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ON THE HEAT CONDUCTION EQUATION AND THE HEAT DISSIPATION FUNCTION IN ANISOTROPIC REACTING FLUID MIXTURES WITH MAGNETIC RELAXATION[*]

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Abstract

In some previous papers, within the framework of the thermodynamics of irreversible processes with internal variables, a linear theory for magnetic relaxation phenomena in anisotropic mixtures, consisting of n reacting fluid components, was developed. In particular, assuming that the macroscopic magnetization $\mathbf{m}$ can be split in two irreversible parts $\mathbf{m} = \mathbf{m}^{(0)} + \mathbf{m}^{(1)}$ a generalized Snoek equation was derived. In this paper we derive for these reacting anisotropic mixtures the heat conduction equation. We show that the heat dissipation function is due to the chemical reactions, the magnetic relaxation, the electric conduction, the viscous, magnetic, temperature fields and the diffusion and the concentrations of the n fluid components. Also, the Snoek and De Groot special cases are studied. The obtained results find applications in nuclear resonance, in biology, in medicine and other fields, where different species of molecules have different magnetic susceptibilities and relaxation times and contribute to the total magnetization.

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1 Introduction

In [1]-[7] a linear theory for magnetic relaxation phenomena in magnetizable continuous media was developed, based on the thermodynamics of irreversible processes with internal variables [8]-[13]. In particular, in [1], in linear approximation Kluitenberg, assuming that the total specific magnetization is given by the sum of one reversible part and one irreversible part, derived for magnetizable isotropic media the *classical Snoek equation* describing magnetic relaxation phenomena [14]. Subsequently, in [2], assuming that the total specific magnetization $\mathbf{m}$ is composed of two irreversible parts, i.e.,

$$\mathbf{m} = \mathbf{m}^{(0)} + \mathbf{m}^{(1)}, \tag{1}$$

Kluitenberg obtained a more general magnetic relaxation equation. In [4], in the assumption that an arbitrary number n of microscopic phenomena give rise to the total specific magnetization $\mathbf{m}$, that can be split in $n+1$ irreversible parts, i.e.

$$\mathbf{m} = \mathbf{m}^{(0)} + \sum_{k=1}^{n} \mathbf{m}^{(k)}, \tag{2}$$

in the isotropic case a generalized Snoek equation was obtained by Kluitenberg and one of the authors, having the form of a linear relation among the magnetic field $\mathbf{B}$, the first n time derivatives of this field, the total magnetization $\mathbf{M} = \varrho\mathbf{m}$, with ϱ the mass density, and the first $n+1$ time derivatives of $\mathbf{M}$, being n the number of phenomena giving rise to the magnetization. In [5] and [6]) reviews about the results obtained in [4] were done.

In [7] the behaviour of anisotropic reacting fluid mixtures with magnetic relaxation was investigated. The irreversible microscopic phenomena giving rise to magnetic relaxation are described, assuming that the total specific magnetization $\mathbf{m}$ given by two irreversible parts $\mathbf{m}^{(0)}$ and $\mathbf{m}^{(1)}$ as in (1), and in the linear case the magnetic relaxation equation was derived.

In [15], [16] analogous studies for dielectric relaxation phenomena in polarizable media with internal variables were performed by using the same methods of the classical thermodynamics of irreversible processes with internal variables.

In [17] mixtures of n reacting fluid components, heat conducting and presenting magnetic and dielectric relaxation, were described within the same thermodynamic framework.

In this paper we derive the heat conduction equation and the heat dissipation function for anisotropic mixtures, constituted by n reacting fluid components, presenting magnetic relaxation. As in [7] we assume that the total specific magnetization can be split in two irreversible parts (see (1)), and we introduce $\mathbf{m}^{(1)}$ as internal variable in the thermodynamic state space.

The paper is organized as follows.

In Sections 2-4 and Section 6 we give, adding further considerations on the fundamentals of thermodynamics of irreversible processes, a review of the model of the considered media, derived by the authors in [7]. We present the governing equations describing all the processes inside them, the entropy balance equation, the phenomenological equations, the Onsager-Casimir relations and the laws of state. In Section 5 the entropy production is worked out, taking into account the phenomenological equations. Finally, in Section 7 the derivation of the field $\mathbf{B}^{(1)}$, conjugated to the internal specific magnetization $\mathbf{m}^{(1)}$, is obtained.

In Section 8 the specific internal energy and the specific entropy are worked out.

In Section 9 the heat conduction equation and the heat dissipation function are derived and in Sections 10 and 11 the Snoek and De Groot cases are studied.

The derived results find applications in several fields as in medicine, in biology, in nuclear resonance, where different species of molecules have different magnetic susceptibilities and relaxation times and contribute to the total magnetization.

In [18] the heat conduction equation and the heat dissipation function were derived for anisotropic magnetizable media with relaxation, in the assumption (1).

In [19] and [20] the heat conduction equation and the heat dissipation function were derived for anisotropic and isotropic polarizable media, respectively, with dielectric relaxation in the linear case, within the same thermodynamic framework.

A continuum phenomenological theory with internal variables for magnetizable media with relaxation phenomena and constitued by n different ionic species was developed by Maugin in [21], [22] to explain the internal mechanisms in these media (see also [23], [24]).

2 Balance equations

In this Section we present the equations governing the behaviour of mixtures consisting of n reacting fluid components, in a current configuration $\mathcal{K}_t$ [12]. We use the Cartesian tensor notation in a rectangular coordinate system. The model for these media was developed by the authors within the framework of classical irreversible thermodynamics with internal variables in [7]. All the processes occurring inside the considered mixtures are governed by the balance equations and the Maxwell equations in Galilean approximation.

The *conservation of mass* is given by

$$\frac{d\varrho}{dt} + \varrho \nabla \cdot \mathbf{v} = 0, \tag{3}$$

where ϱ is the total mass density field, given by $\varrho = \sum_{k=1}^{n} \varrho^{(k)}$, the symbol "$\nabla \cdot$ " denotes the divergence operator, $\frac{d}{dt}$ is the material time derivative, defined by $\frac{d}{dt} = \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i}$, where Einstein convention for repeated indices is used, $\frac{\partial}{\partial t}$ is the time partial derivative, $\frac{\partial}{\partial x_i}$ is the spatial partial derivative and v_i $(i = 1, 2, 3)$ are the cartesian components of the barycentric velocity $\mathbf{v}$ of the fluid mixture under consideration, defined by

$$\mathbf{v} = \frac{1}{\varrho} \sum_{k=1}^{n} \varrho^{(k)} \mathbf{v}^{(k)}, \tag{4}$$

being $\varrho^{(k)}$ the mass density and $\mathbf{v}^{(k)}$ the velocity of the k-th fluid component.

