vertex 0 is globally reachable in $\bar{\mathcal{G}}$. If*

$$\|H\|^2\|P\| + 2\|PF\|\sqrt{8\tau^2[(k \vee 1)^2 + \|F\|^2] + 2\tau\|H\|^2} < \lambda_{\min}(Q), \quad (10)$$

where $k \vee 1 := \max\{k, 1\}$, P and Q are two symmetric positive definite matrices such that $P(E + F) + (E + F)^T P = -Q$. Then, by using algorithms (4)-(6), we have

$$r(t) - r_0(t)1 \rightarrow 0, \quad v(t) - v_0(t)1 \rightarrow 0$$

almost surely, as $t \rightarrow \infty$. Here, r_0 and v_0 are given as above.

Proof. Clearly, it suffices to prove the trivial solution $\varepsilon(t; 0) = 0$ of (9) is almost surely exponential stable.

Let $\{\lambda_i : i = 1, \dots, n\}$ be the set of eigenvalues of $-L - B$. Since vertex 0 is globally reachable in $\overline{\mathcal{G}}$, from Lemma 2 it follows that $-L - B$ is a stable matrix, that is, $\text{Re}(\lambda_i) < 0$ for all i .

Let μ be an eigenvalue of matrix $E + F$ and $\varphi = (\varphi_1^T, \varphi_2^T)^T$ be an associated eigenvector. We thus have

$$\begin{pmatrix} 0 & I_n \\ -kI_n & -L - B \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \mu \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$

which yields $(-L - B)\varphi_1 = \frac{\mu^2 + k}{\mu}\varphi_1$ and $\varphi_1 \neq 0$. Hence μ satisfies $\mu^2 - \lambda_i\mu + k = 0$. The $2n$ eigenvalues of $E + F$ are shown to be given by $\mu_{i\pm} = \frac{\lambda_i \pm \sqrt{\lambda_i^2 - 4k}}{2}$ for $i = 1, \dots, n$. Since $\text{Re}(\lambda_i) < 0$, we get $\text{Re}(\mu_{i-}) = \text{Re}\left(\frac{\lambda_i - \sqrt{\lambda_i^2 - 4k}}{2}\right) < 0$ for $i = 1, \dots, n$. From $\mu_{i+}\mu_{i-} = k$ it follows that μ_{i+} and μ_{i-} are symmetric with respect to the real axis in the complex plane. Accordingly, $\text{Re}(\mu_{i+}) < 0$ for $i = 1, \dots, n$; furthermore, $E + F$ is a stable matrix. By Lyapunov theorem, for all symmetric positive definite matrix Q there exists a unique symmetric positive definite matrix P such that

$$P(E + F) + (E + F)^T P = -Q. \quad (11)$$

On the other hand, we have $\text{trace}(\varepsilon^T H^T H \varepsilon) \leq \|H\|^2 \|\varepsilon\|^2$. Therefore, (8) holds with $\alpha = 0$ and $\beta = \|H\|^2$. Note that $\|E\| = k \vee 1$. We then complete our proof by employing Lemma 1. \square

Remark 3. *Note that the result of Theorem 1 is dependent of the choice of matrices P and Q . From computational points of view, the solution to Lyapunov matrix equation (11) may be expressed by using Kronecker product; $\|H\| = \|L_\sigma + B_\sigma\|$ and $\|F\| = \|L + B\|$ hold.*

Remark 4. *The algorithms (4)-(6) can also be applied to synchronized motion coordination of multi-agent systems, as indicated in [23] (Section 5).*

When deviations between oscillator states exist, we may exploit the following algorithm to take the place of Equation (5):

$$\begin{aligned} dv(t) &= [-k(r(t) - \delta) - (L + B)v(t - \tau) + Bv_0(t - \tau)\mathbf{1}]dt \\ &\quad + [- (L_\sigma + B_\sigma)v(t - \tau) + B_\sigma v_0(t - \tau)\mathbf{1}]dW, \end{aligned} \quad (12)$$

where $\delta = (\delta_1, \dots, \delta_n)^T$ is a constant vector denoting the deviations. Similarly, we obtain the following result.

Corollary 1. *Suppose that vertex 0 is globally reachable in $\bar{\mathcal{G}}$, and condition (10) holds, then by using algorithms (4), (6) and (12), we have*

$$r(t) - \delta - r_0(t)1 \rightarrow 0, \quad v(t) - v_0(t)1 \rightarrow 0$$

almost surely, as $t \rightarrow \infty$. Here, r_0 and v_0 are defined as in Theorem 1.

