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DUALITY: DETECTABILITY VERSUS STABILIZABILITY IN THE STOCHASTIC CONTEXT*

Vasile Drăgan†

Abstract

The aim of this paper is to extend to the stochastic framework the well known duality relation between the detectability property and the stabilizability property of a finite dimensional, linear time invariant, deterministic system. We consider continuous time linear stochastic systems having the state space representation described by a system of Itô differential equations with periodic coefficients possibly subject to some stochastic perturbations modeled by a standard homogeneous Markov process with a finite number of states. It will be seen that in the case when the given system is affected by jump Markov perturbations a state space representation of the dual triple may be rigorously defined if and only if the transition probability matrix of the Markov process is double stochastic (in the sense that the sums of all elements from its each row and each column are equal 1).

Keywords: continuous time linear stochastic systems, stochastic detectability, stochastic stabilizability, duality, periodic coefficients.

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†Vasile.Dragan@imar.ro "Simion Stoilow" Institute of Mathematics of the Romanian Academy, P.O.Box 1-764, RO-014700, The Academy of the Romanian Scientists, Str. Ilfov, 3, 050044, Bucharest, Romania

1 Introduction

The complete controlability and the complete observability of a finite dimensional time invariant linear deterministic system are powerful tools which allow the designing of the state space representation of Kalman filters at the beginning of '60 years [16, 17, 18, 21]. Roughly speaking, the complete controlability (complete observability) of a finite dimensional linear time invariant system, is equivalent to the possibility of designing a control law in a linear state feedback form (in a linear output based feedback form, respectively) such that all the eigenvalues of the closed-loop system, known as **closed-loop poles**, be arbitrarily located in the complex plane with the constraint that the complex poles to be in a complex conjugate pairs. If all closed-loop poles are located in the left half plane, the resulting system is exponentially stable. See for example [3, 9, 25]. Some years later, the requirement of arbitrary assignments of all poles of the closed-loop system was relaxed to the requirement that only a part of the closed-loop poles to be arbitrarily placed in the complex plane with the condition that the closed-loop poles which cannot be modified by a control law in a linear state feedback form (a linear output based feedback form, respectively) to be already located in the left half plane. A finite dimensional, linear time invariant, deterministic system for which the poles that cannot be modified by a control law in a linear state feedback form (a linear output based feedback form, respectively) is named **stabilizable system** (**detectable system**, respectively). For the reader convenience, we refer to [19] and references therein, for the case of continuous time linear systems and [1, 2, 15] and references therein, for the case of the discrete time linear systems.

Starting with [26] the detectability and stabilizability concepts were extended to the stochastic framework. Here we refer to [8, 22, 24] for the case of linear stochastic systems subject to state multiplicative white noise perturbations and [6, 7] for the case of stochastic systems affected by jump Markov perturbations. An unified approach of the stochastic stabilizability and stochastic detectability for linear stochastic systems simultaneously affected by state multiplicative white noise perturbations and jump Markov perturbations was done in [11, 12, 13, 23].

It is well known that in the deterministic framework there exists a duality between the stabilizability concept and the detectability concept, see for example [1].

Our aim is to study if there exists a duality relationship between the properties of stochastic stabilizability and stochastic detectability in the case of the linear stochastic systems having the state space representation described by a system of Itô differential equations with periodic coefficients possible affected by a standard homogeneous Markov process with a finite number of states. It will be seen that in the case when the coefficients of the system under consideration are subject to jump Markov perturbations, the state space representation of the dual of such a system can be rigourously described if and only if the transition probability matrix of the Markov process is double stochastic, that is the sums of all elements on each row and each column is 1.

The rest of the paper is organized as follows: Section 2 contains the model description of the stochastic systems under consideration together with the duality problem setting. Section 3 contains a list of useful Lyapunov type criteria for exponential stability in mean square involved in the derivation of the main results which are stated and proved in Section 4.

Applications of the concept of duality studied in this work will be presented in a future paper.

2 The problem

2.1 Model description

We consider the stochastic linear system having the state space representation described by:

$$\begin{aligned} dx(t) &= (A_0(t,\eta(t))x(t) + B_0(t,\eta(t))u(t))dt \\ &\quad + \sum_{k=1}^{r}(A_k(t,\eta(t))x(t) + B_k(t,\eta(t))u(t))dw_k(t) \end{aligned} \tag{1a}$$

$$dy(t) = C_0(t,\eta(t))x(t)dt + \sum_{k=1}^{r} C_k(t,\eta(t))x(t)dw_k(t), \tag{1b}$$

$t \in \mathbb{R}_+ = [0,\infty)$, where $x(t) \in \mathbb{R}^n$ are the state parameters of the system, $u(t) \in \mathbb{R}^m$ are the input vectors which may include control parameters and/or exogenous disturbances, while $y(t) \in \mathbb{R}^p$ are vectors which are describing outputs providing information about the behaviour of the system.

In (1), $\{w(t)\}_{t\geq 0}$ $(w(t) = (w_1(t)\ w_2(t)\ ...w_r(t))^\top$ is a r-dimensional standard Wiener process and $\{\eta(t)\}_{t\geq 0}$ is a finite states standard homogeneous Markov process defined on a given probability space $(\Omega, \mathfrak{F}, \mathcal{P})$. For rigourous definitions and usual properties of the standard Wiener processes and of the standard homogeneous Markov processes with a finite number of states we refer to [4, 5, 10, 14, 20]. Here we recall only that in the case of a standard homogeneous Markov process with N states, the transition semigroup $P(t)$ is of the form $P(t) = e^{Qt}$, $t > 0$, where $Q \in \mathbb{R}^{N\times N}$ is a matrix (often named generator matrix) whose entries q_{ij} have the properties:

$$q_{ij} \geq 0 \; if \; i \neq j \tag{2a}$$

$$\sum_{k=1}^{N} q_{ik} = 0 \tag{2b}$$

for all $i, j \in \mathcal{N} = \{1, 2, ..., N\}$.

In the sequel, we shall write $A_k(t,i), B_k(t,i), C_k(t,i)$, $0 \leq k \leq r$, whenever the Markov process is in the mode $i \in \mathcal{N}$.

The developments in this work are done under the following assumption:

H1) a) $\{w(t)\}_{t\geq 0}$, $\{\eta(t)\}_{t\geq 0}$ are independent stochastic processes;

b) $(A_k(\cdot,i), B_k(\cdot,i)) : \mathbb{R} \to \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ and $C_k(\cdot,i) : \mathbb{R} \to \mathbb{R}^{p\times n}$, $0 \leq k \leq r$, $i \in \mathcal{N}$ are continuous matrix valued functions which are periodic with period θ. □

For each $t \geq 0$, $\mathcal{F}_t \subset \mathfrak{F}$ stands for the smallest σ-algebra with respect to which the random vectors $w(s)$, $0 \leq s \leq t$ are measurable augmented by all subsets $\mathfrak{A} \in \mathfrak{F}$ with $\mathcal{P}(\mathfrak{A}) = 0$. Also we denote $\mathcal{G}_t \subset \mathfrak{F}$ the smallest σ-algebra with respect to which the random variables $\eta(s)$, $0 \leq s \leq t$ are measurable.

Note: The assumption **H1**) a) is equivalent to the fact that the σ-algebras $\mathcal{F}_t$ and $\mathcal{G}_t$ are independent for any $t \geq 0$. □

Throughout in this work, $\mathcal{H}_t \subset \mathfrak{F}$ denotes the smallest σ-algebra containing the σ-algebras $\mathcal{F}_t$ and $\mathcal{G}_t$. If $J \subset \mathbb{R}_+$ is an interval, $\mathcal{L}^2_{\mathcal{H}}\{J, \mathbb{R}^d\}$ denotes the linear space of d-dimensional stochastic processes $\mathbf{z} = \{z(t)\}_{t\in J}$ which are nonanticipative with respect to the family of σ-algebras $\mathcal{H} = \{\mathcal{H}_t\}_{t\in J}$ and satisfy the condition $\mathbb{E}[\int_J |z(t)|^2 dt] < +\infty$.

Here and in the sequel, $\mathbb{E}[\cdot]$ stands for the mathematical expectation.