Let us introduce the mass fractions $c^{(k)}$ of the n fluid components by

$$c^{(k)} = \frac{\varrho^{(k)}}{\varrho} \quad (k = 1, \ldots, n), \quad \text{with} \quad \sum_{k=1}^{n} c^{(k)} = 1 \tag{5}$$

(see $(4)_1$).

Let us define the diffusion flux of the k-th fluid component with respect to the barycentric motion by the expression

$$\mathbf{J}^{(k)}_{(diff)} = \varrho^{(k)}(\mathbf{v}^{(k)} - \mathbf{v}) \quad (k = 1, 2 \ldots, n) \quad \text{with} \quad \sum_{k=1}^{n} \mathbf{J}^{(k)}_{(diff)} = \mathbf{0}, \tag{6}$$

(see (4)), i.e. only $n - 1$ of the n diffusion fluxes are independent.

The *balance equations for the mass fractions* $c^{(k)}$ have the following form [12]:

$$\varrho \frac{dc^{(k)}}{dt} = -\nabla \cdot \mathbf{J}^{(k)}_{(diff)} + \sum_{h=1}^{r} \nu^{(kh)} J^{(h)}_{(chem)} \quad (k = 1, \dots, n), \tag{7}$$

where $\nu^{(kh)} J^{(h)}_{(chem)}$ is the production of k-th fluid component per unit volume and per unit time by the h-th chemical reaction, $J^{(h)}_{(chem)}$ is the chemical reaction rate of the h-th chemical reaction and the quantity $\nu^{(kh)}$ divided by the molecular mass $\mathcal{M}^{(k)}$ of the k-th fluid component is proportional to the stoichiometric coefficient with which the k-th fluid component appears in the h-th chemical reaction.

From equations (7) we can obtain the following relation, by multiplying both sides by $\dfrac{\mu^{(k)}}{T}$, summing over k and introducing the h-th quantity $A^{(h)}$, representing the opposite of the so-called chemical affinity of each h-th reaction, i.e., $A^{(h)} = -\sum\limits_{k=1}^{n} \mu^{(k)} \nu^{(kh)} \quad (h = 1, \dots, r)$,

$$\begin{aligned} \frac{\varrho}{T} \sum_{k=1}^{n} \mu^{(k)} \frac{dc^{(k)}}{dt} &= -\nabla \cdot \left(\frac{1}{T} \sum_{k=1}^{n} \mu^{(k)} \mathbf{J}^{(k)}_{(diff)} \right) \\ &+ \sum_{k=1}^{n} \mathbf{J}^{(k)}_{(diff)} \cdot \nabla \left(\frac{\mu^{(k)}}{T} \right) - \frac{1}{T} \sum_{h=1}^{r} A^{(h)} J^{(h)}_{(chem)}. \end{aligned} \tag{8}$$

Maxwell's equations, in Galilean approximation, for magnetizable media (in the rationalized Gauss system) keep the form

$$\begin{cases} \nabla \times \mathbf{H} - \dfrac{1}{c} \dfrac{\partial \mathbf{E}}{\partial t} = \dfrac{1}{c} \mathbf{I}, \\ \nabla \cdot \mathbf{E} = \varrho^{(el)}, \\ \nabla \times \mathbf{E} + \dfrac{1}{c} \dfrac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \\ \nabla \cdot \mathbf{B} = 0, \end{cases} \tag{9}$$

where c is the light velocity, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field strengths, respectively, $\mathbf{H}$ is the magnetic displacement field, $\varrho^{(el)}$ is the electric charge per unit volume (electric charge density) and $\mathbf{I}$ is the density of the total electric current.

For magnetizable media the polarization vector, defined by $\mathbf{P} = \mathbf{D} - \mathbf{E}$, with $\mathbf{D}$ the electric displacement field, is null, then $\mathbf{P} = \mathbf{0}$ and $\mathbf{D} = \mathbf{E}$ in equations $(9)_1$ and $(9)_2$. Furthermore, we define the magnetization $\mathbf{M}$ and the specific magnetization $\mathbf{m}$ by

$$\mathbf{M} = \mathbf{B} - \mathbf{H}, \qquad \mathbf{m} = \frac{1}{\varrho}\mathbf{M}, \tag{10}$$

that are axial vectors.

The total charge e per unit of mass of the system is given by

$$e = \frac{1}{\varrho}\sum_{k=1}^{n} \varrho^{(k)} e^{(k)} = \sum_{k=1}^{n} c^{(k)} e^{(k)}, \tag{11}$$

being $e^{(k)}$ the charge per unit of mass of the k-th fluid component (see $(4)_1$).

In equations $(9)_{1,2}$ $\mathbf{I}$ and $\varrho^{(el)}$ are defined by

$$\mathbf{I} = \sum_{k=1}^{n} \varrho^{(k)} e^{(k)} \mathbf{v}^{(k)}, \qquad \varrho^{(el)} = \varrho e = \sum_{k=1}^{n} \varrho^{(k)} e^{(k)}, \tag{12}$$

and they satisfy the following *charge conservation law*

$$\frac{\partial \varrho^{(el)}}{\partial t} = -\nabla \cdot \mathbf{I}. \tag{13}$$

Equation $(12)_1$ can be written in the form

$$\mathbf{I} = \varrho^{(el)}\mathbf{v} + \sum_{k=1}^{n} e^{(k)} \mathbf{J}^{(k)}_{(diff)} \tag{14}$$

(see $(6)_1$ and $(12)_2$).