4 Numerical examples

In this section, we provide numerical simulations to illustrate our results.

We consider a network $\bar{\mathcal{G}}$ consisting of five coupled harmonic oscillators including one leader indexed by 0 and four followers as shown in Fig. 1. We assume that $a_{ij} = 1$ if $j \in \mathcal{N}_i$ and $a_{ij} = 0$ otherwise; $b_i = 1$ if $0 \in \mathcal{N}_i$ and $b_i = 0$ otherwise. Note that vertex 0 is globally reachable in $\bar{\mathcal{G}}$. For simplicity, we take the noise intensity matrices $L_\sigma = 0.1L$ and $B_\sigma = 0.1B$. We take $Q = I_8$ with $\lambda_{\min}(Q) = 1$. By straightforward calculation, it is obtained that $\|H\| = 0.2466$ and $\|F\| = 2.4656$. Two different gains k are explored as follows:

Firstly, we take $k = 0.6$ such that $\|E\| = 1 > k$. We solve P from Equation (11) and get $\|P\| = 8.0944$ and $\|PF\| = 4.1688$. Hence the condition (10) in Theorem 1 is satisfied by taking time delay $\tau = 0.002$. Thus, the oscillator states are synchronized successfully as shown in Fig. 2 and Fig. 3 with initial values given by $\varepsilon(0) = (-5, 1, 4, -3, -8, 2, -1.5, 3)^T$.

Secondly, we take $k = 2$ such that $\|E\| = k > 1$. In this case we obtain $\|P\| = 8.3720$, $\|PF\| = 7.5996$ and the condition (10) is satisfied by taking time delay $\tau = 0.001$. Thereby the oscillator states are synchronized successfully as shown in Fig. 4 and Fig. 5 with the same initial values given as above.

We see that the value of k not only has an effect on the magnitude and frequency of the synchronized states (as implied in Theorem 1), but also affects the shapes of synchronization error curves $\|r^*\|$ and $\|v^*\|$.

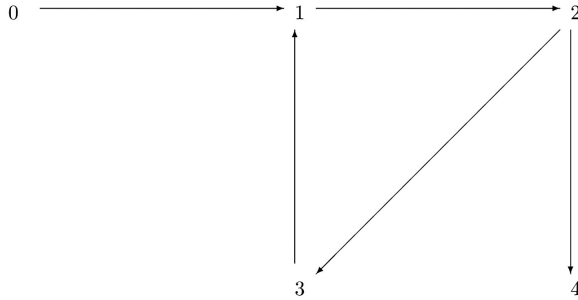


Figure 1: Directed network $\bar{\mathcal{G}}$ for five coupled harmonic oscillators involving one leader. $\bar{\mathcal{G}}$ has 0 – 1 weights.

5 Conclusion

This paper is concerned with synchronization of coupled harmonic oscillators with stochastic perturbation and time delays. Based on the stability theory of stochastic differential delay equations, we have shown that the coupled second-order linear harmonic oscillators are synchronized (i.e. follow the leader) with probability one provided the leader is globally reachable and the time delay is less than a certain critical value. Numerical simulations are presented to illustrate our theoretical results. Since we only investigate the case when the time delay is constant and the network topology is fixed, how to consider the time-varying delay and topology is our future research.

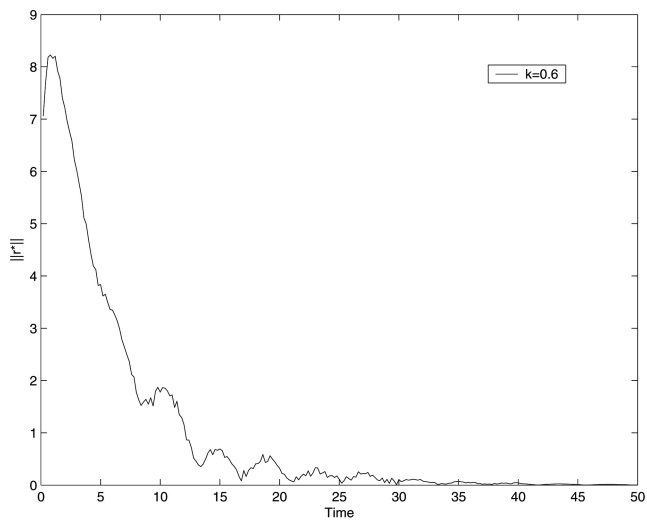


Figure 2: Synchronization error $\|r^*\|$ for $k = 0.6$ and $\tau = 0.002$.

