Annals of the Academy of Romanian Scientists
Series on Mathematics and its Applications
ISSN 2066 - 6594 Volume 3, Number 2 / 2011

AROUND AN INEQUALITY, OR TWO, OF KY FAN*

Charles Horvath†

Abstract

In 1957 Ky Fan gave in [5] a necessary and sufficient condition, known as Fan's Consistency Condition, for a finite system of convex inequalities to have a solution. This result has been somewhat overshadowed by the famous Fan's Inequality which is equivalent to Brouwer's Fixed Point Theorem. Another result which bears Fan's name, but which is not due to him, is Fan's Lopsided Inequality which Aubin and Ekeland prove in [1] using Fan's Inequality.
We first prove a fairly general, but elementary result, Theorem 2.1.1, from which we derive both Fan's Theorem for finite systems of convex inequalities and Fan's Lopsided Inequality whose proof, therefore, does not require Brouwer's Fixed Theorem. We show that Theorem 2.1.1 is equivalent to Fan's Theorem for finite systems of convex inequalities; consequently, the Lopsided Inequality is a consequence of Fan's Theorem for finite systems of convex inequalities.
A number of well known and important results are proved along the way. The paths leading from Fan's 1957 theorem to those results are, we hope, simple enough to demonstrate that it deserves to be as well known as its younger and powerful cousin, Fan's Inequality.

MSC: 46A22, 46N10, 49J35, 49J40.

Keywords: convex inequalities, Fan's consistency condition. fixed points, Mazur-Orlicz.

*Accepted for publication in revised form on May 10, 2011

†Université de Perpignan, département de mathématiques, horvath@univ-perp.fr.

1 Introduction

Apart from Theorem 2.1.1, very little that is not already very well known will be found in this note. From Theorem 2.1.1 one proves Fan's Theorem on systems of inequalities for convex functions, Theorem 2.2.3. This result of Ky Fan is over half a century hold and it has been somewhat left aside after the appearance of Ky Fan's Inequality which, in its many different forms, has become a standard tool from mathematical economics to partial differential equations. But, as long as one does not deal with results that are at least as strong Brouwer's Fixed Point Theorem, Fan's result on finite systems of convex inequalities can be very versatile. We give rather simple proofs, starting from Theorem 2.2.3, or an equivalent formulation, of such results as the Kakutani Fixed Point Theorem for commutative families of continous affine maps, from which one can derive Day's Theorem on the existence of invariant means on compact topological semigroups and the Mazur-Orlicz Theorem. Proposition 2.2.6, whose proof from Fan's theorem is short and direct, leads to simple proofs of Stamppachia's and Lax-Milgram's Theorems (details are left to the reader).
In the last section, we give a proof of Fan's Lopsided Inequality, Theorem 3.0.6 using Theorem 2.1.1, and therefore Fan's Theorem.

2 A lopsided minsup inequality

2.1 The main result

Theorem 2.1.1 *Let X be a compact topological space and $f : X \times X \to \mathbb{R}$ such that:*

(1) *for all $y \in X$ $x \mapsto f(x, y)$ is lower semicontinuous;*

(2) *for all nonempty finite subset $S \subset X$ and for all $(x_1, x_2) \in X \times X$ there exists $x_3 \in X$ such that*

$$\forall y \in S \quad f(x_3, y) \leq \frac{1}{2} f(x_1, y) + \frac{1}{2} f(x_2, y).$$

(3) *for all $(x, y) \in X \times X$*

$$f(x, y) + f(y, x) \leq 0.$$

Then

$$\min_{x\in X}\sup_{y\in X} f(x,y) \le 0.$$

Proof. Let us begin with two remarks:

(A) The set $\mathbb{D}_n$ of dyadic elements of the standard n-dimentional simplex Δ_n is dense in Δ_n, where by $\mathbb{D}_n$ we mean the set of elements $(d_0, \cdots, d_n) \in \Delta_n$ such that each d_i is of the form $k_i/2^{m_i}$ where k_i and m_i are positive integers.

(B) Hypothesis (2) can be generalized as follows: for all nonempty finite subset $S \subset X$, for all $(x_0, \cdots, x_n) \in X^{n+1}$ and for all $(d_0, \cdots, d_n) \in \mathbb{D}_n$ there exists $x_{n+1} \in X$ such that, for all $y \in S$,

$$f(x_{n+1}, y) \le \sum_{i=0}^{n} d_i f(x_i, y). \tag{2.1}$$

To prove (2.1) one can proceed by induction starting with $n = 1$: we have to see that if d is dyadic number then there exists $x_3 \in X$ such that, for all $y \in S$, $f(x_3, y) \le df(x_1, y) + (1-d)f(x_2, y)$.
Let $\mathcal{D}$ be the set of dyadic numbers in the interval $[0, 1]$ and let $\mathcal{D}_i$, $i \in \mathbb{N}$ be those dyadic numbers which can be written as $\frac{k}{2^i}$ with k being an integer not greater than 2^i; we have $\mathcal{D} = \cup_{i\in\mathbb{N}}\mathcal{D}_i$. Since $\mathcal{D}_0 = \{0, 1\}$ there is nothing to prove if $d \in \mathcal{D}_0$. Also, since $\mathcal{D}_1 = \left\{0, \frac{1}{2}, 1\right\}$ the existence of x_3 either holds trivially or by hypothesis (2). If $d \in \mathcal{D}_{k+1}$ but $d \notin \mathcal{D}_k$ we can write $d = \frac{1}{2}d_1 + \frac{1}{2}d_2$ with d_1 and d_2 in $\mathcal{D}_k$. By the induction hypothesis we can find $x_{3,1}$ and $x_{3,2}$ such that, for all $y \in S$, $f(x_{3,i}, y) \le d_i f(x_1, y) + (1-d_i)f(x_2, y)$; by hypothesis (2) there exists x_3 such that, for all $y \in S$, $f(x_3, y) \le \frac{1}{2}f(x_{3,1}, y) + \frac{1}{2}f(x_{3,2}, y)$. This concludes the proof of (2.1) for $n = 1$.