On the right hand side of (14) the first term $\varrho^{(el)}\mathbf{v}$ is the electric current due to convection, the second one is the electric current due to the relative motion of the n fluid components, called conduction current $\mathbf{j}^{(el)}$ and given by

$$\mathbf{j}^{(el)} = \sum_{k=1}^{n} e^{(k)} \mathbf{J}^{(k)}_{(diff)}. \tag{15}$$

The *first law of thermodynamics* for magnetizable media in an electromagnetic field, in Galilean approximation, see [12], has the form

$$\varrho\frac{du}{dt} = -\nabla \cdot \mathbf{J}^{(q)} + \tau_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt} + \mathbf{j}^{(el)} \cdot \mathbf{E} + \varrho\mathbf{B} \cdot \frac{d\mathbf{m}}{dt}, \tag{16}$$

where u is the specific internal energy of the system, $\mathbf{J}^{(q)}$ is the heat flux density, $\tau_{\alpha\beta}$ is the symmetric mechanical stress tensor and $\frac{d\varepsilon_{\alpha\beta}}{dt}$ is the small strain rate tensor, given by

$$\frac{d\varepsilon_{\alpha\beta}}{dt} = \frac{1}{2}\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right) \quad (\alpha, \beta = 1, 2, 3). \tag{17}$$

On the right hand side of equation (16) there are present four contributions given by the heat supply, the work done by the mecanical stress, the Joule heat and the work done by the magnetic field to change the magnetization, respectively.

In (16) all the quantities are per unit of volume and per unit of time.

We note that (16) does not satisfy the requirements of the relativity theory. However, we assume that the velocity of the medium under consideration with respect to the observer is small compared with the velocity of the light. In this case equation (16) is a good approximation for the first law of thermodynamics (Galilean approximation). For the exact relativistic formulation of the first law for viscous, magnetizable simple fluid mixtures influenced by an electromagnetic field the reader may consult Kluitenberg [25] and Kluitenberg and De Groot [26], [27].

In the following Section 3 we will introduce the *entropy balance equation* and *the phenomenological equations*, derived in [7].

3 Choice of the state space variables, entropy production and phenomenological equations

In this Section we give a review of some results obtained in [7], with some considerations on the foundations of the irreversible processes thermodynamics. Let us suppose that the total specific magnetization $\mathbf{m}$ is given by two irreversible contributions, $\mathbf{m}^{(0)}$ and $\mathbf{m}^{(1)}$, due to two different types of irreversible microscopic phenomena, that give rise to the magnetic relaxation and let us introduce $\mathbf{m}^{(1)}$ in the thermodynamic state space as internal variable to describe these phenomena. Thus, we assume that the specific entropy s (the entropy per unit of mass) is a function of the specific internal energy u, the strain tensor $\varepsilon_{\alpha\beta}$, the specific magnetization $\mathbf{m}^{(1)}$, the concentrations $c^{(k)}$ of the n fluid components ($k = 1, \ldots, n$) (see [7])

$$s = s\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right). \tag{18}$$

The internal variables are a powerful tool to describe complex media. In this paper we work within the classical irreversible thermodynamics (CIT), see [8]-[13], where gradients and time derivatives of the physical fields, dissipative fluxes are not included in the thermodynamic state space and the *local equilibrium hypothesis* for the system is assumed (see[28]-[30]), i.e. out of the equilibrium each point of the medium is considered as a thermodynamic cell where the reversible thermodynamics is applicable. For some remarks about the internal variables and some versions of non-equilibrium thermodynamics see Section 2 of Reference [31]. We define the equilibrium temperature T, the equilibrium stress tensor $\tau^{(eq)}_{\alpha\beta}$, the equilibrium magnetic field $\mathbf{B}^{(eq)}$, the thermodynamic affinity $\mathbf{B}^{(1)}$, conjugate to the internal variable $\mathbf{m}^{(1)}$, and the thermodynamic or chemical potential $\mu^{(k)}$ of the k-th fluid component, respectively, by

$$\begin{cases} T^{-1} = \dfrac{\partial}{\partial u} s\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}, c^{(1)}, \dots, c^{(n)}\right), \\ \tau^{(eq)}_{\alpha\beta} = \varrho T \dfrac{\partial}{\partial \varepsilon_{\alpha\beta}} s\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}, c^{(1)}, \dots, c^{(n)}\right), \\ \mathbf{B}^{(eq)} = -T \dfrac{\partial}{\partial \mathbf{m}} s\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}, c^{(1)}, \dots, c^{(n)}\right), \\ \mathbf{B}^{(1)} = T \dfrac{\partial}{\partial \mathbf{m}^{(1)}} s\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}, c^{(1)}, \dots, c^{(n)}\right), \\ \mu^{(k)} = -T \dfrac{\partial}{\partial c^{(k)}} s\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}, c^{(1)}, \dots, c^{(n)}\right) \quad (k = 1, \dots, n). \end{cases}$$

Considering very small deviations with respect to a local equilibrium state, we expand the entropy (18) into Taylor's series with respect to this state, and confining our consideration to the linear terms, we obtain the differential of the entropy s in a point of the thermodynamic phase space (see[32], [33]), i.e. in a local position in a current configuration K_t, in the following form, called Gibbs relation,

$$T ds = du - \frac{1}{\varrho}\tau^{(eq)}_{\alpha\beta} d\varepsilon_{\alpha\beta} - \mathbf{B}^{(eq)} \cdot d\mathbf{m} + \mathbf{B}^{(1)} \cdot d\mathbf{m}^{(1)} - \sum_{k=1}^{n} \mu^{(k)} dc^{(k)}, \quad (19)$$

where we have used Eqs. $(3)_2$-$(3)_5$. From (19) it follows that the time derivative of the entropy s in the considered point of the thermodynamic phase space takes the form

$$\frac{ds}{dt} = \frac{du}{dt} - \frac{1}{\varrho}\tau^{(eq)}_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt} - \mathbf{B}^{(eq)} \cdot \frac{d\mathbf{m}}{dt} + \mathbf{B}^{(1)} \cdot \frac{d\mathbf{m}^{(1)}}{dt} - \sum_{k=1}^{n} \mu^{(k)} \frac{dc^{(k)}}{dt}. \quad (20)$$