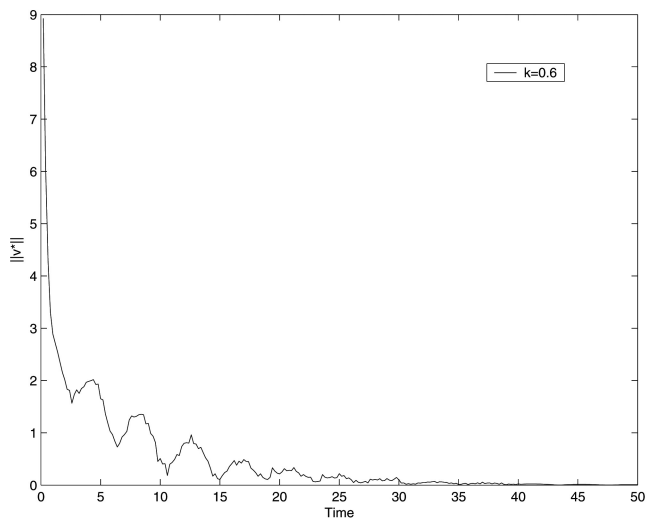


Figure 3: Synchronization error $\|v^*\|$ for $k = 0.6$ and $\tau = 0.002$.

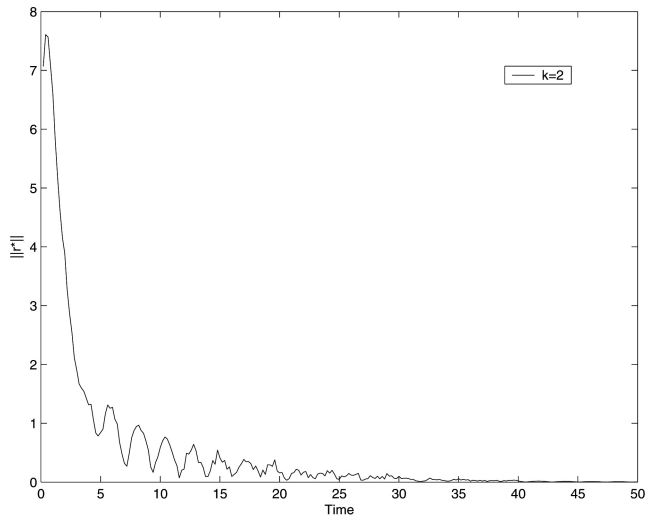


Figure 4: Synchronization error $\|r^*\|$ for $k = 2$ and $\tau = 0.001$.

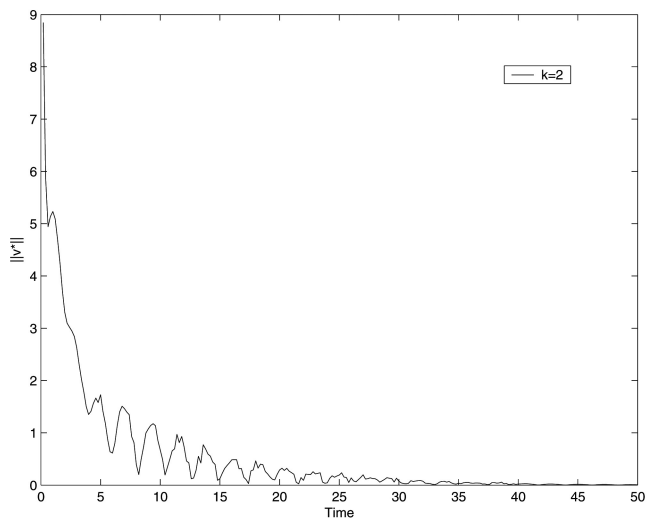


Figure 5: Synchronization error $\|v^*\|$ for $k = 2$ and $\tau = 0.001$.