For $n = m + 1$ we can assume that $d_{m+1} \ne 1$ and we set for $i \le m$, $d'_i = \frac{d_i}{\sum_{j=0}^m d_j}$ and we find $x'_{m+1} \in X$ such that; for all $y \in S$, $f(x'_{m+1}, y) \le \sum_{i=0}^m d'_i f(x_i, y)$. Since a sum of dyadic numbers is a dyadic number we can find $x_{m+2} \in X$ such that, for all $y \in S$,
$f(x_{m+2}, y) \le \left(\sum_{j=0}^m d_j\right) f(x'_{m+1}, y) + d_{m+1}f(x_{m+1}, y)$.

Let us now proceed with the proof. Let $S = \{y_0, \cdots, y_n\}$ be an arbitrary nonempty finite subset of X and define on the standard n-dimensional simplex Δ_n a bilinear form as follows: $B_S(u, v) = \sum_{i,j} u_i f(y_i, y_j) v_j$. From condition (3) we have,

$$\forall u \in \Delta_n \quad B_S(u, u) \leq 0. \tag{2.2}$$

From Von Neumann's Minimax Theorem for bilinear forms, there exists a saddle point $(u_S, v_S) \in \Delta_n \times \Delta_n$ for B_S. From

$$\forall\, (u, v) \in \Delta_n \times \Delta_n \quad B_S(u_S, v) \leq B_S(u, v_S)$$

and

$$B_S(v_S, v_S) \leq 0$$

we obtain

$$\forall v \in \Delta_n \quad B_S(u_S, v) \leq 0 \tag{2.3}$$

Let $\varepsilon > 0$ be an arbitrary positive number. Since the elements of $\mathbb{D}_n$ are dense in Δ_n one can find $\bar{u}_S \in \mathbb{D}_n$ such that

$$\forall v \in \Delta_n \quad B_S(\bar{u}_S, v) \leq \varepsilon \tag{2.4}$$

which can also be written as

$$\forall j \in \{0, \cdots, n\} \quad \sum_{i=0}^{n} \bar{u}_{S,i} f(y_i, y_j) \leq \varepsilon. \tag{2.5}$$

From (B) with $x_i = y_i$ there exists $x_{S,\varepsilon} \in X$ such that,

$$\forall y \in S \quad f(x_{S,\varepsilon}, y) \leq \varepsilon. \tag{2.6}$$

To complete the proof let, for all $\varepsilon > 0$ and all $y \in S$,

$$[f(-, y) \leq \varepsilon] = \{x \in X : f(x, y) \leq \varepsilon\}.$$

We have shown that the family of sets $\big\{[f(-, y) \leq \varepsilon] : y \in X\big\}$ has the finite intersection property; by hypothesis (1) all the sets in question are closed. By compactness of X the set $\cap_{y\in X}[f(-, y) \leq \varepsilon]$ is not empty, and also compact. For $0 \leq \varepsilon' \leq \varepsilon$ we obviously have $\cap_{y\in X}[f(-, y) \leq \varepsilon'] \subset \cap_{y\in X}[f(-, y) \leq \varepsilon]$ and consequently $\cap_{\varepsilon>0} \cap_{y\in X} [f(-, y) \leq \varepsilon] \neq \emptyset$. This concludes the proof. $\square$

An equivalent, but more general, formulation of Theorem 2.1.1 can be given without hypothesis (3).
Let $\lambda = \sup_{(x,y)\in X\times X} \dfrac{f(x,y)+f(y,x)}{2}$ and, to obtain a non trivial result, assume that $\lambda \neq +\infty$. Now let $g(x,y) = f(x,y) - \lambda$ and notice that (1) and (2) hold for g if they hold for f and that (3) holds for g.

Theorem 2.1.2 *Let X be a compact topological space and $f : X \times X \to \mathbb{R}$ such that:*

(1) *for all $y \in X$ $x \mapsto f(x,y)$ is lower semicontinuous;*

(2) *for all nonempty finite subset $S \subset X$ and for all $(x_1,x_2) \in X \times X$ there exists $x_3 \in X$ such that*

$$\forall y \in S \quad f(x_3,y) \leq \frac{1}{2}f(x_1,y) + \frac{1}{2}f(x_2,y).$$

Then

$$\exists x_0 \in X \text{ such that } \forall y \in X \quad f(x_0,y) \leq \sup_{(x,y)\in X\times X} \frac{f(x,y)+f(y,x)}{2}.$$

Let us say that a function $f : X \times Y \to \mathbb{R}$ defined on the product of two arbitrary sets X and Y is **finitely midconvex in its first variable** if condition (2) of Theorem 2.1.1 holds; one can similarly define what it means to be **finitely midconcave in its second variable**.

A given function $f : X \times Y \to \mathbb{R}$ is finitely midconvex in its first variable exactly if the family $\{S(x_1,x_2 : y) : y \in Y\}$ has the finite intersection property, where

$$S(x_1,x_2;y) = \{x \in X : f(x,y) \leq \tfrac{1}{2}f(x_1,y) + \tfrac{1}{2}f(x_2,y)\}.$$

Furthermore, if X is a compact topological space and if $f : X \times Y \to \mathbb{R}$ is lower semicontinuous in its first variable then, for all $y \in Y$ and for all $x_1, x_2 \in X$, the set $S(x_1,x_2 : y)$ is compact.
In conclusion, if X is a compact topological space and if $f : X \times Y \to \mathbb{R}$ is finitely midconvex and lower semicontinuous in its first variable then $\cap_{y\in Y} S(x_1,x_2;y) \neq \emptyset$, that is, there exists $x_3 \in X$ such that, for all $y \in Y$, $f(x_3,y) \leq \frac{1}{2}f(x_1,y) + \frac{1}{2}f(x_2,y)$.