From (20), using (8) and (16) we have

$$\varrho\frac{ds}{dt}=-\nabla\cdot\mathbf{J}^{(s)}+\frac{1}{T}\left(-\frac{1}{T}\mathbf{J}^{(q)}\cdot\nabla T+\tau^{(vi)}_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt}+\varrho\mathbf{B}^{(ir)}\cdot\frac{d\mathbf{m}}{dt}\right.$$

$$\left.+\varrho\mathbf{B}^{(1)}\cdot\frac{d\mathbf{m}^{(1)}}{dt}+\sum_{h=1}^{r}A^{(h)}J^{(h)}_{(chem)}\right)-\sum_{k=1}^{n}\mathbf{J}^{(k)}_{(diff)}\cdot\nabla\left(\frac{\mu^{(k)}}{T}\right)+\frac{1}{T}\mathbf{j}^{(el)}\cdot\mathbf{E},\quad(21)$$

where $\mathbf{J}^{(s)}$, $\tau^{(vi)}_{\alpha\beta}$ and $B^{(ir)}$ are the entropy flux, the viscous stress tensor and the irreversible magnetic field, respectively, defined by

$$\mathbf{J}^{(s)}=\frac{1}{T}\left(\mathbf{J}^{(q)}-\sum_{k=1}^{n}\mu^{(k)}\mathbf{J}^{(k)}_{(diff)}\right),\quad(22)$$

$$\tau^{(vi)}_{\alpha\beta}=\tau_{\alpha\beta}-\tau^{(eq)}_{\alpha\beta}\quad(\alpha,\beta=1,2,3),\qquad\mathbf{B}^{(ir)}=\mathbf{B}-\mathbf{B}^{(eq)}.\quad(23)$$

Defining the thermodynamic force $\chi^{(k)}$, conjugate to the diffusion flux $\mathbf{J}^{(k)}_{(diff)}$, by

$$\chi^{(k)}=-\left[T\,\nabla\left(\frac{\mu^{(k)}}{T}\right)-e^{(k)}\mathbf{E}\right]\quad(k=1,\ldots,n),\quad(24)$$

and taking into account equation (15) we have

$$-\sum_{k=1}^{n}\mathbf{J}^{(k)}_{(diff)}\cdot\nabla\left(\frac{\mu^{(k)}}{T}\right)+\frac{1}{T}\mathbf{j}^{(el)}\cdot\mathbf{E}=\frac{1}{T}\sum_{k=1}^{n}\mathbf{J}^{(k)}_{(diff)}\cdot\chi^{(k)}=$$

$$=\frac{1}{T}\sum_{k=1}^{n-1}\mathbf{J}^{(k)}_{(diff)}\cdot\left(\chi^{(k)}-\chi^{(n)}\right)=\frac{1}{T}\sum_{k=1}^{n-1}\mathbf{J}^{(k)}_{(diff)}\cdot\mathbf{X}^{(k)},\quad(25)$$

where we have used (6), (15) and defined $\mathbf{X}^{(k)}=\chi^{(k)}-\chi^{(n)}(k=1,\ldots,n-1)$.

Using (25), *the entropy balance equation* (21) can be written in the following form

$$\varrho\frac{ds}{dt}=-\nabla\cdot\mathbf{J}^{(s)}+\sigma^{(s)},\quad(26)$$

where $\sigma^{(s)}$ is *the entropy production* per unit volume and per unit time, given by

$$\sigma^{(s)}=\frac{1}{T}\left(\varrho\mathbf{B}^{(ir)}\cdot\frac{d\mathbf{m}}{dt}+\varrho\mathbf{B}^{(1)}\cdot\frac{d\mathbf{m}^{(1)}}{dt}-\frac{1}{T}\mathbf{J}^{(q)}\cdot\nabla T\right.$$

$$+\sum_{k=1}^{n-1}\mathbf{J}^{(k)}_{(diff)}\mathbf{X}^{(k)}+\sum_{h=1}^{r}A^{(h)}J^{(h)}_{(chem)}+\tau^{(vi)}_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt}\Bigg). \tag{27}$$

In the expression (27) $\sigma^{(s)}$ is the intrinsic entropy production defined as non-negative quantity by the *second law of thermodynamics*: $\sigma^{(s)} \geq 0$.

$\sigma^{(s)} = 0$ when the system is in *thermodynamic equilibrium*, i.e. when irreversible processes inside the system are not present. In thermodynamic equilibrium the sources (the body force, the energy source, the external entropy production) are null, the velocity of the medium is constant or null, the fluxes of physical quantities, the gradients and the time derivatives of the fields are null, the internal variables "become dependent on the variables of the equilibrium subspace, but they are not determined by them, i.e., the equilibrium subspace is many-valued" [34].

If we assume that the specific entropy s is a constitutive function of the independent variables of the thermodynamic state space $C = C\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}^{(0)}, \mathbf{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right)$, i.e.

$$s = s\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}^{(0)}, \mathbf{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right), \tag{28}$$

in (26) $\sigma^{(s)}$ assumes the following form

$$\sigma^{(s)} = \frac{1}{T}\Bigg[-\frac{1}{T}\mathbf{J}^{(q)}\cdot\nabla T+\tau^{(vi)}_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt}+\varrho\mathbf{B}^{(ir)}\cdot\frac{d\mathbf{m}^{(0)}}{dt}+$$

$$+\varrho\left(\mathbf{B}^{(1)}+\mathbf{B}^{(ir)}\right)\cdot\frac{d\mathbf{m}^{(1)}}{dt}+\sum_{h=1}^{r}A^{(h)}J^{(h)}_{(chem)}+\sum_{k=1}^{n-1}\mathbf{J}^{(k)}_{(diff)}\cdot\mathbf{X}^{(k)}\Bigg]. \tag{29}$$

It is seen from (29) that if the magnetic field $\mathbf{B}$ equals the equilibrium magnetic field $\mathbf{B}^{(eq)}$, $\mathbf{B}^{(ir)}$ vanishes and only the specific partial magnetization $\mathbf{m}^{(1)}$ contributes to the entropy production, i.e. changes in $\mathbf{m}^{(0)}$ become reversible and we are in the Snoek case (see Section 10). Thus, $\mathbf{m}^{(0)}$ is an irreversible part of the total magnetization only when the magnetizable medium is not in a thermodynamic equilibrium state. Also, from (29) we can obtain the De Groot-Mazur case (see Section 11), when $\mathbf{m} = \mathbf{m}^{(0)}$, i.e. there is no internal variable $\mathbf{m}^{(1)}$.

From (27) it is seen that the intrinsic entropy production

$$\sigma^{(in)} = T\sigma^{(s)}$$

is a bilinear form composed of a sum of m terms, where each term is a product of the components of a flux J_i, scalar, vector, tensor, and the components of an affinity, or thermodynamic force Y_i (conjugate to the flux J_i), scalar, vector, tensor, i.e.

$$\sigma^{(in)} = \sum_{i=1}^{m} J_i. \tag{30}$$

From (30), using the procedures of the classical irreversible thermodynamics, the phenomenological equations can be written in the following linear form (see [12], [13] and also [35]))

$$J_i = \sum_{k=1}^{m} L_{ik} Y_k \quad (i = 1, \ldots, m), \tag{31}$$

where the quantities L_{ik} are called phenomenological coefficients and are constant. In equations (31) it is seen that a thermodynamic flux does not depend only on the corresponding force, but also on other forces.