Based on Theorem 1.1 Chapter 5 in [14] we obtain:

Proposition 2.1. For any $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ and all $\mathbf{u} = \{u(t)\}_{t\in\mathbb{R}_+}$ the stochastic linear differential equation (1a) has a unique solution $x(\cdot; t_0, x_0, \mathbf{u}) : [t_0, \infty) \to \mathbb{R}^n$ which is a stochastic process having the properties:

a) $x(\cdot; t_0, x_0, \mathbf{u})$ is continuous a.s. at any $t \geq t_0$;

b) $x(\cdot; t_0, x_0, \mathbf{u}) \in \mathcal{L}^2_{\mathcal{H}}\{[t_0, T], \mathbb{R}^n\}$ for all $T > t_0$;

c) $x(t_0; t_0, x_0, \mathbf{u}) = x_0$. □

Among the particular cases of the system (1) is that when $\mathcal{N} = \{1\}$, that is, the Markov process has only one state. In this case, the system (1) takes the form:

$$dx(t) = (A_0(t)x(t) + B_0(t)u(t))dt + \sum_{k=1}^{r}(A_k(t)x(t) + B_k(t)u(t))dw_k(t) \tag{3a}$$

$$dy(t) = C_0(t)x(t)dt + \sum_{k=1}^{r} C_k(t)x(t)dw_k(t), \tag{3b}$$

$t \geq 0$. To ease the presentation of the results and to make the analogy with the deterministic case more obvious, we say sometimes that the system (1) and (3), respectively, are defined by the triples $(\mathbf{C}(\cdot), \mathbf{A}(\cdot), \mathbf{B}(\cdot))$ and $(\mathcal{C}(\cdot), \mathcal{A}(\cdot), \mathcal{B}(\cdot))$, respectively, where $\mathbf{C}(\cdot) = \begin{pmatrix} \mathbb{C}_0(\cdot) & \mathbb{C}_1(\cdot) & ... & \mathbb{C}_r(\cdot) \end{pmatrix}$, $\mathbf{A}(\cdot) = \begin{pmatrix} \mathbb{A}_0(\cdot) & \mathbb{A}_1(\cdot) & ... & \mathbb{A}_r(\cdot) \end{pmatrix}$, $\mathbf{B}(\cdot) = \begin{pmatrix} \mathbb{B}_0(\cdot) & \mathbb{B}_1(\cdot) & ... & \mathbb{B}_r(\cdot) \end{pmatrix}$, with the convention that $\mathbb{M}_k(\cdot) = (M_k(\cdot, 1), M_k(\cdot, 2), ..., M_k(\cdot, N))$ with $\mathbb{M}_k(\cdot) \in \in \{\mathbb{C}_k(\cdot), \mathbb{A}_k(\cdot), \mathbb{B}_k(\cdot)\}$, $M_k(\cdot, i) \in \{C_k(\cdot, i), A_k(\cdot, i), B_k(\cdot, i)\}$, $0 \leq k \leq r$, $C_k(\cdot, i), A_k(\cdot, i), B_k(\cdot, i)$ are the matrix valued functions which are describing the coefficients of (1) and $\mathcal{C}(\cdot) = (C_0(\cdot), C_1(\cdot), ..., C_r(\cdot))$, $\mathcal{A}(\cdot) = (A_0(\cdot), A_1(\cdot), ..., A_r(\cdot))$, $\mathcal{B}(\cdot) = (B_0(\cdot), B_1(\cdot), ..., B_r(\cdot))$, $A_k(\cdot), B_k(\cdot), C_k(\cdot)$ are the matrix valued functions which are describing the coefficients of (3).

The system (1) will be named **the state space representation** of the triple $(\mathbf{C}(\cdot), \mathbf{A}(\cdot), \mathbf{B}(\cdot))$, while the system (3) is **the state space representation** of the triple $(\mathcal{C}(\cdot), \mathcal{A}(\cdot), \mathcal{B}(\cdot))$.

2.2 Basic definitions

Definition 2.1. a) We say that the linear stochastic system (1a) is **stochastic stabilizable** or equivalently, the pair $(\mathbf{A}(\cdot), \mathbf{B}(\cdot))$ is stabilizable if there exist continuous and θ-periodic matrix valued functions $F(\cdot, i) : \mathbb{R}_+ \to \mathbb{R}^{m \times n}$, $i \in \mathcal{N}$ with the property that the closed-loop stochastic linear differential equation:

$$\begin{aligned} dx(t) &= (A_0(t, \eta(t)) + B_0(t, \eta(t))F(t, \eta(t)))x(t)dt \\ &+ \sum_{k=1}^{r}(A_k(t, \eta(t)) + B_k(t, \eta(t))F(t, \eta(t)))x(t)dw_k(t), \end{aligned} \tag{4}$$

$t \geq 0$ is exponentially stable in mean square (ESMS).

b) We say that the stochastic linear system (3a) is **stochastic stabilizable** or equivalently $(\mathcal{A}(\cdot), \mathcal{B}(\cdot))$ is stabilizable if there exists a continuous and θ-periodic matrix valued function $F(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{m\times n}$ with the property that the closed-loop stochastic linear differential equation

$$dx(t) = (A_0(t) + B_0(t)F(t))x(t)dt + \sum_{k=1}^{r}(A_k(t) + B_k(t)F(t))x(t)dw_k(t), \quad (5)$$

$t \geq 0$, is ESMS. □

Definition 2.2. a) We say that the linear stochastic system (1) with $B_k(t,i) \equiv 0$, $0 \leq k \leq r$, is **stochastic detectable** or equivalently the pair $(\mathbf{C}(\cdot), \mathbf{A}(\cdot))$ is detectable if there exist θ-periodic continuous matrix valued functions $K(\cdot, i) : \mathbb{R}_+ \to \mathbb{R}^{n\times p}$, $i \in \mathcal{N}$, with the property that the following closed-loop linear stochastic differential equation:

$$\begin{aligned} dx(t) &= (A_0(t,\eta(t)) + K(t,\eta(t))C_0(t,\eta(t)))x(t)dt \\ &+ \sum_{k=1}^{r}(A_k(t,\eta(t)) + K(t,\eta(t))C_k(t,\eta(t)))x(t)dw_k(t), \end{aligned} \quad (6)$$

$t \geq 0$, is ESMS.

b) We say that the linear stochastic system (3) with $B_k(t) \equiv 0$, $0 \leq k \leq r$, is **stochastic detectable** or equivalently the pair $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$ is detectable if there exist continuous and θ-periodic matrix valued functions $K(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times p}$ such that the closed-loop linear differential equation

$$dx(t) = (A_0(t) + K(t)C_0(t))x(t)dt + \sum_{k=1}^{r}(A_k(t) + K(t)C_k(t))x(t)dw_k(t), \quad (7)$$

$t \geq 0$, is ESMS.

2.3 Duality: stochastic stabilizability versus stochastic detectability

The problem which we want to study in this work requires that for a linear stochastic system of the form (1) or equivalently for the associated triple $(\mathbf{C}(\cdot), \mathbf{A}(\cdot), \mathbf{B}(\cdot))$ to construct a triple $(\mathbf{C}^\sharp(\cdot), \mathbf{A}^\sharp(\cdot), \mathbf{B}^\sharp(\cdot))$ with the property that $(\mathbf{C}(\cdot), \mathbf{A}(\cdot))$ is detectable if and only if the pair $(\mathbf{A}^\sharp(\cdot), \mathbf{B}^\sharp(\cdot))$ is stabilizable and the pair $(\mathbf{A}(\cdot), \mathbf{B}(\cdot))$ is stabilizable if and only if the pair $(\mathbf{C}^\sharp(\cdot), \mathbf{A}^\sharp(\cdot))$ is detectable.

The triple $(\mathbf{C}^\sharp(\cdot), \mathbf{A}^\sharp(\cdot), \mathbf{B}^\sharp(\cdot))$ with these properties will be named the dual of the triple $(\mathbf{C}(\cdot), \mathbf{A}(\cdot), \mathbf{B}(\cdot))$ and the state space representation of the dual triple will be named the dual of the system (1).

Similarly, in the case of the linear stochastic system (3) or equivalently for the associated triple $(\mathcal{C}(\cdot), \mathcal{A}(\cdot), \mathcal{B}(\cdot))$ to construct a triple $(\mathcal{C}^\sharp(\cdot), \mathcal{A}^\sharp(\cdot), \mathcal{B}^\sharp(\cdot))$ with the property that $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$ is detectable if and only if $(\mathcal{A}^\sharp(\cdot), \mathcal{B}^\sharp(\cdot))$ is stabilizable and $(\mathcal{A}(\cdot), \mathcal{B}(\cdot))$ is stabilizable if and only if $(\mathcal{C}^\sharp(\cdot), \mathcal{A}^\sharp(\cdot))$ is detectable. The state space representation of the dual triple $(\mathcal{C}^\sharp(\cdot), \mathcal{A}^\sharp(\cdot), \mathcal{B}^\sharp(\cdot))$ will be named the dual of the linear stochastic system (3).

In Section 4 we shall see that in the case of a triple $(\mathcal{C}(\cdot), \mathcal{A}(\cdot), \mathcal{B}(\cdot))$ associated to a linear stochastic system (3), the dual triple $(\mathcal{C}^\sharp(\cdot), \mathcal{A}^\sharp(\cdot), \mathcal{B}^\sharp(\cdot))$ may be defined in a natural way together with its state space representation, while in the case of a linear stochastic system as (1) the association of a dual triple and of a dual system is possible only under an additional assumption regarding the Markov process.