Assume now that f is both lower semicontinuous and finitely midconvex in its first variable.

Take an arbitrary real number $t \in [0,1]$ and a sequence $(d_n)_{n\in\mathbb{N}}$ of dyadic numbers in $[0,1]$ which converges to t; for all n there exists $x_{3,n} \in X$ such that, for all $y \in Y$, $f(x_{3,n}, y) \leq (1-d_n)f(x_1, y)+d_n f(x_2, y)$ and therefore, by compactness of X and lower semicontinuity of $f(-, y)$, there exists $x_{3,t} \in X$ such that, for all $y \in Y$, $f(x_{3,t}, y) \leq (1-t)f(x_1, y) + tf(x_2, y)$. In other words, f is **convexlike in its first variable** that is; for all $x_1, x_2 \in X$ and for all $t \in [0,1]$, there exists $x_3 \in X$ such that, for all $y \in Y$, $f(x_3, y) \leq (1-t)f(x_1, y) + tf(x_2, y)$. In conclusion, *assuming compactness of X and lower semicontinuity in the first variable, being finitely midconvex in the first variable or being convexlike in the first variable are equivalent conditions and these are in turn equivalent to*

$$\forall n \in \mathbb{N} \quad \forall (x_0, \cdots, x_n) \in X^{n+1} \quad \forall u \in \Delta_n \quad \exists \hat{x} \in X$$
$$\text{such that}$$
$$\forall y \in X \quad f(\hat{x}, y) \leq \sum_{i=0}^{n} u_i f(x_i, y). \tag{2.7}$$

One could similarly define what it means for f to be concave like in its second variable and reach a similar conclusion with respect to functions which are finitely midconcave in the second variable.

2.2 Some results that can be derived from the Main Theorem

Proposition 2.2.1 *Let X and Y be two compact topological spaces and $f, g : X \times Y \to \mathbb{R}$ two functions such that:*

(1) *f is lower semicontinuous and finitely midconvex in its first variable;*

(2) *g is upper semicontinuous and finitely midconcave in its second variable;*

(3) *$\forall (x, y) \in X \times Y$ $f(x, y) \leq g(x, y)$.*

Then

$$\exists (x_0, y_0) \in X \times Y \quad \textit{such that} \quad \forall (x, y) \in X \times Y \quad f(x_0, y) \leq g(x, y_0).$$

Proof. Apply Theorem 2.1.1 to the compact topological space $Z = X \times Y$ and the function $F\big((x_1, y_1), (x_2, y_2)\big) = f(x_1, y_2) - g(x_2, y_1)$. □

Taking $f = g$ in Proposition 2.2.1 one obtains Proposition 2.2.2 below. On the one hand, Proposition 2.2.2 clearly implies Von Neumann's Minimax

Theorem, on the other hand, Theorem 2.1.1 was derived from Von Neumann's Minimax Theorem . Proposition 2.2.2 and Von Neumann's Minimax Theorem are therefore equivalent.

Proposition 2.2.2 *Let X and Y be two compact topological spaces and let $f : X \times Y \to \mathbb{R}$ be a function which is lower semicontinuous and finitely midconvex in its first variable and upper semicontinuous and finitely midconcave in its second variable. Then,*

$$\exists (x_0, y_0) \in X \times Y \quad \textit{such that} \quad \forall (x, y) \in X \times Y \quad f(x_0, y) \leq f(x_0, y_0) \leq f(x, y_0).$$

Propositions 2.2.1 and 2.2.2, in a somewhat more general version involving 4 functions, are due to Granas and Liu [8].

In [5] Fan proved the following existence theorem for finite systems of inequalities:

Theorem 2.2.3 (Fan's Theorem) *Let $f_i : X \to \mathbb{R}$, $i \in \{0, \cdots, n\}$, be a finite family of lower semicontinuous functions defined on a compact convex subset of a linear topological vector space.*
Assume that the following condition holds:

$$\forall u \in \Delta_n \quad \exists x \in X \quad \textit{such that} \quad \sum_{i=0}^{n} u_i f_i(x) \leq 0 \tag{2.8}$$

then

$$\exists x_0 \in X \quad \textit{such that} \quad \forall i \in \{0, \cdots, n\} \quad f_i(x_0) \leq 0.$$

The proof of Fan's Theorem can be found on page 41 of [7]. Fan's Theorem follows from Theorem 2.1.1. We prove a somewhat more general result, which is implicitely contained in Fan's paper. First, let us say that a family of functions $f_i : X \to \mathbb{R}$, $i \in I$ is a **finitely midconvex family** if the function $F : X \times I \to \mathbb{R}$ defined by $F(x, i) = f_i(x)$ is finitely midconvex in its first variable. In case I is a finite set the adjective "finitely" is dropped. Let us say that **Fan's consistency condition** holds for the family $\mathcal{F} = \{f_i : i \in I\}$ if for all finite subsets $\{f_0, \cdots, f_n\}$ of $\mathcal{F}$ condition (2.8) of Fan's Theorem holds.