In equation (27) $\sigma^{(in)}$ is the sum of six terms, thus the phenomenological equations, in which the irreversible fluxes are *linear functions* of the thermodynamic forces, have the following form (see [7])

$$\begin{aligned} B_\alpha^{(ir)} = {} & \varrho L^{(0,0)}_{(M)\alpha\beta} \frac{dm_\beta}{dt} + L^{(0,1)}_{(M)\alpha\beta} B^{(1)}_\beta - \frac{1}{T} L^{(0,q)}_{(M)\alpha\beta} \frac{\partial T}{\partial x_\beta} + \sum_{k=1}^{n-1} L^{(0,k)}_{(MD)\alpha\beta} X^{(k)}_\beta \\ & + \sum_{h=1}^{r} L^{(0,h)}_{(MC)\alpha} A^{(h)} + L^{(0,vi)}_{(M)\alpha\beta\gamma} \frac{d\varepsilon_{\beta\gamma}}{dt}, \end{aligned} \tag{32}$$

$$\begin{aligned} \varrho \frac{dm^{(1)}_\alpha}{dt} = {} & \varrho L^{(1,0)}_{(M)\alpha\beta} \frac{dm_\beta}{dt} + L^{(1,1)}_{(M)\alpha\beta} B^{(1)}_\beta - \frac{1}{T} L^{(1,q)}_{(M)\alpha\beta} \frac{\partial T}{\partial x_\beta} + \sum_{k=1}^{n-1} L^{(1,k)}_{(MD)\alpha\beta} X^{(k)}_\beta \\ & + \sum_{h=1}^{r} L^{(1,h)}_{(MC)\alpha} A^{(h)} + L^{(1,vi)}_{(M)\alpha\beta\gamma} \frac{d\varepsilon_{\beta\gamma}}{dt}, \end{aligned} \tag{33}$$

$$\begin{aligned} J^{(q)}_\alpha = {} & \varrho L^{(q,0)}_{(M)\alpha\beta} \frac{dm_\beta}{dt} + L^{(q,1)}_{(M)\alpha\beta} B^{(1)}_\beta - \frac{1}{T} L^{(q,q)}_{\alpha\beta} \frac{\partial T}{\partial x_\beta} + \sum_{k=1}^{n-1} L^{(q,k)}_{(D)\alpha\beta} X^{(k)}_\beta \\ & + \sum_{h=1}^{r} L^{(q,h)}_{(C)\alpha} A^{(h)} + L^{(q,vi)}_{\alpha\beta\gamma} \frac{d\varepsilon_{\beta\gamma}}{dt}, \end{aligned} \tag{34}$$

$$\begin{aligned} J^{(j)}_{(diff)\alpha} = {} & \varrho L^{(j,0)}_{(DM)\alpha\beta} \frac{dm_\beta}{dt} + L^{(j,1)}_{(DM)\alpha\beta} B^{(1)}_\beta - \frac{1}{T} L^{(j,q)}_{(D)\alpha\beta} \frac{\partial T}{\partial x_\beta} + \sum_{k=1}^{n-1} L^{(j,k)}_{(DD)\alpha\beta} X^{(k)}_\beta \\ & + \sum_{h=1}^{r} L^{(j,h)}_{(DC)\alpha} A^{(h)} + L^{(j,vi)}_{(D)\alpha\beta\gamma} \frac{d\varepsilon_{\beta\gamma}}{dt} \quad (j = 1, \ldots, n-1), \end{aligned} \tag{35}$$

$$J^{(l)}_{(chem)} = \varrho L^{(l,0)}_{(CM)\beta}\frac{dm_\beta}{dt} + L^{(l,1)}_{(CM)\beta}B^{(1)}_\beta - \frac{1}{T}L^{(l,q)}_{(C)\beta}\frac{\partial T}{\partial x_\beta} + \sum_{k=1}^{n-1} L^{(l,k)}_{(CD)\beta}X^{(k)}_\beta$$

$$+ \sum_{h=1}^{r} L^{(l,h)}_{(CC)}A^{(h)} + L^{(l,vi)}_{(C)\beta\gamma}\frac{d\varepsilon_{\beta\gamma}}{dt} \quad (l = 1, ..., r), \tag{36}$$

$$\tau^{(vi)}_{\alpha\beta} = \varrho L^{(vi,0)}_{(M)\alpha\beta\gamma}\frac{dm_\gamma}{dt} + L^{(vi,1)}_{(M)\alpha\beta\gamma}B^{(1)}_\gamma - \frac{1}{T}L^{(vi,q)}_{\alpha\beta\gamma}\frac{\partial T}{\partial x_\gamma} + \sum_{k=1}^{n-1} L^{(vi,k)}_{(D)\alpha\beta\gamma}X^{(k)}_\gamma$$

$$+ \sum_{h=1}^{r} L^{(vi,h)}_{(C)\alpha\beta}A^{(h)} + L^{(vi,vi)}_{\alpha\beta\gamma\delta}\frac{d\varepsilon_{\gamma\delta}}{dt}. \tag{37}$$