In Section 3 we will briefly recall several known aspects regarding the Lyapunov type linear differential equations arising in stochastic control and several criteria for the exponential stability in mean square of a system of stochastic linear differential equations which will be involved in the derivation of the main results from this paper.

3 Some intermediate facts

Throughout this work $\mathcal{S}_n \subset \mathbb{R}^{n\times n}$ denotes the linear space of symmetric matrices of size $n \times n$ and $\mathcal{S}_n^N := \mathcal{S}_n \times \mathcal{S}_n \times ... \times \mathcal{S}_n$.

The elements of $\mathcal{S}_n^N$ are finite sequences of symmetric matrices, that is $\mathbf{X} = (X(1), X(2), ..., X(N))$.

On $\mathcal{S}_n^N$ we introduce the inner product

$$< \mathbf{X}, \mathbf{Y} >= \sum_{i=1}^{N} Tr[X(i)Y(i)] \tag{8}$$

for all $\mathbf{X} = (X(1), X(2), ..., X(N))$, $\mathbf{Y} = (Y(1), Y(2), ..., Y(N))$ from $\mathcal{S}_n^N$. In (8), $Tr[\cdot]$ stands for the trace of a matrix. On the linear space $\mathcal{S}_n^N$ we introduce the ordering relation "$\succeq$" induced by the convex cone

$$\mathcal{S}_n^{N+} = \{\mathbf{X} = (X(1), X(2), ..., X(N)) \in \mathcal{S}_n^N | X(i) \geq 0, 1 \leq i < N\}.$$

Here $X(i) \geq 0$ means that $X(i)$ is a positive semidefinite matrix. In the particular case $N = 1$, $\mathcal{S}_n^1$ is just $\mathcal{S}_n$ and the cone $\mathcal{S}_n^{1+}$ is the convex cone of the symmetric semi-positive matrices $\mathcal{S}_n^+$.

If $M_k(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $M_k(\cdot, i) : \mathbb{R} \to \mathbb{R}^{n\times n}$, $i \in \mathcal{N}$, $0 \leq k \leq r$ are continuous matrix valued functions, periodic of period θ, we define the operator valued functions $\mathcal{L}(\cdot) : \mathbb{R} \to \mathcal{B}[\mathcal{S}_n]$ and $\mathbf{L}(\cdot) : \mathbb{R} \to \mathcal{B}[\mathcal{S}_n^N]$ as:

$$\mathcal{L}(t)[X] = M_0(t)X + XM_0^\top(t) + \sum_{k=1}^{r} M_k(t)XM_k^\top(t) \tag{9}$$

for all $X \in \mathcal{S}_n$, $t \in \mathbb{R}$ and

$$\mathbf{L}(t)[\mathbf{X}] = (L_1(t)[\mathbf{X}], L_2(t)[\mathbf{X}], ..., L_N(t)[\mathbf{X}]),$$

where

$$\begin{aligned} L_i(t)[\mathbf{X}] &= M_0(t,i)X(i) + X(i)M_0^\top(t,i) + \sum_{k=1}^{r} M_k(t,i)X(i)M_k^\top(t,i) \\ &\quad + \sum_{j=1}^{N} q_{ji}X(j), \end{aligned} \tag{10}$$

for all $\mathbf{X} = (X(1), X(2), ..., X(N)) \in \mathcal{S}_n^N$, $t \in \mathbb{R}$, $i \in \mathcal{N}$.

Note: If $\mathcal{H}_1, \mathcal{H}_2$ are vector spaces, then $\mathcal{B}[\mathcal{H}_1, \mathcal{H}_2]$ stands for the vector space of the linear operators $T : \mathcal{H}_1 \to \mathcal{H}_2$. When $\mathcal{H}_1 = \mathcal{H}_2$ we shall write $\mathcal{B}[\mathcal{H}_1]$ instead of $\mathcal{B}[\mathcal{H}_1, \mathcal{H}_1]$. □

In (10) q_{ij} are the real numbers which are satisfying a condition of type (2). By direct calculation one obtains that the adjoint operator $\mathcal{L}^*(t)[\cdot]$ of the operator $\mathcal{L}(t)[\cdot]$ is

$$\mathcal{L}^*(t)[X] = M_0^\top(t)X + XM_0(t) + \sum_{k=1}^{r} M_k^\top(t)XM_k(t) \tag{11}$$

for all $(t, X) \in \mathbb{R} \times \mathcal{S}_n$. The adjoint operator $\mathbf{L}^*(t)[\cdot]$ of the operator $\mathbf{L}(t)[\cdot]$ defined in (10) is

$$\mathbf{L}^*(t)[\mathbf{X}] = (L_1^*(t)[\mathbf{X}],\ L_2^*(t)[\mathbf{X}],\ ...\ , L_N^*(t)[\mathbf{X}]),$$

with

$$\begin{aligned} L_i^*(t)[\mathbf{X}] &= M_0^\top(t,i)X(i) + X(i)M_0(t,i) + \sum_{k=1}^{r} M_k^\top(t,i)X(i)M_k(t,i) \\ &\quad + \sum_{j=1}^{N} q_{ij}X(j) \end{aligned} \tag{12}$$

for all $t \in \mathbb{R}$, $\mathbf{X} = (X(1), X(2), ..., X(N)) \in \mathcal{S}_n^N$.

Based on the operator valued functions from (9)-(10) we define the linear differential equations on the Hilbert spaces $\mathcal{S}_n$ and $\mathcal{S}_n^N$, respectively, as:

$$\dot{X}(t) = \mathcal{L}(t)[X(t)], \quad t \in \mathbb{R} \tag{13}$$

and

$$\dot{\mathbf{X}}(t) = \mathbf{L}(t)[\mathbf{X}(t)], \quad t \in \mathbb{R}. \tag{14}$$

Let $\mathcal{T}(t,t_0) : \mathcal{S}_n \to \mathcal{S}_n$ and $\mathbf{T}(t,t_0) : \mathcal{S}_n^N \to \mathcal{S}_n^N$ be the linear evolution operators on $\mathcal{S}_n$ and $\mathcal{S}_n^N$, respectively, defined by the linear differential equations (13), (14) respectively.

By definition $\mathcal{T}(t,t_0)[X_0] = X(t;t_0,X_0), \quad \forall t,t_0 \in \mathbb{R}, \quad X_0 \in \mathcal{S}_n$ and $\mathbf{T}(t,t_0)[\mathbf{X}_0] = \mathbf{X}(t;t_0,\mathbf{X}_0), \ \forall t,t_0 \in \mathbb{R}, \quad \mathbf{X}_0 \in \mathcal{S}_n^N$.

Here, $X(\cdot;t_0,X_0)$ is the solution of the differential equation (13) with initial condition $X(t_0;t_0,X_0) = X_0$ and $\mathbf{X}(\cdot;t_0,\mathbf{X}_0)$ is the solution of the differential equation (14) with the initial value $\mathbf{X}(t_0;t_0,\mathbf{X}_0) = \mathbf{X}_0$.

Applying Theorem 2.6.1 from [12] in the case of the differential equations (13), (14), respectively, we obtain:

Corollary 3.1. The linear differential equation (13) defines a positive evolution on the ordered Hilbert space $(\mathcal{S}_n, \mathcal{S}_n^+)$ while the linear differential equation (14) defines a positive evolution on the ordered Hilbert space $(\mathcal{S}_n^N, \mathcal{S}_n^{N+})$. This means that $\mathcal{T}(t,t_0)[X_0] \succcurlyeq 0, \ \forall t \geq t_0$ whenever $X_0 \in \mathcal{S}_n^+$ and $\mathbf{T}(t,t_0)[\mathbf{X}_0] \succcurlyeq 0, \ \forall t \geq t_0$ whenever $\mathbf{X}_0 \in \mathcal{S}_n^{N+}$. □

Definition 3.1. a) We say that the linear differential equation (13) defines an exponentially stable evolution on the Hilbert space $\mathcal{S}_n$ if

$$\|\mathcal{T}(t,t_0)\| \leq \beta_1 e^{-\alpha_1(t-t_0)}, \quad \forall t \geq t_0, \ \ t,t_0 \in \mathbb{R}. \tag{15}$$

b) We say that the linear differential equation (14) defines an exponentially stable evolution on the Hilbert space $\mathcal{S}_n^N$ if

$$\|\mathbf{T}(t,t_0)\| \leq \beta_2 e^{-\alpha_2(t-t_0)}, \quad \forall t \geq t_0, \ \ t,t_0 \in \mathbb{R}, \tag{16}$$

where $\alpha_k > 0$, $\beta_k \geq 1$, $k = 1,2$. □

In (15), $\|\cdot\|$ denotes any operator norm on the space $\mathcal{B}[\mathcal{S}_n]$, while, in (16), $\|\cdot\|$ can be any operator norm on the space $\mathcal{B}[\mathcal{S}_n^N]$.