Theorem 2.2.4 *Let $\mathcal{F}$ be a finitely midconvex family of lower semicontinuous functions defined on a compact topological space. If Fan's consistency condition holds then*

$$\exists x_0 \in X \quad \textit{such that} \quad \sup_{f\in\mathcal{F}} f(x_0) \leq 0.$$

Proof. For all $f \in \mathcal{F}$ let $[f \leq 0] = \{x \in X : f(x) \leq 0\}$. Since X is compact and the elements of $\mathcal{F}$ are lower semicontinuous we have to show that the family $\{[f \leq 0] : f \in \mathcal{F}\}$ has the finite intersection property. Given a finite subfamily $\{f_0, \cdots, f_n\}$ of $\mathcal{F}$ the function $\varphi : X \times \Delta_n \to \mathbb{R}$ defined by $\varphi(x, u) = \sum_{i=0}^n u_i f_i(x)$ is finitely midconvex and lower semicontinuous in its first variable and finitely midconcave and upper semicontinuous in its second variable.
By Proposition 2.2.2 there exists $(x_0, u_0) \in X \times \Delta_n$ such that, for all (x, u) in $X \times \Delta_n$, $\varphi(x_0, u) \leq \varphi(x_0, u_0) \leq \varphi(x, u_0)$.
From Fan's consistency condition, there exists $x^\star \in X$ such that $\varphi(x^\star, u_0) \leq 0$ and therefore, $\varphi(x_0, u_0) \leq 0$; we have shown that $\sup_{u\in\Delta_n} \varphi(x_0, u) \leq 0$, that is $x_0 \in \cap_{i=0}^n [f_i \leq 0]$. □

To close this circle of ideas let us see that Theorem 2.1.1 can be deduced from Theorem 2.2.4.

Given $f : X \times X \to \mathbb{R}$ as in Theorem 2.1.1 take X itself as the set of indices and let $f_y(x) = f(x, y)$. If Fan's Consistency Condition holds for the family $\{f_y : y \in X\}$ we are done.
If Fan's Consistency Condition does not hold then there exists a finite subset $\{y_0, \cdots, y_n\}$ of X and there exists $u \in \Delta_n$ such that,

$$\forall x \in X \quad \sum_{i=0}^n u_i f(x, y_i) > 0. \tag{2.9}$$

From (2.9) and $f(x, y_i) + f(y_i, x) \leq 0$ we have

$$\forall x \in X \quad \sum_{i=0}^n u_i f(y_i, x) < 0 \tag{2.10}$$

and from (2.7), there exists $\hat{y} \in X$ such that

$$\forall x \in X \quad f(\hat{y}, x) \leq \sum_{i=0}^n u_i f(y_i, x) \tag{2.11}$$

and therefore, from (2.10),

$$\forall x \in X \quad f(\hat{y}, x) < 0. \tag{2.12}$$

But (2.12) clearly implies that Fan's Consistency Condition holds (and it also implies the conclusion of Theorem 2.1.1). We have reached a contradiction and therefore Fan's Consistency Condition holds. □

Fan's Theorem is a non linear version of Fourier's Theorem on systems of linear inequalities, a classical result of linear programming, from which one can derive Von Neumann's Minimax Theorem for bilinear forms, or the well known Farkas Lemma; all these results are equivalent, in the sense that they can all be derived from any given one of them. Fourier's Theorem can be proved in a completely elementary way, as in [11]. Here is a short proof from Fan's Theorem.

Theorem 2.2.5 (Fourier) *Let A be an $m \times n$ matrix and $B \in \mathbb{R}^m$ then, either the system of linear inequalities $AX \geq B$ has a solution or there exists $Y \in \mathbb{R}^m_+$ such that $A^t Y = 0$ and $Y^t B > 0$.*

To see that Theorem 2.2.5 follows from Fan's Theorem, let $f_i(X) = b_i - \sum_{j=1}^n a_{ij}x_j$, $i \in \{1, \cdots, n\}$. If there is no solution in $\mathbb{R}^n$ to the system of inequalities $f_i(X) \leq 0$ then, for all integer $k > 0$, there exists $Y_k \in \Delta_{n-1}$ such that, for all $X \in \mathbb{R}^n$ of norm not exceeding k, $\sum_{i=1}^n y_{n,k} f_i(X) > 0$, that is, $Y_k^t B - Y_k^t AX > 0$. We can assume that the sequence $(Y_k)_{k \in \mathbb{N}}$ converges to some $Y^\star \in \Delta_{n-1}$. For any given $X \in \mathbb{R}^n$, we will have, for $k > \|X\|$, $Y_k^t B - Y_k^t AX > 0$; and consequently $Y^{\star t} B - Y^{\star t} AX \geq 0$. We must have $Y^{\star t} A = 0$, otherwise we can choose X such that $Y^{\star t} AX > 0$, and therefore $Y^{\star t} A(rX) > 0$ for all $r > 0$. This proves Fourier's Theorem.

Von Neumann's Minimax Theorem for bilinear forms is easily derived from Fourier's Theorem. There are many elementary proofs of the Von Neumann's Minimax Theorem for bilinear forms.

Proposition 2.2.6 (Weak Fan Inequality) *Let $f : C \times C \to \mathbb{R}$ be a function defined on a compact convex subset of some linear space. Assume that the following conditions hold:*

(1) $\forall y \in C \quad x \mapsto f(x, y)$ *is lower semicontinuous and convex on C;*

(2) $\forall x \in C \quad y \mapsto f(x,y)$ *is concave on* C*;*

(3) $\forall x \in C \quad f(x,x) \leq 0.$

Then, there exists $x_0 \in C$ *such that, for all* $y \in C$, $f(x_0,y) \leq 0$.

Proof. Let us see that Fan's Consistency Condition holds for the family $\mathcal{F} = \{f(-,y) : y \in C\}$. Otherwise, there exists $y_0, \cdots, y_n \in C$ and $(u_0, \cdots, u_n) \in \Delta_n$ such that, for all $x \in C$,

$$\sum_{i=0}^{n} u_i f(x, y_i) > 0. \tag{2.13}$$

Let $\hat{x} = \sum_{i=0}^{n} u_i y_i$ and, in (2.13), take $x = \hat{x}$. From the second hypothesis we then have $f(\hat{x}, \hat{x}) > 0$ which is contradiction with hypothesis (3). $\square$

Along with Theorem 4.2 of [7] page 65 (which can be seen as an elementary proof of the weak compactness of closed convex subsets of a Hilbert space.), Proposition 2.2.6 can be used to easily prove such results as the Stampacchia or the Lax-Milgram theorems. In [7] these results are derived from a weak form of the KKM Lemma.