In principle, all irreversible phenomena described by (32)-(37) can influence each other. For instance, the third, fourth, fifth and sixth term on the right-hand side of (33) describe the influences of the heat flux, diffusion fluxes, chemical reactions and viscous flow on magnetic relaxation. These phenomena are called cross effects. Equations (32) and (33) describe irreversible changes in the magnetization, the phenomenological equations (34)-(36) describe the irreversible processes of the heat flux, diffusion fluxes and chemical reactions. Equation (37) is a generalization of Newton's law for viscous fluid flow. The quantities $L^{(0,0)}_{(MM)\alpha\beta}, L^{(0,1)}_{(MM)\alpha\beta}, L^{(0,q)}_{(M)\alpha\beta}, \ldots$, that are present in (32)-(37) are called phenomenological tensors. $L^{(0,0)}_{(MM)\alpha\beta}$ and $L^{(0,1)}_{(MM)\alpha\beta}$ are polar tensors of second order connected with the magnetic relaxation, $L^{(0,vi)}_{(M)\alpha\beta\gamma}$ and $L^{(1,vi)}_{(M)\alpha\beta\gamma}$ are pseudotensors of third order connected with the influence of the viscous flow on the magnetic relaxation. $L^{(q,q)}_{\alpha\beta}$ is the heat conductivity polar tensor of order two, $L^{(vi,vi)}_{\alpha\beta\gamma\delta}$ is the viscosity polar tensor of order four, $L^{(j,k)}_{(DD)\alpha\beta}$ $(j, k = 1, \ldots, n-1)$ is a polar tensor of order two connected with the diffusion flow of the k-th fluid component, $L^{(l,h)}_{(CC)}$ is a scalar connected with the chemical affinity of the reaction h $(l, h = 1, \ldots, r)$, having taken into account that $\varrho\frac{d\mathbf{m}}{dt}$, $\varrho\frac{d\mathbf{m}^{(1)}}{dt}$, $\mathbf{B}^{(ir)}$ and $\mathbf{B}^{(1)}$ are pseudovectors (or axial vectors) while $T^{-1}\nabla T$, $\mathbf{X}^{(k)}$ $(k = 1, \ldots, n-1)$ and $\tau^{(vi)}_{\alpha\beta}$, $\mathbf{J}^{(q)}$, $\mathbf{J}^{(k)}_{(diff)}$, $(k = 1, \ldots, n-1)$, $\frac{d\varepsilon_{\alpha\beta}}{dt}$ are polar tensors and $J^{(l)}$, $A^{(l)}$ $(l = 1, \ldots, r)$, T are scalar quantities.

4 Onsager-Casimir reciprocity relations and symmetry relations

The macroscopic quantities, which are present in a phenomenological theory come from statistical averages of functions of the coordinates and momenta of the microscopic particles constituting the system at mesoscopic level (see [13]). From the microscopic point of view a macroscopic quantity is called even (or odd), if it is a even (or odd) function of the speed of the microscopic particles. Macroscopic quantities which do not depend on the velocity of these microscopic particles are considered even, the thermodynamic force conjugated to an even flux is an odd quantity, while the affinity conjugate to an odd flux is an even quantity. The heat flux, the time derivative of the small strain tensor, the time derivative of an even quantity, the spatial derivative of an odd quantity are examples of odd functions; the temperature, the mass density, the kinetic energy, the stress tensor, the time derivative of an odd quantity, the spatial derivative of an even quantity are examples of even functions. From the macroscopic point of view we distinguish the macroscopic quantities in even and odd functions when they are even or odd under time reversal. Introducing the symbols "$^{(o)}$" and "$^{(e)}$" to indicate odd and even macroscopic quantities, respectively, the intrinsic entropy production (30) can be written in the form

$$\sigma^{(in)} = \sum_{i=1}^{p} J_i^{(o)} Y_i^{(e)} + \sum_{k=1}^{q} J_k^{(e)} Y_k^{(o)}, \quad \text{with} \quad p + q = m, \tag{38}$$

where the odd quantities $J_i^{(o)}$ and the even quantities $J_k^{(e)}$ are components of vectors, tensors, which represent the fluxes, while the even quantities $Y_i^{(e)}$ and the odd quantities $Y_k^{(o)}$ are components of vectors, tensors, which represent the thermodynamic forces conjugated to the corresponding fluxes. From (29), since T is an even function, $\sigma^{(in)}$ is an odd quantity. By virtue of (38) the phenomenological equations (31) take the following expression

$$J_i^{(o)} = \sum_{j=1}^{p} L_{ij}^{(o,e)} Y_j^{(e)} + \sum_{k=1}^{q} L_{ik}^{(o,o)} Y_k^{(o)} \qquad (i = 1, \dots, p), \tag{39}$$

$$J_l^{(e)} = \sum_{j=1}^{p} L_{lj}^{(e,e)} Y_j^{(e)} + \sum_{k=1}^{q} L_{lk}^{(e,o)} Y_k^{(o)} \qquad (l = 1, \dots, q), \tag{40}$$

with $p + q = m$.

In (27) $\varrho\dfrac{d\mathbf{m}}{dt}$, $\varrho\dfrac{d\mathbf{m}^{(1)}}{dt}$, $A^{(h)}$ $(h=1,\dots,r)$, $T^{-1}\nabla T$, $\tau^{(vi)}_{\alpha\beta}$ and $\mathbf{X}^{(k)}$ $(k=1,\dots,n-1)$ are even functions of the microscopic particle velocities, while $\mathbf{B}^{(ir)}$, $\mathbf{B}^{(1)}$, $\mathbf{J}^{(q)}$, $\mathbf{J}^{(k)}_{(diff)}$ $(k=1,\dots,r)$, $J^{(l)}$ $(l=1,\dots,r)$, $\dfrac{d\varepsilon_{\alpha\beta}}{dt}$ are odd functions of these velocities.

Thus, we can write

$$\begin{aligned}
\left(B^{(ir)}_{\alpha}\right)^{(o)} &= \varrho\left(L^{(0,0)}_{(M)\alpha\beta}\right)^{(o,e)}\left(\frac{dm_\beta}{dt}\right)^{(e)} + \left(L^{(0,1)}_{(M)\alpha\beta}\right)^{(o,o)}\left(B^{(1)}_{\beta}\right)^{(o)} \\
&-\frac{1}{T}\left(L^{(0,q)}_{(M)\alpha\beta}\right)^{(o,e)}\left(\frac{\partial T}{\partial x_\beta}\right)^{(e)} + \sum_{k=1}^{n-1}\left(L^{(0,k)}_{(MD)\alpha\beta}\right)^{(o,e)}\left(X^{(k)}_{\beta}\right)^{(e)} \\
&+\sum_{h=1}^{r}\left(L^{(0,h)}_{(MC)\alpha}\right)^{(o,e)}\left(A^{(h)}\right)^{(e)} + \left(L^{(0,vi)}_{(M)\alpha\beta\gamma}\right)^{(o,o)}\left(\frac{d\varepsilon_{\beta\gamma}}{dt}\right)^{(o)}, \qquad (41)
\end{aligned}$$