The linear operators $\mathcal{L}(t)$ and $\mathbf{L}(t)$, introduced in (9), (10), respectively, are named Lyapunov type operators from stochastic control and the linear differential equations (13) and (14) are known as Lyapunov type linear differential equations of stochastic control.

To make clearer the relationship of these equations with the stochastic framework, let us consider the following stochastic linear differential equations (SLDEs):

$$dx(t) = M_0(t)x(t)dt + \sum_{k=1}^{r} M_k(t)x(t)dw_k(t), \quad t \geq 0 \tag{17}$$

and

$$d\mathbf{x}(t) = M_0(t,\eta(t))\mathbf{x}(t)dt + \sum_{k=1}^{r} M_k(t,\eta(t))\mathbf{x}(t)dw_k(t), \quad t \geq 0. \tag{18}$$

In (17), $M_k(\cdot)$ are the matrix valued functions which are involved in (9) and in (18), $M_k(\cdot, i)$ are the matrix valued functions occurring in (10).
The stochastic processes $\{w(t)\}_{t\geq 0}$, $\{\eta(t)\}_{t\geq 0}$ are those which are appearing in (1) and satisfy the assumption **H1**).

Let $x(t;t_0,x_0)$ be a solution of the SLDE (17) satisfying $x(t_0;t_0,x_0) = x_0$ for an arbitrary initial pair $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$.

We set $X(t) := \mathbb{E}[x(t;t_0,x_0)x^\top(t;t_0,x_0)]$. By direct calculations based on Itô's formula one obtains that $t \to X(t)$ is the solution of the following initial value problem (IVP):

$$\dot{X}(t) = \mathcal{L}(t)[X(t)], \quad t \geq t_0, \tag{19a}$$

$$X(t_0) = x_0 x_0^\top. \tag{19b}$$

Let $\mathbf{x}(t;t_0,x_0)$ be the solution of the SLDE (18) satisfying $\mathbf{x}(t_0;t_0,x_0) = x_0$ for an arbitrary initial pair $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. We set $X(t,i) = \mathbb{E}[\mathbf{x}(t;t_0,x_0)\mathbf{x}^\top(t;t_0,x_0)\chi_{\{\eta(t)=i\}}]$, $1 \leq i \leq N$, $\chi_{\{\eta(t)=i\}}$ being the indicator function of the event $\{\eta(t) = i\}$, that is $\chi_{\{\eta(t)=i\}}(\omega) = \begin{cases} 1, & if \quad \eta(t,\omega) = i \\ 0, & if \quad \eta(t,\omega) \neq i. \end{cases}$

By direct calculations, involving Itô's formula one obtains that $t \to \mathbf{X}(t) = (X(t,1), X(t,2), ..., X(t,N))$ is the solution of the following IVP on $\mathcal{S}_n^N$:

$$\dot{\mathbf{X}}(t) = \mathbf{L}(t)[\mathbf{X}(t)], \quad t \geq t_0, \tag{20a}$$

$$\mathbf{X}(t_0) = (\pi_1(t_0)x_0x_0^\top, \pi_2(t_0)x_0x_0^\top, ..., \pi_N(t_0)x_0x_0^\top), \tag{20b}$$

where $\pi_i(t_0) = \mathcal{P}\{\eta(t_0) = i\}$, $1 \leq i \leq N$.

Definition 3.2. a) We say that the SLDE (17) is exponentially stable in mean square (ESMS) if its solutions $x(\cdot;t_0,x_0)$ are satisfying the condition

$$\mathbb{E}[|x(t;t_0,x_0)|^2] \leq \beta e^{-\alpha(t-t_0)}|x_0|^2 \tag{21}$$

for all $t \geq t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, where $\alpha > 0$, $\beta \geq 1$ do not depend upon t, t_0, x_0.

b) We say that the SLDE (18) is:

(i) exponentially stable in mean square with conditioning (ESMS-C) if its solutions are satisfying

$$\mathbb{E}[|\mathbf{x}(t; t_0, x_0)|^2 | \eta(t_0) = j] \leq \beta_1 e^{-\alpha_1(t-t_0)} |x_0|^2, \tag{22}$$

$\forall t \geq t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and for any initial probability distribution $\pi(0)$ of the Markov process;

(ii) exponentially stable in mean square (ESMS) if its solutions $\mathbf{x}(\cdot; t_0, x_0)$ are satisfying

$$\mathbb{E}[|\mathbf{x}(t; t_0, x_0)|^2] \leq \beta_2 e^{-\alpha_2(t-t_0)} |x_0|^2 \tag{23}$$

for all $t \geq t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and for any initial probability distribution $\pi(0)$ of the Markov process.

In (22) and (23) $\alpha_k > 0, \beta_k \geq 1$, $k = 1, 2$ are not depending upon $t, t_0, \pi(0)$.□

Note: a) The initial probability distributions of the Markov process $\{\eta(t)\}_{t \geq 0}$ are defined by $\pi(0) = (\pi_1(0), \pi_2(0), ..., \pi_N(0))$ with $\pi_k(0) := \mathcal{P}\{\eta(0) = k\}$, $1 \leq k \leq N$.

b) In general, the property of ESMS-C of a SLDE of type (18) implies the property of ESMS of it. However, when the coefficients of (18) are periodic functions the property of ESMS-C is equivalent to the property of ESMS, see Theorem 3.2.5 from [12] applied in the case of the SLDE (18). □

From Theorem 2.2.2 in [12] we obtain:

Corollary 3.2. The following equivalences hold:

(i) The Lyapunov type linear differential equation (13) generates an exponentially stable evolution on the Hilbert space $\mathcal{S}_n$ if and only if the accompanying SLDE (17) is ESMS.

(ii) The Lyapunov type linear differential equation (14) generates an exponentially stable evolution on the Hilbert space $\mathcal{S}_n^N$ if and only if the accompanying SLDE (18) is ESMS-C. □

Applying Corollary 2.3.8 and Corollary 2.4.4 from [12] in the case of the operator valued function $\mathcal{L}(\cdot)$ defined in (9) together with the equivalence (i) from Corollary 3.2 (above), we obtain:

Proposition 3.3: If the matrix valued functions $M_k(\cdot)$ are continuous and periodic of period θ, the following are equivalent:

(i) the SLDE (17) is ESMS;

(ii) the Lyapunov type linear differential equation (13) generates an exponentially stable evolution on the Hilbert space $\mathcal{S}_n$;

(iii) for any continuous, θ-periodic, uniform positive matrix valued function $H(\cdot) : \mathbb{R} \to \mathcal{S}_n^+$, the non-homogeneous backward Lyapunov type matrix differential equation

$$\dot{Y}(t) + M_0^\top(t)Y(t) + Y(t)M_0(t) + \sum_{k=1}^{r} M_k^\top(t)Y(t)M_k(t) + H(t) = 0 \quad (24)$$

has a solution $Y(\cdot) : \mathbb{R} \to \mathcal{S}_n^+$ with the property that $Y(\cdot)$ is a θ-periodic and uniform positive matrix valued function;

(iv) there exists a continuous, θ-periodic uniform positive matrix valued function $\tilde{H}(\cdot) : \mathbb{R} \to \mathcal{S}_n^+$ with the property that the corresponding non-homogeneous backward Lyapunov type matrix differential equation of the form (24) has a θ-periodic and uniform positive solution $\tilde{Y}(\cdot) : \mathbb{R} \to \mathcal{S}_n^+$;

(v) for any continuous, θ-periodic uniform positive matrix valued function $H(\cdot) : \mathbb{R} \to \mathcal{S}_n^+$ the non-homogenous forward Lyapunov type matrix differential equation

$$\dot{X}(t) = M_0(t)X(t) + X(t)M_0^\top(t) + \sum_{k=1}^{r} M_k(t)X(t)M_k^\top(t) + H(t) \quad (25)$$

has a θ-periodic and uniform positive solution $X(\cdot) : \mathbb{R} \to \mathcal{S}_n^+$;

(vi) there exists a continuous, θ-periodic, uniform positive matrix valued function $\tilde{H}(\cdot) : \mathbb{R} \to \mathcal{S}_n^+$ with the property that the corresponding non-homogeneous forward Lyapunov type matrix differential equation of the form (25) has a θ-periodic and uniform positive solution $\tilde{X}(\cdot) : \mathbb{R} \to \mathcal{S}_n^+$. □

Further, Corollary 2.3.8 and Corollary 2.4.4 from [12] applied in the case of the operator valued function $\mathbf{L}(\cdot)$ defined in (10) together with the equivalence (ii) from Corollary 3.2 give:

Proposition 3.4: If the matrix valued functions $M_k(\cdot, i) : \mathbb{R} \to \mathbb{R}^{n \times n}$, $0 \leq k \leq r$, $i \in \mathcal{N}$, are continuous and θ-periodic, the following are equivalent:

(i) the SLDE (18) is ESMS;