2.3 Fan's Theorem and Fixed Points

Lemma 2.3.1 (Markov's Theorem) *A linear map* $X \mapsto PX$ *from* Δ_n *to itself has a fixed point.*

Proof. For $(X,Y) \in \Delta_n \times \Delta_n$ let $f_Y(X) = X^t\big(P^t - I\big)Y$; if Fan's Consistency Condition holds for the family $\mathcal{F} = \{f_Y : Y \in \Delta_n\}$ then there exists $X_0 \in \Delta_n$ such that, for all $Y \in \Delta_n$, $X_0^t\big(P^t - I\big)Y \leq 0$ which is equivalent to $X_0^t\big(P^t - I\big) \leq 0$ or, $PX_0 - X_0 \leq 0$. Since both PX_0 and X_0 belong to Δ_n equality must hold.
For a contradiction, assume that Fan's Consistency Condition does not hold. Then, there exists $Y_0, \cdots, Y_k \in \Delta_n$ and $u \in \Delta_k$ such that, for all $X \in \Delta_n$, $\sum_{i=0}^{k} u_k X^t\big(P^t - I\big)Y_k > 0$. With $\hat{Y} = \sum_{i=0}^{k} u_k Y_k$ we obtain

$$\forall X \in \Delta_n \quad X^t\big(P^t - I\big)\hat{Y} > 0 \tag{2.14}$$

which is equivalent to

$$\forall i \in \{1, \cdots, n\} \quad \sum_{j=1}^{n} p_{j,i}\hat{y}_j > \hat{y}_i. \tag{2.15}$$

Let $\left\|\hat{Y}\right\| = \max\{\hat{y_1}, \cdots, \hat{y_n}\}$ and choose i_0 such that $\hat{y_{i_0}} = \left\|\hat{Y}\right\|$. Since the entries of P are non negative and since each column sums up to 1 we obtain from (2.15) $\left\|\hat{Y}\right\| > \left\|\hat{Y}\right\|$. □

A similar elementery proof of Markov's theorem based on Farkas Theorem can be found in [6]. Farkas Theorem is easily derived from Fourier's Theorem.

Theorem 2.3.2 (Kakutani) *Let C be a convex compact subspace of a locally convex topological vector space and let $\mathcal{F}$ be a commutative family of continuous affine maps from C into itself. Then the members of $\mathcal{F}$ have a common fixed point.*

Proof. (a) Let T be an arbitrary, but fixed, element of $\mathcal{F}$. We show that T has a fixed point. Let W be an arbitrary convex neighborhood of the origin. By compactness there is a finite subset $\{x_0, \ldots, x_n\}$ of C such that $C \subset \bigcup_{i=0}^{n}(x_i + W)$. For each index $i \in \{0, \cdots, n\}$ choose an index $\varphi(i) \in \{0, \cdots, n\}$ such that

$$T(x_i) \in x_{\varphi(i)} + W. \tag{2.16}$$

Let T_W be the unique affine map from Δ_n to itself such that $T_W(e_i) = e_{\varphi(i)}$ where $e_0, \ldots, e_n$ are the vertices of Δ_n and let $p_w = \sum_{i=0}^{n} \mu_i e_i$ be a fixed point of T_W. From $p_w = T_W(p_w)$ we have $\sum_{i=0}^{n} \mu_i e_i = \sum_{i=0}^{n} \mu_i e_{\varphi(i)}$. Let $U_W : \Delta_n \to C$ be the unic affine function such that $U(e_i) = x_i$; from $U_W(p_w) = U_W\big(T_W(p_w)\big)$ it follows that

$$\sum_{i=0}^{n} \mu_i x_i = \sum_{i=0}^{n} \mu_i x_{\varphi(i)}. \tag{2.17}$$

Since W is convex we have from (2.16)

$$\sum_{i=0}^{n} \mu_i \left(T(x_i) - x_{\varphi(i)}\right) \in W \tag{2.18}$$

and since T is affine

$$\sum_{i=0}^{n} \mu_i \left(T(x_i) - x_{\varphi(i)}\right) = T\Big(\sum_{i=0}^{n} \mu_i x_i\Big) - \sum_{i=0}^{n} \mu_i x_{\varphi(i)} \tag{2.19}$$

$$= T\Big(\sum_{i=0}^{n} \mu_i x_i\Big) - \sum_{i=0}^{n} \mu_i x_i \tag{2.20}$$

We have shown that for any neighborhood W of the origin there is a point $x \in C$ such that $T(x) - x \in W$. By the compactness of C and the continuity of T we can infer that T has a fixed point.

For each element T of $\mathcal{F}$ let $\mathrm{Fix}(T)$ be the set of fixed points of T. Each of these sets is closed in C and therefore compact, they are also convex since each map T is affine and we have shown that they are not empty.

The proof can now be completed as in [7]. The commutativity of $\mathcal{F}$ implies that for all finite subsets $\{T_1, \ldots, T_n\}$ and all T_0 of $\mathcal{F}$ the inclusion $T_0\left(\bigcap_{i=1}^n \mathrm{Fix}\,(T_i)\right) \subset \left(\bigcap_{i=1}^n \mathrm{Fix}\,(T_i)\right)$ holds. Now a straightforward induction shows that the family $\{\mathrm{Fix}\,(T) : T \in \mathcal{F}\}$ has the finite intersection property and therefore by compactness the set $\bigcap \{\mathrm{Fix}\,(T) : T \in \mathcal{F}\}$ is not empty. □

A simple proof of Theorem 2.3.2 making explicit use of Fan's Theorem can be found on page 43 of [7].