$$\begin{aligned}
\varrho\left(\frac{dm^{(1)}_{\alpha}}{dt}\right)^{(e)} &= \varrho\left(L^{(1,0)}_{(M)\alpha\beta}\right)^{(e,e)}\left(\frac{dm_\beta}{dt}\right)^{(e)} + \left(L^{(1,1)}_{(M)\alpha\beta}\right)^{(e,o)}\left(B^{(1)}_{\beta}\right)^{(o)} \\
&-\frac{1}{T}\left(L^{(1,q)}_{(M)\alpha\beta}\right)^{(e,e)}\left(\frac{\partial T}{\partial x_\beta}\right)^{(e)} + \sum_{k=1}^{n-1}\left(L^{(1,k)}_{(MD)\alpha\beta}\right)^{(e,e)}\left(X^{(k)}_{\beta}\right)^{(e)} \\
&+\sum_{h=1}^{r}\left(L^{(1,h)}_{(MC)\alpha}\right)^{(e,e)}\left(A^{(h)}\right)^{(e)} + \left(L^{(1,vi)}_{(M)\alpha\beta\gamma}\right)^{(e,o)}\left(\frac{d\varepsilon_{\beta\gamma}}{dt}\right)^{(o)}, \qquad (42)
\end{aligned}$$

$$\begin{aligned}
\left(J^{(q)}_{\alpha}\right)^{(o)} &= \varrho\left(L^{(q,0)}_{(M)\alpha\beta}\right)^{(o,e)}\left(\frac{dm_\beta}{dt}\right)^{(e)} + \left(L^{(q,1)}_{(M)\alpha\beta}\right)^{(o,o)}\left(B^{(1)}_{\beta}\right)^{(o)} \\
&-\frac{1}{T}\left(L^{(q,q)}_{\alpha\beta}\right)^{(o,e)}\left(\frac{\partial T}{\partial x_\beta}\right)^{(e)} + \sum_{k=1}^{n-1}\left(L^{(q,k)}_{(D)\alpha\beta}\right)^{(o,e)}\left(X^{(k)}_{\beta}\right)^{(e)} \\
&+\sum_{h=1}^{r}\left(L^{(q,h)}_{(C)\alpha}\right)^{(o,e)}\left(A^{(h)}\right)^{(e)} + \left(L^{(q,vi)}_{\alpha\beta\gamma}\right)^{(o,o)}\left(\frac{d\varepsilon_{\beta\gamma}}{dt}\right)^{(o)}, \qquad (43)
\end{aligned}$$

$$\begin{aligned}
\left(J^{(j)}_{(diff)\alpha}\right)^{(o)} &= \varrho\left(L^{(j,0)}_{(DM)\alpha\beta}\right)^{(o,e)}\left(\frac{dm_\beta}{dt}\right)^{(e)} + \left(L^{(j,1)}_{(DM)\alpha\beta}\right)^{(o,o)}\left(B^{(1)}_{\beta}\right)^{(o)} \\
&-\frac{1}{T}\left(L^{(j,q)}_{(D)\alpha\beta}\right)^{(o,e)}\left(\frac{\partial T}{\partial x_\beta}\right)^{(e)} + \sum_{k=1}^{n-1}\left(L^{(j,k)}_{(DD)\alpha\beta}\right)^{(o,e)}\left(X^{(k)}_{\beta}\right)^{(e)}
\end{aligned}$$

$$+\sum_{h=1}^{r}\left(L^{(j,h)}_{(DC)\alpha}\right)^{(o,e)}\left(A^{(h)}\right)^{(e)}+\left(L^{(j,vi)}_{(D)\alpha\beta\gamma}\right)^{(o,o)}\left(\frac{d\varepsilon_{\beta\gamma}}{dt}\right)^{(o)} \tag{44}$$

$$(j=1,\ldots,n-1),$$

$$\left(J^{(l)}_{(chem)}\right)^{(o)}=\varrho\left(L^{(l,0)}_{(CM)\beta}\right)^{(o,e)}\left(\frac{dm_{\beta}}{dt}\right)^{(e)}+\left(L^{(l,1)}_{(CM)\beta}\right)^{(o,o)}\left(B^{(1)}_{\beta}\right)^{(o)}$$

$$-\frac{1}{T}\left(L^{(l,q)}_{(C)\beta}\right)^{(o,e)}\left(\frac{\partial T}{\partial x_{\beta}}\right)^{(e)}+\sum_{k=1}^{n-1}\left(L^{(l,k)}_{(CD)\beta}\right)^{(o,e)}\left(X^{(k)}_{\beta}\right)^{(e)}$$

$$+\sum_{h=1}^{r}\left(L^{(l,h)}_{(CC)}\right)^{(o,e)}\left(A^{(h)}\right)^{(e)}+\left(L^{(l,vi)}_{(C)\beta\gamma}\right)^{(o,o)}\left(\frac{d\varepsilon_{\beta\gamma}}{dt}\right)^{(o)}\quad(l=1,\ldots,r), \tag{45}$$

$$\left(\tau^{(vi)}_{\alpha\beta}\right)^{(e)}=\varrho\left(L^{(vi,0)}_{(M)\alpha\beta\gamma}\right)^{(e,e)}\left(\frac{dm_{\gamma}}{dt}\right)^{(e)}+\left(L^{(vi,1)}_{(M)\alpha\beta\gamma}\right)^{(e,o)}\left(B^{(1)}_{\gamma}\right)^{(o)}$$

$$-\frac{1}{T}\left(L^{(vi,q)}_{\alpha\beta\gamma}\right)^{(e,e)}\left(\frac{\partial T}{\partial x_{\gamma}}\right)^{(e)}+\sum_{k=1}^{n-1}\left(L^{(vi,k)}_{(D)\alpha\beta\gamma}\right)^{(e,e)}\left(X^{(k)}_{\gamma}\right)^{(e)}$$

$$+\sum_{h=1}^{r}\left(L^{(vi,h)}_{(C)\alpha\beta}\right)^{(e,e)}\left(A^{(h)}\right)^{(e)}+\left(L^{(vi,vi)}_{\alpha\beta\gamma\delta}\right)^{(e,o)}\left(\frac{d\varepsilon_{\gamma\delta}}{dt}\right)^{(o)}. \tag{46}$$