References

- [1] J. A. Acebrón, L. L. Bonilla, C. J. Pérez-Vicente, F. Ritort, R. Spigler. The Kuramoto model: A simple paradigm for synchronization phenomena. *Rev. Mod. Phys.* 77:137-185, 2005.
- [2] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, C. Zhou. Synchronization in complex networks. *Physics Reports* 469:93-153, 2008.
- [3] L. Ballard, W. Ren. Experiments with coupled harmonic oscillators with local interaction. *IEEE/ASME International Conference on Advanced Intelligent Mechatronics* 716-721 (art. no. 4601748), 2008.
- [4] R. Carli, A. Chiuso, L. Schenato, S. Zampieri. A PI consensus controller for networked clocks synchronization. *Proc. 17th IFAC World Congress*, Seoul, Korea, 2008.
- [5] C. Cutts, J. Speakman. Energy savings in formation flight of Pink-Footed geese. *Journal of Experimental Biology* 189:251-261, 1994.
- [6] C. Godsil, G. Royle. *Algebraic Graph Theory*. Springer-Verlag, New York, 2001.
- [7] J. K. Hale, S. M. V. Lunel. *Introduction to Functional Differential Equations*. Springer-Verlag, New York, 1993.
- [8] Y. Hong, G. Chen, L. Bushnell. Distributed observers design for leader-following control of multi-agent networks. *Automatica* 44:846-850, 2008.
- [9] Y. Hong, J. Hu, L. Gao. Tracking control for multi-agent consensus with an active leader and variable topology. *Automatica* 42:1177-1182, 2006.
- [10] J. Hu, Y. Hong. Leader-following coordination of multi-agent systems with coupling time delays. *Physica A* 374:853-863, 2007.
- [11] O. Ilaya, C. Bil, M. Evans. Control design for unmanned aerial vehicle swarming. *Journal of Aerospace Engineering* 222:549-567, 2008
- [12] Y. Kuramoto. *Chemical Oscillations, Waves, and Turbulence*. Springer-Verlag, New York, 1984.
- [13] P. Li, Z. Yi. Synchronization of Kuramoto oscillators in random complex networks. *Physica A* 387:1669-1674, 2008.

- [14] P. Lin, Y. Jia. Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies. *Automatica* 45:2154-2158, 2009
- [15] X. Mao. Robustness of exponential stability of stochastic differential delay equations. *IEEE Trans. on Autom. Control* 41:442-447, 1996.
- [16] S. Mu, T. Chu, L. Wang. Coordinated collective motion in a motile particle group with a leader. *Physica A* 351:211-226, 2005.
- [17] U. Münz, A. Papachristodoulou, F. Allgöwer. Consensus reaching in multi-agent packet-switched networks with non-linear coupling. *International Journal of Control* 82:953-969, 2009.
- [18] R. Olfati-Saber. Flocking for multi-agent dynamic systems: algorithms and theory. *IEEE Trans. on Autom. Control* 51:401-420, 2006
- [19] R. Olfati-Saber, J. A. Fax, R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proc. IEEE* 95:215-233, 2007.
- [20] G. V. Osipov, J. Kurths, C. Zhou. *Synchronization in Oscillatory Networks*. Springer-Verlag, Berlin, 2007.
- [21] A. Papachristodoulou, A. Jadbabaie. Synchronization in oscillator networks with heterogeneous delays, switching topologies and nonlinear dynamics. *IEEE Proc. 45th Conference on Decision and Control*, 4307-4312, 2006.
- [22] K. Peng, Y. Yang. Leader-following consensus problem with a varying-velocity leader and time-varying delays. *Physica A* 388:193-208, 2009.
- [23] W. Ren. Synchronization of coupled harmonic oscillators with local interaction. *Automatica* 44:3195-3200, 2008.
- [24] W. Ren. On consensus algorithms for double-integrator dynamics. *IEEE Trans. on Autom. Control* 53:1503-1509, 2008.
- [25] W. Ren, R. W. Beard. Consensus seeking in multi-agent systems under dynamically changing interaction topologies. *IEEE Trans. on Autom. Control* 50:655-661, 2005.

- [26] Y. Shang. Emergence in random noisy environments. *International Journal of Mathematical Analysis* 4:1205-1215, 2010.
- [27] R. Solis, V. Borkar, P. R. Kumar. A new distributed time synchronization protocol for multihop wireless networks. *IEEE Proc. 45th Conference on Decision and Control*, 2734-2739, 2006.
- [28] S. Strogatz. *SYNC: The Emerging Science of Spontaneous Order*. Hyperion Press, New York, 2003.
- [29] B. Veselić, Č. Milosavljević. Sliding mode based harmonic oscillator synchronization. *International Journal of Electronics* 90:553-570, 2003.
- [30] G. Xie, L. Wang. Consensus control for a class of networks of dynamic agents. *International Journal of Robust and Nonlinear Control* 17:941-959, 2007.
- [31] W. Yang, A. L. Bertozzi, X. Wang. Stability of a second-order consensus algorithm with time delay. *IEEE Proc. 47th Conference on Decision and Control*, 2926-2931, 2008.
- [32] W. Yu, G. Chen, M. Cao. On second-order consensus in multi-agent dynamical systems with directed topologies and time delays. *IEEE Proc. 48th Conference on Decision and Control*, 3709-3714, 2009.