2.4 Invariant means and the Mazur-Orlicz Theorem

Day's theorem on the existence of invariant means on compact topological semigroups is usually proved via the Hahn-Banach theorem, [2], [10]. It is obtained here as a direct consequence of Kakutani's Theorem, and therefore, indirectly, as a consequence of Fan's Theorem. From Day's Theorem we derive, following [3] with a slight adaptation, the Mazur-Orlicz Theorem. There is a very short step from the Mazur-Orlicz theorem to the Hahn-Banach theorem.
We give the theorem of Mazur-Orlicz a somewhat geometrical formulation which is readily seen to be equivalent to the standard formulation.

Let G be an abelian semigroup and let $B(G)$ be the space of all bounded real valued functions on G. An invariant mean on G is a real valued linear function m on $B(G)$ such that

$$m(1) = 1,$$

$$m(f) \geq 0 \quad \text{if} \quad f \geq 0$$

and

$$m(f_g) = m(f)$$

for all $f \in B(G)$ and all $g \in G$, where $f_g(x) = f(gx)$.

Theorem 2.4.1 (Day) *If G is an abelian semigroup then there is an invariant mean on G.*

Proof. With the norm $\|f\| = \sup_{x\in G} \mid f(x) \mid$ the space of bounded functions on G is a Banach space. Let E be the Banach space of bounded linear functionals on $B(G)$ and let C be the subset of the unit ball of E consisting of positive functionals taking the value 1 on the constant function 1 of $B(G)$. If $g \in G$ and $f \in B(G)$ then $f \mapsto f(g)$ defines an element of C. Consequently C is not empty and it is obviously a closed and convex subset of the unit ball of E. For the weak topology C is therefore compact.

Now for $g \in G$, $L \in C$ and $f \in B(G)$ let $T_g(L)(f) = L(f_g)$. Then $\{T_g : g \in G\}$ is a commutative family of continuous affine maps on C. By Kakutani's theorem there is an element m of C such that for each $g \in G$ one has $T_g(m) = m$. This m is an invariant mean on G. □

Theorem 2.4.2 *Let $p : G \to \mathbb{R}$ be a subadditive map defined on an abelian semigroup G (respectively, an abelian group G) and let $C \subseteq G \times \mathbb{R}$ be an additive subset (i.e. if $(x, r), (x', r') \in C$ then $(x+x', r+r') \in C$). Then, there exists an additive function (respectively, a group homomorphism) $f : G \to \mathbb{R}$ such that*

$$(i)\ \forall x \in G\ \ f(x) \leq p(x)$$

and

$$(ii)\ \forall (x, r) \in C\quad r \leq f(x)$$

if and only if

$$\forall (x, r) \in C \quad r \leq p(x).$$

The necessity of the condition is obvious. Let us show that this condition is sufficient.

For all $x \in G$ let $P(x) = \inf \{p(x + y) - r : (y, r) \in C\}$. From the subadditivity of p it follows that,

$$\forall x, x' \in G \quad P(x + x') - P(x') \leq p(x). \tag{2.21}$$

Since C is an additive subset of $G \times \mathbb{R}$ we have, from the definition of P,

$$\forall x \in G \quad \forall (y, r), (y', r') \in C \quad P(x) + r \leq p(x + y + y') - r'. \tag{2.22}$$

Taking the infimum over $(y', r') \in C$ gives

$$\forall x \in G \quad \forall (y, r) \in C \quad r \leq P(x + y) - P(x). \tag{2.23}$$

Let m be an invariant mean on G. For $x \in G$ let $f(x) = m\,(P_x - P)$ where, for all $y \in E$, $P_x(y) = P(x+y)$.

From (2.21) and (2.23) we have

$$\forall x \in G \quad f(x) \leq p(x) \quad \text{and} \quad \forall (y,r) \in C \quad r \leq f(x) \tag{2.24}$$

For all $x, x' \in E$ one has

$$\begin{aligned} f(x) &= m\,(P_x - P) \\ &= m\,((P_x - P)_{x'}) \\ &= m(P_{(x+x')} - P) - m\,(P_{x'} - P) = f(x+x') - f(x'). \end{aligned}$$

We have shown that f is additive and consequently, a group homorphism if G is a group. □

In Theorem 2.4.2 one does not have to assume that C is an additive subset of $G \times \mathbb{R}$ since, for an arbitrary $S \subset G \times \mathbb{R}$, Theorem 2.4.2 holds with S instead of C if and only if it holds with C being the additive subset of $G \times \mathbb{R}$ spanned by S.

Theorem 2.4.3 (Mazur-Orlicz) *Let $p : E \to \mathbb{R}$ be a subadditive and positively homogeneous map defined on real vector space E and let $C \subseteq E \times \mathbb{R}$ a convex cone. Then there is a linear function $f : E \to \mathbb{R}$ such that*

$$(i)\ \forall x \in E\ \ f(x) \leq p(x)$$

and

$$(ii)\ \forall (x,r) \in C \quad r \leq f(x)$$

if and only if

$$\forall (x,r) \in C \quad r \leq p(x).$$

Proof A convex cone in $E \times \mathbb{R}$ is an additive subset. The function $f : E \to \mathbb{R}$ defined in the proof Theorem 2.4.2 is a group homomorphism. Lemma 2.4.4 below shows that f is linear. □

Lemma 2.4.4 *Let $p : E \to \mathbb{R}$ be a subadditive and positively homogeneous map defined on real vector space E. If $f : E \to \mathbb{R}$ is an additive map such that*

$$\forall x \in E \quad f(x) \leq p(x)$$

then f is linear.

Proof. Since f is a group homomorphism, we have, for all $(x, r) \in E \times \mathbb{Q}$, $f(rx) = rf(x)$.