(ii) the Lyapunov type linear differential equation (14) generates an exponentially stable evolution on the Hilbert space $\mathcal{S}_n^N$;

(iii) for any continuous θ-periodic uniform positive matrix valued functions $H(\cdot,i):\mathbb{R}\to\mathcal{S}_n$, $i\in\mathcal{N}$, the non-homogeneous backward Lyapunov type matrix differential equation on the Hilbert space $\mathcal{S}_n^N$:

$$\dot{Y}(t,i)+M_0^\top(t,i)Y(t,i)+Y(t,i)M_0(t,i)+\sum_{k=1}^{r}M_k^\top(t,i)Y(t,i)M_k(t,i)$$
$$+\sum_{j=1}^{N}q_{ij}Y(t,j)+H(t,i)=0,\quad i\in\mathcal{N},\quad t\in\mathbb{R},\ (26)$$

has a θ-periodic and uniform positive solution

$$\mathbf{Y}(\cdot)=(Y(\cdot,1),Y(\cdot,2),...,Y(\cdot,N)):\mathbb{R}\to\mathcal{S}_n^N;$$

(iv) there exist continuous, θ-periodic, uniform positive matrix valued functions $\tilde{H}(\cdot,i):\mathbb{R}\to\mathcal{S}_n^N$ with the property that the corresponding non-homogeneous backward Lyapunov type matrix differential equation of the form (26) has a θ-periodic and uniform positive solution

$$\tilde{\mathbf{Y}}(\cdot)=(\tilde{Y}(\cdot,1),\tilde{Y}(\cdot,2),...,\tilde{Y}(\cdot,N)):\mathbb{R}\to\mathcal{S}_n^N;$$

(v) for any continuous, θ-periodic, uniform positive matrix valued functions $H(\cdot,i):\mathbb{R}\to\mathcal{S}_n$, the non-homogeneous forward Lyapunov type matrix differential equation on $\mathcal{S}_n$:

$$\begin{aligned}\dot{X}(t,i)\ &=M_0(t,i)X(t,i)+X(t,i)M_0^\top(t,i)+\textstyle\sum_{k=1}^{r}M_k(t,i)X(t,i)M_k^\top(t,i)\\&\quad+\textstyle\sum_{j=1}^{N}q_{ji}X(t,j)+H(t,i),\quad i\in\mathcal{N}\end{aligned}\qquad(27)$$

has a θ-periodic and uniform positive solution $\mathbf{X}(\cdot)=(X(\cdot,1),...,X(\cdot,N)):\mathbb{R}\to\mathcal{S}_n^N$;

(vi) there exist continuous, θ-periodic, uniform positive matrix valued functions $\tilde{H}(\cdot,i):\mathbb{R}\to\mathcal{S}_n$, $i\in\mathcal{N}$, with the property that the corresponding non-homogenous forward matrix differential equation of the form (27) has a θ-periodic and uniform positive solution $\tilde{\mathbf{X}}(\cdot)=(\tilde{X}(\cdot,1),\tilde{X}(\cdot,2),...,\tilde{X}(\cdot,N)):\mathbb{R}\to\mathcal{S}_n^N$. □

4 The main results

4.1 The case of the linear stochastic systems free of jump Markov perturbations

A. Let us assume that the system (3a) is stochastic stabilizable, or equivalently the pair $(\mathcal{A}(\cdot), \mathcal{B}(\cdot))$ is stabilizable. According to Definition 2.1 (b) there exists a continuous and θ-period matrix valued function $F(\cdot) : \mathbb{R} \to \mathbb{R}^{m\times n}$ with the property that the corresponding stochastic linear differential equation (SLDE) (5) is ESMS. Employing the implications $(i) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i)$ from Proposition 3.3 with the updates $M_k(t) \leftarrow A_k(t)+B_k(t)F(t)$, $0 \le k \le r$, $t \in \mathbb{R}$ we may infer that $(\mathcal{A}(\cdot), \mathcal{B}(\cdot))$ is stabilizable if and only if there exists a continuous and θ-periodic matrix valued function $F(\cdot)$ with the property that the corresponding non-homogeneous forward Lyapunov type matrix differential equation:

$$\begin{aligned}\dot{X}(t) \quad &= (A_0(t) + B_0(t)F(t))X(t) + X(t)(A_0(t) + B_0(t)F(t))^\top \\ &+ \sum_{k=1}^{r} (A_k(t) + B_k(t)F(t))X(t)(A_k(t) + B_k(t)F(t))^\top + I_n, \end{aligned} \tag{28}$$

$t \in \mathbb{R}$, has a solution $\tilde{X}(\cdot) : \mathbb{R} \to \mathcal{S}_n$ which is a θ-periodic and uniform positive function satisfying

$$\tilde{X}(t) \ge \delta^2 I_n, \quad \forall t \in \mathbb{R}. \tag{29}$$

By direct calculations one obtains that the function $\tilde{Y}(\cdot) : \mathbb{R} \to \mathcal{S}_n$ defined by $\tilde{Y}(t) \triangleq \tilde{X}(-t)$ is a θ-periodic solution of the following non-homogeneous backward Lyapunov type matrix differential equation:

$$\begin{aligned}&\dot{\tilde{Y}}(t) + (\tilde{A}_0(t) + \tilde{K}(t)\tilde{C}_k(t))^\top \tilde{Y}(t)(\tilde{A}(t) + \tilde{K}(t)\tilde{C}_k(t)) \\ &+ \sum_{k=1}^{r} (\tilde{A}_k(t) + \tilde{K}(t)\tilde{C}_k(t))^\top \tilde{Y}(t)(\tilde{A}_k(t) + \tilde{K}(t)\tilde{C}_k(t)) + I_n = 0 \qquad (30a)\\ &\tilde{Y}(t) \ge \delta^2 I_n, \qquad (30b)\end{aligned}$$

$t \in \mathbb{R}$, where

$$\tilde{A}_k(t) := A_k^\top(-t),\ \tilde{C}_k(t) := B_k^\top(-t),\ 0 \le k \le r,\ \tilde{K}(t) := F^\top(-t),\ t \in \mathbb{R}. \tag{31}$$

The equivalence $(iv) \Leftrightarrow (i)$ from Proposition 3.3 together with the updates $M_k(t) \leftarrow \tilde{A}_k(t) + \tilde{K}(t)\tilde{C}_k(t)$, $0 \le k \le r$, $t \in \mathbb{R}$ guarantees that the feasibility of (30) is equivalent to the exponential stability in mean square of the SLDE:

$$dx(t) = (\tilde{A}_0(t)+\tilde{K}(t)\tilde{C}_0(t))x(t)dt+\sum_{k=1}^{r}(\tilde{A}_k(t)+\tilde{K}(t)\tilde{C}_k(t))x(t)dw_k(t), t \in \mathbb{R}_+. \tag{32}$$

Further, the exponential stability in mean square of the SLDE (32) is equivalent to the stochastic detectability of the stochastic linear system:

$$dx(t) = \tilde{A}_0(t)x(t)dt + \sum_{k=1}^{r} \tilde{A}_k(t)x(t)dw_k(t) \tag{33a}$$

$$dy(t) = \tilde{C}_0(t)x(t)dt + \sum_{k=1}^{r} \tilde{C}_k(t)x(t)dw_k(t). \tag{33b}$$

According to Definition 2.2 a) applied in the case of the system (33) we may conclude that the system (33) is stochastic detectable, if and only if the pair $(\mathcal{C}^\sharp(\cdot), \mathcal{A}^\sharp(\cdot))$ is detectable, where

$$\mathcal{C}^\sharp(\cdot) = (\tilde{C}_0(\cdot), \tilde{C}_1(\cdot), ..., \tilde{C}_r(\cdot)), \tag{34a}$$

$$\mathcal{A}^\sharp(\cdot) = (\tilde{A}_0(\cdot), \tilde{A}_1(\cdot), ..., \tilde{A}_r(\cdot)), \tag{34b}$$

$\tilde{C}_k(\cdot), \tilde{A}_k(\cdot)$ being introduced in (31).