Take $(x, t) \in E \times \mathbb{R}$ and assume that $f(x) \geq 0$; which implies $p(x) \geq 0$. Then

$$\begin{aligned} tf(x) - f(tx) &= \inf\{rf(x) : r > t, r \in \mathbb{Q}\} - f(tx) \\ &= \inf\{rf(x) - f(tx) : r > t, r \in \mathbb{Q}\} \\ &= \inf\{f((r-t)x) : r > t, r \in \mathbb{Q}\} \\ &\leq \inf\{p((r-t)x) : r > t, r \in \mathbb{Q}\} \\ &= \inf\{(r-t)p(x) : r > t, r \in \mathbb{Q}\} = 0. \end{aligned}$$

We have shown that

$$\forall (x, t) \in E \times \mathbb{R} \text{ such that } f(x) \geq 0 \text{ one has } tf(x) \leq f(tx) \tag{2.25}$$

which also shows that if $t \geq 0$ and $f(x) \geq 0$ then $f(tx) \geq 0$. Therefore, in (2.25), for $t > 0$, we can replace x by tx and t by $1/t$ to obtain, $f(tx) \leq tf(x)$. Since $f(0) = 0$ we have

$$\forall (x, t) \in E \times \mathbb{R}_+ \text{ such that } f(x) \geq 0 \text{ one has } tf(x) = f(tx). \tag{2.26}$$

Finally, $f(-x) = -f(x)$ implies $f(tx) = tf(x)$ for all $(x, t) \in E \times \mathbb{R}$. □

Another proof of the Mazur-Orlicz Theorem using Kakutani's Fixed Point Theorem for commuting families of affine maps can be found in both [7] on pages 70 to 73, and also in [12]. Those proofs make explicit use of Tychonov's Theorem on the compactness of an arbitrary product of compact spaces.

Proposition 2.2.2 appears in [12] as Theorem 2.1 under the additional hypothesis that X is a compact convex subset of some topological vector space

3 Ky Fan's lopsided inequality

Lemma 3.0.5 *Let $\varphi : C \times C \to \mathbb{R}$ be a function defined on a convex subset C of some topological vector space and assume that the following conditions hold:*

(1) $\forall x \in C$ *the partial map* $\varphi(x, -)$ *is concave;*

(2) $\forall (x, y) \in C \times C$ *the map* $t \mapsto \varphi\big((1-t)y + tx, y\big)$ *is lower semicontinuous on* $[0, 1]$*;*

(3) $\forall x \in C \quad \varphi(x, x) \leq 0$.

Then, for all $x_0 \in C$ *such that* $\inf_{y \in C} \varphi(y, x_0) \geq 0$ *we also have* $\sup_{y \in C} \varphi(x_0, y) \leq 0$.

Proof. Assume that $\inf_{y \in C} \varphi(y, x_0) \geq 0$. Take an arbitrary element $y \in C$ and let $\eta(t) = (1-t)x_0 + ty$ for $t \in [0, 1]$. From $0 \leq \varphi(\eta(t), x_0)$ and (3) we obtain, for all $0 \leq t < 1$,

$$0 \leq \varphi(\eta(t), x_0) - \frac{1}{1-t}\varphi(\eta(t), \eta(t)). \tag{3.1}$$

Since $\varphi(\eta(t), -)$ is concave we obtain from (3.1)

$$\forall t \in [0, 1[\quad 0 \leq -\frac{t}{1-t}\varphi(\eta(t), y) \tag{3.2}$$

or, equivalently,

$$\forall t \in [0, 1[\quad \varphi(\eta(t), y) \leq 0 \tag{3.3}$$

and finally, since $t \mapsto \varphi(\eta(t), y)$ is lower semicontinuous on $[0, 1]$, $\varphi(x_0, y) \leq 0$. □

Theorem 3.0.6 (Fan's Lopsided Inequality) *Let C be a convex subset of a topological vector space and let $\varphi : C \times C \to \mathbb{R}$ be a function such that:*

(1) $\forall x \in C \quad \varphi(x, -)$ *is upper semicontinuous and concave on* C*;*

(2) $\forall (x, y) \in C^2 \quad t \mapsto \varphi\big((1-t)y + tx, y\big)$ *is lower semicontinuous on* $[0, 1]$,

(3) $\forall (x, y) \in C^2 \quad 0 \leq \varphi(x, y) + \varphi(y, x)$*;*

(4) $\forall x \in C \quad \varphi(x, x) \leq 0$*;*

(5) $\exists y_0 \in C$ *such that* $\{x \in C : \varphi(x, y_0) \leq 0\}$ *is contained in a compact and convex subset* K_0 *of* C.

Then

$$\exists x_0 \in C \ \text{ such that } \ \sup_{y \in C} \varphi(x_0, y) \leq 0.$$

Proof. Let $\mathcal{F}$ be the family of compact and convex subsets of C containing K_0. To each $K \in \mathcal{F}$ associate the function $f_K : K \times K \to \mathbb{R}$ defined by $f_K(x, y) = -\varphi(y, x)$. By Theorem 2.1.1 there exists $x_K \in K$ such that $\inf_{y \in K} \varphi(y, x_K) \geqslant 0$. By Lemma 3.0.5 we also have $\sup_{y \in K} \varphi(x_K, y) \leqslant 0$.

Let $A_K = \{x \in K : \sup_{y \in K} \varphi(x, y) \leqslant 0\}$ and $F_K = \overline{A_K}$; from (4) we have $y_0 \in K_0$ and from (5) $A_K \subset K_0$. F_K is therefore a nonempty compact subset of C. Notice also that if $K \subset K'$ then $A_{K'} \subset A_K$ since, if $x \in K'$ and $\sup_{y \in K'} \varphi(x, y) \leqslant 0$ then $x \in K_0 \subset K$. Consequently, if $K \subset K'$ then $F_{K'} \subset F_K$.

Let us see that the family $\{F_K : K \in \mathcal{F}\}$ has the finite intersection property. Let Δ_m be the standard m-dimensional simplex and, given $K_0, \cdots, K_m$ in $\mathcal{F}$, let

$$K = \left\{ \sum_{i=0}^{m} t_i x_i : (t_0, \cdots, t_m) \in \Delta_m \text{ and } (x_0, \cdots, x_m) \in \prod_{i=0}^{m} K_i \right\}.$$

Since K is compact and convex and, for all $i \in \{0, \cdots, m\}$, $K_i \subset K$ we have $K \in \mathcal{F}$ and $F_K \subset \cap_{i=0}^{m} F_{K_i}$.

We have shown that $\bigcap_{K \in \mathcal{F}} F_K \neq \emptyset$. Let $x^\star$ be an arbitrary point of $\bigcap_{K \in \mathcal{F}} F_K$.

For all $y \in C$ let $K(y) = \{(1-t)y + tx : x \in K_0\}$ and fix an arbitrary $\bar{y}$ in C.

From $K(\bar{y}) \in \mathcal{F}$ and $x^\star \in F_{K(\bar{y})}$ we have, for all neighborhood U of $x^\star$ in C,

$$U \cap \{x \in K(\bar{y}) : \sup_{y \in K(\bar{y})} \varphi(x, y) \leqslant 0\} \neq \emptyset.$$

Hypothesis (3) implies that $\inf_{y \in K(\bar{y})} \varphi(y, x_U) \geqslant 0$ for all $x_U \in U$ such that $\sup_{y \in K(\bar{y})} \varphi(x_U, y) \leqslant 0$. Since U is an arbitrary neigborhood of $x^\star$ we have shown that $x^\star$ belongs to the closure of $\{x \in K(\bar{y}) : \inf_{y \in K(\bar{y})} \varphi(y, x) \geqslant 0\}$.

By hypothesis, for all $y \in K(\bar{y})$, $\varphi(y,-)$ is upper semicontinuous on $K(\bar{y})$ and therefore $\{x \in K(\bar{y}) : \inf_{y\in K(\bar{y})} \varphi(y,x) \geqslant 0\}$ is closed in C.

We have shown that $\inf_{y\in K(\bar{y})} \varphi(y,x^\star) \geqslant 0$; another application of Lemma 3.0.5 yields $\sup_{y\in K(\bar{y})} \varphi(x^\star,y) \leqslant 0$ and in particular $\varphi(x^\star,\bar{y}) \leqslant 0$. Since $\bar{y}$ was an arbitrary element of C this concludes the proof. □

Ky Fan's Inequality, fixed point theorems and variational inequalities are all closely related. Let us give, without proofs, two classical results that can be derived without much difficulty from Theorem 3.0.6; the first is the Minty-Browder Theorem on the surjectivity of monotone operators, the second is the Hilbert space version of the fixed point theorem for nonexpansive maps of Browder-Goehde-Kirk.

Theorem 3.0.7 *Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ a lower semicontinuous function and $A : E \to E^\star$ such that:*

(1) *A is weakly continuous on the finite dimensional subspaces of E;*

(2) $\forall (x,y) \in E \times E$ $\langle Ax - Ay, x - y\rangle \geq 0$*;*

(3) $\exists y_0 \in E$ *such that* $\lim\limits_{\|x\|\to\infty} \dfrac{\langle Ax, x - y_0\rangle + g(x)}{\|x\|} = \infty$

Then, for all $y^\star \in E^\star$ there exists $x_0 \in E$ such that

$$\forall y \in E \quad \langle A(x_0) - y^\star, x_0 - y\rangle + g(x_0) \leq g(y).$$

Proof. Apply Theorem 3.0.6 to $\varphi(x,y) = \langle A(x) - y^\star, x - y\rangle + g(x) - g(y)$. □

Theorem 3.0.8 *Let $f : C \to C$ be a function defined on a closed bounded convex subset C of a Hilbert space H. If f is nonexpansive, that is,*

$\forall (x,y) \in C \times C \quad \|f(x) - f(y)\| \leq \|x - y\|$, *then f has a fixed point.*

Proof. Apply Theorem 3.0.6 to $\varphi(x,y) = \langle x - f(x), x - y\rangle$. □

Acknowledgement. A la mémoire de Ky Fan avec respect et gratitude.

References

[1] J.P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley & Sons, 1984.

[2] S. K. Berberian, *Lectures in Functional Analysis and Operator Theory*, G.T.M. Springer Verlag, 1974.

[3] W. Chojnacki, *Sur un Théorème de Day, un Théorème de Mazur-Orlicz et une Généralisation de Quelques Théorèmes de Silverman* , Colloquium Mathematicum, vol. L, Fasc. 2 (1986), 257–262.

[4] W. Chojnacki, *Erratum à l'article "Sur un Théorème de Day, un Théorème de Mazur-Orlicz et une Généralisation de Quelques Théorèmes de Silverman"* , Colloquium Mathematicum, vol. LXII, Fasc. 1 (1992).

[5] K. Fan *Existence theorems and extreme solutions for inequalities concerning convex functions or linear transformations*, Math. Z., Vol. 68 (1957), 205-217.

[6] J. Franklin, *Mathematical Methods of Economics*, Amer. Math. Monthly, Vol. 90, nb. 4 (1983), 229–244.

[7] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer 2003.

[8] A. Granas and F. C. Liu, *Some Minimax Theorems without Convexity*, in Nonlinear and Convex Analysis, proceedings in honour of Ky Fan, Edited by Bor-Luh Lin and Stephen Simons, Lectures notes in pure and applied mathematics, Vol. 107, Marcel Dekker, 1987.

[9] G. Maltese, *A Simple Proof of the Fundamental Theorem of Finite Markov's Chains*, Amer. Math. Monthly, vol. 93, nb. 8 (1986), 629–630.

[10] F. P. Greenleaf, *Invariant Means on Topological Groups*, Van Nostrand Mathematical studies 16, 1969.

[11] A. S. Solodovnikov, *Systems of Linear Inequalities*, Little Mathematical Library, Mir Publishers, Moscow, 1979.

[12] W. Takahashi, *Fan's existence theorem for inequalities concerning convex functions and its applications*, Minimax Theory and its Applications, B. Ricceri and S. Simons (eds) (1998), 241-260.