Summarizing the previous developments we can state:

Proposition 4.1. Under the assumption **H1**) a) with $N = 1$, if $\mathcal{A}(\cdot), \mathcal{B}(\cdot)$ are associated to the system (3), then the following are equivalent:
(i) $(\mathcal{A}(\cdot), \mathcal{B}(\cdot))$ is stabilizable;
(ii) $(\mathcal{C}^\sharp(\cdot), \mathcal{A}^\sharp(\cdot))$ is detectable. □

Note: In the sequel, the pair $(\mathcal{C}^\sharp(\cdot), \mathcal{A}^\sharp(\cdot))$ introduced via (34) will be named the **dual pair** of the pair $(\mathcal{A}(\cdot), \mathcal{B}(\cdot))$ and (33) will be named the **dual system** of the system (3a). □

B. Let us consider now the system

$$dx(t) = A_0(t)x(t)dt + \sum_{k=1}^{r} A_k(t)x(t)dw_k(t) \tag{35a}$$

$$dy(t) = C_0(t)x(t)dt + \sum_{k=1}^{r} C_k(t)x(t)dw_k(t), \tag{35b}$$

obtained from (3) taking $B_k(t) \equiv 0$, $0 \leq k \leq r$, $t \in \mathbb{R}$. Employing the implications $(i) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$ from Proposition 3.3 in the case of the closed-loop SLDE (7) we obtain that the linear stochastic system (35) is stochastic detectable, or equivalently the pair $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$ is detectable if and only if the non-homogeneous backward Lyapunov type matrix differential equation:

$$\dot{Y}(t) + (A_0(t) + K(t)C_0(t))^\top Y(t) + Y(t)(A_0(t) + K(t)C_0(t))$$
$$+\sum_{k=1}^{r}(A_k(t) + K(t)C_k(t))^\top Y(t)(A_k(t) + K(t)C_k(t)) + I_n = 0 \qquad (36)$$

has a uniform positive solution $\hat{Y}(\cdot) : \mathbb{R} \to \mathcal{S}_n$ which is a θ-periodic function. Setting $\hat{X}(t) \triangleq \hat{Y}(-t)$, $t \in \mathbb{R}$, we obtain by direct calculation that $\hat{X}(\cdot)$ solves:

$$\dot{X}(t) = (\tilde{A}_0(t) + \tilde{B}_0(t)F(t))X(t) + X(t)(\tilde{A}_0(t) + \tilde{B}_0(t)F(t))^\top$$
$$+\sum_{k=1}^{r}(\tilde{A}_k(t) + \tilde{B}_k(t)F(t))X(t)(\tilde{A}_k(t) + \tilde{B}_k(t)F(t))^\top + I_n \qquad (37a)$$
$$\nu I_n \geq X(t) \geq \gamma^2 I_n, \qquad (37b)$$

$t \in \mathbb{R}$, where

$$\tilde{B}_k(t) := C^\top(-t), \quad F(t) := K^\top(-t), \quad 0 \leq k \leq r, \;\; t \in \mathbb{R} \qquad (38)$$

and $\tilde{A}_k(t), 0 \leq k \leq r$, $t \in \mathbb{R}$ are those defined in (31).

The equivalence $(vi) \Leftrightarrow (i)$ from Proposition 3.3 with the updates $M_k(t) \leftarrow \tilde{A}_k(t) + \tilde{B}_k(t)F(t)$, $t \in \mathbb{R}$, $0 \leq k \leq r$, guarantees that the feasibility of (38) is equivalent to the exponential stability in mean square of the following SLDE:

$$dx(t) = (\tilde{A}_0(t)+\tilde{B}_0(t)F(t))x(t)dt+\sum_{k=1}^{r}(\tilde{A}_k(t)+\tilde{B}_k(t)F(t))x(t)dw_k(t). \qquad (39)$$

The property of ESMS of (39) is equivalent to the stochastic stabilizability of the system:

$$dx(t) = (\tilde{A}_0(t)x(t)+\tilde{B}_0(t)u(t))dt+\sum_{k=1}^{r}(\tilde{A}_k(t)x(t)+\tilde{B}_k(t)u(t))dw_k(t), t \in \mathbb{R}_+. \qquad (40)$$

But the stochastic stabilizability of (40) is equivalent to the stabilizability of the pair $(\mathcal{A}^\sharp(\cdot), \mathcal{B}^\sharp(\cdot))$, where $\mathcal{A}^\sharp(\cdot)$ is defined in (34b) and

$$\mathcal{B}^\sharp(\cdot) \triangleq (\tilde{B}_0(\cdot), \tilde{B}_1(\cdot), ..., \tilde{B}_r(\cdot)), \qquad (41)$$

$\tilde{B}_k(\cdot)$ being defined by (38).

Thus, we have proved:

Proposition 4.2. Under the assumptions **H1**) a) for $N = 1$, if the pair $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$ is defined based on the coefficients of the linear stochastic system (35), the following are equivalent:

(i) the pair $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$ is detectable;

(ii) the pair $(\mathcal{A}^\sharp(\cdot), \mathcal{B}^\sharp(\cdot))$ is stabilizable. □

Note: The triple $(\mathcal{C}^\sharp(\cdot), \mathcal{A}^\sharp(\cdot), \mathcal{B}^\sharp(\cdot))$ introduced by (34), (41) is the dual of the triple $(\mathcal{C}(\cdot), \mathcal{A}(\cdot), \mathcal{B}(\cdot))$ whose state space representation is described by (3). The state-space representation of the dual triple is

$$\begin{aligned} d\tilde{x}(t) &= (\tilde{A}_0(t)\tilde{x}(t) + \tilde{B}_0(t)\tilde{u}(t))dt + \sum_{k=1}^{r}(\tilde{A}_k(t)\tilde{x}(t) + \tilde{B}_k(t)\tilde{u}(t))dw_k(t) \\ d\tilde{y}(t) &= \tilde{C}_0(t)\tilde{x}(t)dt + \sum_{k=1}^{r}\tilde{C}_k(t)\tilde{x}(t)dw_k(t), \end{aligned} \tag{42}$$

where $\tilde{A}_k(t), \tilde{B}_k(t), \tilde{C}_k(t)$, $0 \leq k \leq r$ were defined in (31) and (38). □

4.2 The case of linear stochastic systems subject to jump Markov perturbations

In this subsection we study the duality of the detectability versus the stabilizability in the case of the linear stochastic systems of type (1).

C. First, let us assume that the system described by (1a) is stochastic stabilizable, or equivalently, that the pair $(\mathbf{A}(\cdot), \mathbf{B}(\cdot))$ is stabilizable. According to Definition 2.1(a), in this case there exist continuous and θ-periodic matrix valued functions $F(\cdot, i) : \mathbb{R} \to \mathbb{R}^{m\times n}$, $i \in \mathcal{N}$, with the property that the closed-loop SLDE (4) is ESMS.

The chain of implications $(i) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i)$ from Proposition 3.4 with the updates $M_k(t,i) \leftarrow A_k(t,i) + B_k(t,i)F(t,i)$, $i \in \mathcal{N}$, $0 \leq k \leq r$, allows us to infer that $(\mathbf{A}(\cdot), \mathbf{B}(\cdot))$ is stabilizable if and only if there exist continuous and θ-periodic matrix valued functions $F(\cdot, i) : \mathbb{R} \to \mathbb{R}^{m\times n}$, $i \in \mathcal{N}$, with the property that the corresponding non-homogeneous forward

Lyapunov type matrix differential equation on $\mathcal{S}_n^N$:

$$\begin{aligned}\dot{X}(t,i) \quad &= (A_0(t,i)+B_0(t,i)F(t,i))X(t,i)+X(t,i)(A_0(t,i)\\ &+B_0(t,i)F(t,i))^\top+\sum_{k=1}^{r}(A_k(t,i)+B_k(t,i)F(t,i))X(t,i)(A_k(t,i)\\ &+B_k(t,i)F(t,i))^\top+\sum_{j=1}^{N}q_{ji}X(t,j)+I_n, \quad t\in\mathbb{R}, \quad i\in\mathcal{N} \end{aligned} \tag{43}$$

has a solution $\tilde{\mathbf{X}}(\cdot)=(\tilde{X}(\cdot,1),\tilde{X}(\cdot,2),...,\tilde{X}(\cdot,N)):\mathbb{R}\to\mathcal{S}_n^N$ which is a θ-periodic function and satisfies

$$\tilde{X}(t,i)\geq\gamma^2 I_n, \quad \forall(t,i)\in\mathbb{R}\times\mathcal{N}, \tag{44}$$

γ being a constant not depending upon t and i.

We set $\tilde{Y}(t,i):=\tilde{X}(-t,i)$ for all $(t,i)\in\mathbb{R}\times\mathcal{N}$.

By direct calculation, involving (43) one obtains that $t\to\tilde{\mathbf{Y}}(t)=(\tilde{Y}(t,1),\tilde{Y}(t,2),...,\tilde{Y}(t,N)):\mathbb{R}\to\mathcal{S}_n^N$ is a θ-periodic function which satisfies:

$$\begin{aligned}&\frac{d}{dt}\tilde{Y}(t,i)+(\check{A}_0(t,i)+K(t,i)\check{C}_0(t,i))^\top\tilde{Y}(t,i)+\tilde{Y}(t,i)(\check{A}_0(t,i)\\ &+K(t,i)\check{C}_0(t,i))+\sum_{k=1}^{r}(\check{A}_k(t,i)+K(t,i)\check{C}_k(t,i))^\top\tilde{Y}(t,i)(\check{A}_k(t,i)\\ &+K(t,i)\check{C}_k(t,i))+\sum_{j=1}^{N}\check{q}(i,j)\tilde{Y}(t,j)+I_n=0\end{aligned} \tag{45a}$$

$$\nu^2 I_n\geq\tilde{Y}(t,i)\geq\gamma^2 I_n \tag{45b}$$

for all $(t,i)\in\mathbb{R}\times\mathcal{N}$, where

$$\begin{aligned}\check{A}_k(t,i):=A_k^\top(-t,i),\ \check{C}_k(t,i):=B_k^\top(-t,i)\\ 0\leq k\leq r,\ \ K(t,i):=F^\top(-t,i),\ \ \forall(t,i)\in\mathbb{R}\times\mathcal{N}.\end{aligned} \tag{46}$$

In (45a),

$$\check{q}_{ij}:=q_{ji}, \quad \forall i,j\in\mathcal{N}. \tag{47}$$

Let us assume that besides the conditions (2), the component q_{ij} of the generator matrix Q of the Markov process are satisfying the additional condition:

$$\sum_{j=1}^{N}q_{ji}=0, \quad \forall i\in\mathcal{N}. \tag{48}$$

Under these conditions we may invoke the equivalence $(iv) \Leftrightarrow (i)$ from Proposition 3.4 with the updates $M_k(t,i) \leftarrow \check{A}_k(t,i) + K(t,i)\check{C}_k(t,i)$, $i \in \mathcal{N}$, $0 \le k \le r$, $t \in \mathbb{R}$ to deduce that under the condition (48), the feasibility of (45) is equivalent to the exponential stability in mean square of the following SLDE:

$$\begin{aligned} dx(t) &= (\check{A}_0(t)\check{\eta}(t) + K(t,\check{\eta}(t))\check{C}_0(t,\check{\eta}(t)))x(t)dt \\ &+ \sum_{k=1}^{r}(\check{A}(t,\check{\eta}(t)) + K(t,\check{\eta}(t))\check{C}_k(t,\check{\eta}(t)))x(t)dw_k(t), \end{aligned} \tag{49}$$

$t \ge 0$ where $\{\check{\eta}(t)\}_{t\ge 0}$ is a standard homogeneous Markov process defined on the probability space $(\Omega, \mathfrak{F}, \mathcal{P})$ taking values in the finite set $\mathcal{N} = \{1, 2, ..., N\}$ and have the transition semigroup $\{\check{P}(t)\}_{t>0}$, where

$$\check{P}(t) = e^{Q^\top t}. \tag{50}$$

According to Definition 2.2 (a) the exponential stability in mean square of the SLDE (49) is equivalent to the stochastic detactability of the stochastic linear system

$$dx(t) = \check{A}_0(t,\check{\eta}(t))x(t)dt + \sum_{k=1}^{r} \check{A}_k(t,\check{\eta}(t))x(t)dw_k(t) \tag{51a}$$

$$dy(t) = \check{C}_0(t,\check{\eta}(t))x(t)dt + \sum_{k=1}^{r} \check{C}_k(t,\check{\eta}(t))x(t)dw_k(t), \tag{51b}$$

or equivalently with the detectability of the pair $(\mathbf{C}^\sharp(\cdot), \mathbf{A}^\sharp(\cdot))$, where

$$\mathbf{C}^\sharp(\cdot) := (\check{\mathbb{C}}_0(\cdot), \check{\mathbb{C}}_1(\cdot), ..., \check{\mathbb{C}}_r(\cdot) \tag{52a}$$

$$\check{\mathbb{C}}_k(\cdot) = (\check{C}_k(\cdot,1), \check{C}_k(\cdot,2), ..., \check{C}_k(\cdot,N)) \tag{52b}$$

$$\mathbf{A}^\sharp(\cdot) := (\check{\mathbb{A}}_0(\cdot), \check{\mathbb{A}}_1(\cdot), ..., \check{\mathbb{A}}_r(\cdot)) \tag{53a}$$

$$\check{\mathbb{A}}_k(\cdot) = (\check{A}_k(\cdot,1), \check{A}_k(\cdot,2), ..., \check{A}_k(\cdot,N)), \tag{53b}$$

where $\check{C}_k(\cdot,i), \check{A}_k(\cdot,i)$, $0 \le k \le r$, $i \in \mathcal{N}$ are those defined in (46).
So, we proved:

Proposition 4.3. Assume: a) the assumption **H1**) holds;

b) the components q_{ij} of the generator matrix Q of the Markov process are satisfying the condition (48).

Under these conditions, the following are equivalent:

(i) the pair $(\mathbf{A}(\cdot), \mathbf{B}(\cdot))$ having the state space representation described by (1a) is stabilizable;

(ii) the pair $(\mathbf{C}^{\sharp}(\cdot), \mathbf{A}^{\sharp}(\cdot))$ introduced by (52)-(53) is detectable. $\square$

Note: In the sequel, the linear stochastic system (51) will be named the dual of (1a), while $(\mathbf{C}^{\sharp}(\cdot), \mathbf{A}^{\sharp}(\cdot))$ will be called the dual pair of $(\mathbf{A}(\cdot), \mathbf{B}(\cdot))$. It is worth mentioning that the dual system (51) of (1a) can be rigourously defined of the components q_{ij} of the generator matrix Q if the Markov process which affects the coefficients of (1a) are satisfying the additional condition (48). $\square$

D. Let us consider the linear stochastic system:

$$dx(t) = A_0(t, \eta(t))x(t)dt + \sum_{k=1}^{r} A_k(t, \eta(t))x(t)dw_k(t) \tag{54a}$$

$$dy(t) = C_0(t, \eta(t))x(t)dt + \sum_{k=1}^{r} C_k(t, \eta(t))x(t)dw_k(t), \tag{54b}$$

$t \geq 0$, obtained from (1) when $B_k(t, i) \equiv 0$, $0 \leq k \leq r$.

Employing the chain of implications $(i) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$ from Proposition 3.4 in the case of the closed-loop SLDE (6) we may infer that the stochastic linear system (54) is stochastic detectable or equivalently the pair $(\mathbf{C}(\cdot), \mathbf{A}(\cdot))$ is detectable, if and only if there exist continuous and θ-periodic matrix valued functions $K(\cdot, i)$ with the property that the non-homogeneous backward Lyapunov type matrix differential equation on $\mathcal{S}_n^N$:

$$\begin{aligned}&\tfrac{d}{dt}Y(t,i) + (A_0(t,i) + K(t,i)C_0(t,i))^{\top}Y(t,i) + Y(t,i)(A_0(t,i)\\ &+K(t,i)C_0(t,i)) + \sum_{k=1}^{r}(A_k(t,i) + K(t,i)C_k(t,i))^{\top}Y(t,i)(A_k(t,i)\\ &+K(t,i)C_k(t,i)) + \sum_{j=1}^{N} q_{ij}Y(t,j) + I_n = 0\end{aligned} \tag{55}$$

has an uniform positive solution $\check{\mathbf{Y}}(\cdot) = (\check{Y}(\cdot,1), \check{Y}(\cdot,2), ..., \check{Y}(\cdot,N)) : \mathbb{R} \to \mathcal{S}_n^N$ which is a θ-periodic function.

Setting $\check{X}(t,i) := \check{Y}(-t,i)$, $(t,i) \in \mathbb{R} \times \mathcal{N}$, by direct calculation involving

(55) we obtain that $\check{\mathbf{X}}(\cdot) = (\check{X}(\cdot,1), \check{X}(\cdot,2), ..., \check{X}(\cdot,N))$ satisfies

$$\frac{d}{dt}\check{X}(t,i) = (\check{A}_0(t,i) + \check{B}_0(t,i)F(t,i))\check{X}(t,i) + \check{X}(t,i)(\check{A}_0(t,i) + \check{B}_0(t,i)F(t,i))^\top + \sum_{k=1}^{r}(\check{A}_k(t,i) + \check{B}_k(t,i)F(t,i))\check{X}(t,i)(\check{A}_k(t,i) + \check{B}_k(t,i)F(t,i))^\top + \sum_{j=1}^{N}\check{q}_{ji}\check{X}(t,j) + I_n \tag{56a}$$

$$\nu^2 I_n \geq \check{X}(t,i) \geq \gamma^2 I_n, \tag{56b}$$

$\forall (t,i) \in \mathbb{R} \times \mathcal{N}$, where

$$\check{B}_k(t,i) := C_k^\top(-t,i), \;\; 0 \leq k \leq r, \;\; F(t,i) := K^\top(-t,i), \tag{57}$$

$\forall (t,i) \in \mathbb{R} \times \mathcal{N}$, $\check{A}_k(t,i), 0 \leq k \leq r$ being introduced in (46).

In (56a), $\check{q}_{ji} := q_{ij}, \quad \forall i,j \in \mathcal{N}$. Let us assume that the components q_{ij} of the generator matrix Q of the Markov process satisfy the additional condition (48). Under these conditions, we may invoke the equivalence $(vi) \Leftrightarrow (i)$ from Proposition 3.4 with the updates $M_k(t,i) \leftarrow \check{A}_k(t,i) + \check{B}_k(t,i)F(t,i)$, $0 \leq k \leq r$, $(t,i) \in \mathbb{R} \times \mathcal{N}$, to deduce that under condition (48), the fesabiltiy of (56) is equivalent to the exponential stability in mean square of the following SLDE:

$$dx(t) = (\check{A}_0(t,\check{\eta}(t)) + \check{B}_0(t,\check{\eta}(t))F(t,\check{\eta}(t)))x(t)dt + \sum_{k=1}^{r}(\check{A}_k(t,\check{\eta}(t)) + \check{B}_k(t,\check{\eta}(t))F(t,\check{\eta}(t)))x(t)dw_k(t), \tag{58}$$

$t \geq 0$, where $\{\check{\eta}(t)\}_{t\geq 0}$ is a standard homogeneous Markov process taking values in the finite set $\mathcal{N} = \{1,2,...,N\}$ and having the transition semigroups $\{\check{P}(t)\}_{t>0}$ with $\check{P}(t)$ described by (50).

According to Definition 2.1(a), the exponential stability in mean square of the SLDE (58) is equivalent to the stochastic stabilizability of the linear stochastic system:

$$dx(t) = (\check{A}_0(t,\check{\eta}(t))x(t) + \check{B}_0(t,\check{\eta}(t))u(t))dt + \sum_{k=1}^{r}(\check{A}_k(t,\check{\eta}(t))x(t) + \check{B}_k(t,\check{\eta}(t))u(t))dw_k(t), \tag{59}$$

$t \geq 0$, or equivalent to stabilizability of the pair $(\mathbf{A}^\sharp(\cdot), \mathbf{B}^\sharp(\cdot))$, where $\mathbf{A}^\sharp(\cdot)$ was described by (53) and

$$\mathbf{B}^\sharp(\cdot) = (\check{\mathbb{B}}_0(\cdot), \check{\mathbb{B}}_1(\cdot), ..., \check{\mathbb{B}}_r(\cdot)) \tag{60a}$$

$$\check{\mathbb{B}}_k(\cdot) = (\check{B}_k(\cdot,1), \check{B}_k(\cdot,2), ..., \check{B}_k(\cdot,N)), \tag{60b}$$

$\check{B}_k(\cdot, i)$ being defined in (57).

The previous developments allow us to state:

Proposition 4.4. Assume that the assumptions from Proposition 4.3. hold. Under these conditions the following are equivalent:

(i) the pair $(\mathbf{C}(\cdot), \mathbf{A}(\cdot))$ having the state space representation described by (54) is detectable;

(ii) the pair $(\mathbf{A}^\sharp(\cdot), \mathbf{B}^\sharp(\cdot))$ described by (53) and (60) is stabilizable. □

Note: a) The linear stochastic system (59) will be named the **dual system** of (54), while the pair $(\mathbf{A}^\sharp(\cdot), \mathbf{B}^\sharp(\cdot))$ will be named the dual pair of $(\mathbf{C}(\cdot), \mathbf{A}(\cdot))$. It is worth mentioning that the dual system (59) of the system (54) can be rigourously defined if the components q_{ij} of the generator matrix Q satisfy the additional condition (48).

b) Under the condition (48) the triple $(\mathbf{C}^\sharp(\cdot), \mathbf{A}^\sharp(\cdot), \mathbf{B}^\sharp(\cdot))$ introduced via (52), (53) and (60) is the dual of the triple $(\mathbf{C}(\cdot), \mathbf{A}(\cdot), \mathbf{B}(\cdot))$ whose state space representation is described by the linear stochastic system (1).
The state space representation of the dual triple is given by

$$d\check{x}(t) = (\check{A}_0(t, \check{\eta}(t))\check{x}(t) + \check{B}_0(t, \check{\eta}(t))\check{u}(t))dt + \sum_{k=1}^{r}(\check{A}_k(t, \check{\eta}(t))\check{x}(t) + \check{B}_k(t, \check{\eta}(t))\check{u}(t))dw_k(t) \tag{61a}$$

$$d\check{y}(t) = \check{C}_0(t, \check{\eta}(t))\check{x}(t)dt + \sum_{k=1}^{r}\check{C}_k(t, \check{\eta}(t))\check{x}(t)dw_k(t), \tag{61b}$$

$\check{A}_k(t, i), \check{B}_k(t, i), \check{C}_k(t, i), 0 \leq k \leq r$ are defined in (46) and (57).
In (61), $\{\check{\eta}(t)\}_{t\geq 0}$ is a standard homogeneous Markov process defined on the probability space $(\Omega, \mathfrak{F}, \mathcal{P})$ taking values in the finite set $\mathcal{N}$ and having the transition semigroup defined in (50).

c) It remains as an open problem to introduce a concept of duality that covers the cases when condition (48) is violated.

References

[1] B.D.O. Anderson and J.B. Moore, Detectability and stabilizability of time-varying discrete-time linear systems, *SIAM J. Control Optim.* 19 (1) (1981), 20-32.

[2] S. Bittanti and P. Colaneri, *Periodic systems, filtering and control*, Springer-Verlag, London, 2009.

[3] C.T. Chen, A note on pole assigment, *IEEE Trans. Automat. Control* 13 (5) (1968), 597-598.

[4] G. Ciucu, C. Tudor, *Teoria Probabilităţilor şi Aplicaţii*, Editura Ştiinţifică şi Enciclopedică, Bucureşti, 1983.

[5] K.L. Chung, *Markov chains with stationary transition probabilities*, Springer, 1967.

[6] O.L.V. Costa, M.D. Fragoso and R.P. Marques, *Discrete-time Markov jump linear systems*, Springer, New York, 2005.

[7] O.L.V. Costa, M.D. Fragoso and M.G. Todorov, *Continuous-time Markov jump linear systems*, Springer, New-York, 2013.

[8] T. Damm, On detectability of stochastic systems, *Automatica* 43 (2007), 928-933.

[9] E.J. Davison, On pole assignment in multivariable linear systems, *IEEE Trans. Automat. Control* 13 (6) (1968), 747-748.

[10] J.L. Doob, *Stochastic Processes*, Wiley, New York, 1967.

[11] V. Dragan, T. Morozan and A.M. Stoica, *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems*, Springer, New York, 2010.

[12] V. Dragan, T. Morozan and A.M. Stoica, *Mathematical Methods in Robust Control of Linear Stochastic Systems*, second edition, Springer, New York, 2013.

[13] V. Dragan, E.F. Costa, I.L. Popa and S. Aberkane, Exact detectability of discrete-time and continuous-time linear stochastic systems. A unified approach, *IEEE Trans. Automat. Control* 67 (11) (2022), 5730-5745.

[14] A. Friedman, *Stochastic Differential Equations and Applications*, vol. I Academic Press, New York, 1975.

[15] A. Halanay and V. Ionescu, *Time Varying Discrete Linear Systems*, Birkhäuser, Berlin, 1994.

[16] R.E. Kalman, Contributions to the theory of optimal control, *Bol. Soc. Aht. Mexicana* 5 (1960), 102-119.

[17] R.E. Kalman, A new approach to linear filtering and prediction problems, *ASME-Trans.-Part D, J. Basic Engineering* 82 (1960), 34-45.

[18] R.E. Kalman and R.S. Bucy, New results in linear filtering and prediction theory, *ASME Trans.–Part D, J. Basic Engineering* 83 (1961), 95-108.

[19] H. Kwakernaak and R. Sivan, *Linear optimal control systems*, Wiley - Intersicience, New York, 1972.

[20] R. Oxendal, *Stochastic Differential Equations*, Springer, Berlin, 1998.

[21] V.M. Popov, *Hyperstability of control systems*, Editura Academiei - Springer Verlag, Berlin, 1973.

[22] W. Zhang, H. Zhang and B. Chen, Generalized Lyapunov Equation Approach to State-Dependent Stochastic Stabilization/Detectability Criterion, *IEEE Trans. Automat. Control* 53 (7) (2008), 1630-1642.

[23] Y. Ni, W. Zhang and H. Fang, On the observability and detectability of linear stochastic systems with Markov jumps and multiplicative noise, *J. Syst. Sci. Complex.* 23 (2010), 102-115.

[24] W. Zhang, H. Zhang and B. Chen, Detectability, stabilizability, Lyapunov type theorems of linear discrete time-varying stochastic systems, with multiplicative noise, *Int. J. Control* 90 (2017), 2490-2507.

[25] W.M. Wonham, On pole assignment in multi-input controllable linear systems, *IEEE Trans. Automat. Control* 12 (1967), 660-665.

[26] W.M. Wonham, On a matrix Riccati equation of stochastic control, *SIAM J. Control Optim.* 6 (4) (1968), 681-697.