Let us introduce the Onsager symmetry relations for the phenomenological coefficients coming from microscopic considerations [36], [37] (the minus sign which occurs in (49) is called the Casimir minus sign), see also [12], [13]:

$$L^{(o,e)}_{ij}=L^{(o,e)}_{ji}\qquad(i,j=1,\ldots,p), \tag{47}$$

$$L^{(e,o)}_{kl}=L^{(e,o)}_{lk}\qquad(k,l=1,\ldots,q), \tag{48}$$

$$L^{(o,o)}_{ik}=-L^{(e,e)}_{ki}\qquad(i=1,\ldots,p;\ k=1,\ldots,q). \tag{49}$$

Thus, we have the following Onsager-Casimir reciprocity relations for the phenomenological coefficients present in (32)-(37) (see [7] where they were derived)

$$L^{(0,0)}_{(M)\alpha\beta}=L^{(0,0)}_{(M)\beta\alpha},\quad L^{(1,1)}_{(M)\alpha\beta}=L^{(1,1)}_{(M)\beta\alpha},\quad L^{(q,q)}_{\alpha\beta}=L^{(q,q)}_{\beta\alpha},$$

$$L^{(0,1)}_{(M)\alpha\beta}=-L^{(1,0)}_{(M)\beta\alpha},\quad L^{(0,q)}_{(M)\alpha\beta}=L^{(q,0)}_{(M)\beta\alpha},\quad L^{(q,vi)}_{\alpha\beta\gamma}=-L^{(vi,q)}_{\beta\gamma\alpha},$$

$$L^{(0,k)}_{(MD)\alpha\beta}=L^{(k,0)}_{(DM)\beta\alpha},\quad L^{(j,k)}_{(DD)\alpha\beta}=L^{(k,j)}_{(DD)\beta\alpha}\quad(j,k=1,\ldots,n-1),$$

$$\begin{gathered}
L^{(0,h)}_{(MC)\alpha} = L^{(h,0)}_{(CM)\alpha}, \quad L^{(j,h)}_{(DC)\alpha} = L^{(h,j)}_{(CD)\alpha} \quad (j = 1, \ldots, n-1), \ (h = 1, \ldots, r), \\
L^{(1,k)}_{(MD)\alpha\beta} = -L^{(k,1)}_{(MD)\beta\alpha}, \quad L^{(k,vi)}_{(D)\alpha\beta\gamma} = -L^{(vi,k)}_{(D)\beta\gamma\alpha} \quad (k = 1, \ldots, n-1), \\
L^{(1,q)}_{(M)\alpha\beta} = -L^{(q,1)}_{(M)\beta\alpha}, \quad L^{(q,k)}_{(D)\alpha\beta} = L^{(k,q)}_{(D)\beta\alpha} \quad (k = 1, \ldots, n-1) \\
L^{(1,h)}_{(MC)\alpha} = -L^{(h,1)}_{(CM)\alpha}, \quad L^{(l,h)}_{(CC)} = L^{(h,l)}_{(CC)} \quad (l, h = 1, \ldots, r) \\
L^{(q,h)}_{(C)\alpha} = L^{(h,q)}_{(C)\alpha}, \quad L^{(0,vi)}_{(M)\alpha\beta\gamma} = -L^{(vi,0)}_{(M)\beta\gamma\alpha}, \\
L^{(h,vi)}_{(C)\alpha\beta} = -L^{(vi,h)}_{(C)\alpha\beta} \quad (h = 1, \ldots, r), \quad L^{(1,vi)}_{(M)\alpha\beta\gamma} = L^{(vi,1)}_{(M)\beta\gamma\alpha}, L^{(vi,vi)}_{\alpha\beta\gamma\delta} = L^{(vi,vi)}_{\gamma\delta\alpha\beta}.
\end{gathered} \tag{50}$$

The Onsager-Casimir reciprocity relations reduce the number of independent components of the phenomenological tensors.

Because of the symmetry of $\varepsilon_{\alpha\beta}$, $\tau^{(eq)}_{\alpha\beta}$ and the viscous stress $\tau^{(vi)}_{\alpha\beta}$ one has the following symmetry relations

$$\begin{gathered}
L^{(1,vi)}_{(M)\alpha\beta\gamma} = L^{(1,vi)}_{(M)\alpha\gamma\beta}, \qquad L^{(vi,1)}_{(M)\alpha\beta\gamma} = L^{(vi,1)}_{(M)\beta\alpha\gamma}, \\
L^{(q,vi)}_{\alpha\beta\gamma} = L^{(q,vi)}_{\alpha\gamma\beta}, \qquad L^{(vi,q)}_{\alpha\beta\gamma} = L^{(vi,q)}_{\beta\alpha\gamma}, \\
L^{(0,vi)}_{(M)\alpha\beta\gamma} = L^{(0,vi)}_{(M)\alpha\gamma\beta}, \qquad L^{(vi,0)}_{(M)\alpha\beta\gamma} = L^{(vi,0)}_{(M)\beta\alpha\gamma}, \\
L^{(l,vi)}_{(C)\alpha\beta} = L^{(l,vi)}_{(C)\beta\alpha}, \qquad L^{(vi,h)}_{(C)\alpha\beta} = L^{(vi,h)}_{(C)\beta\alpha} \quad (l, h = 1, \ldots, r), \\
L^{(j,vi)}_{(D)\alpha\beta\gamma} = L^{(j,vi)}_{(D)\alpha\gamma\beta}, \qquad L^{(vi,k)}_{(D)\alpha\beta\gamma} = L^{(vi,k)}_{(D)\beta\alpha\gamma} \quad (j, k = 1, \ldots, n-1), \\
L^{(vi,vi)}_{\alpha\beta\gamma\delta} = L^{(vi,vi)}_{\alpha\beta\delta\gamma} = L^{(vi,vi)}_{\beta\alpha\gamma\delta} = L^{(vi,vi)}_{\beta\alpha\delta\gamma}.
\end{gathered}$$

From the last symmetry relations, by virtue of Onsager relations $(50)_3$, we have

$$L^{(vi,vi)}_{\alpha\beta\gamma\delta} = L^{(vi,vi)}_{\alpha\beta\delta\gamma} = L^{(vi,vi)}_{\beta\alpha\gamma\delta} = L^{(vi,vi)}_{\beta\alpha\delta\gamma} = L^{(vi,vi)}_{\gamma\delta\alpha\beta} = L^{(vi,vi)}_{\gamma\delta\beta\alpha} = L^{(vi,vi)}_{\delta\gamma\alpha\beta} = L^{(vi,vi)}_{\delta\gamma\beta\alpha}.$$

5 Derivation of the intrinsic entropy production

In this section we work out the intrinsic entropy production $\sigma^{(in)} = T\sigma^{(s)}$ as a bilinear form.

From (27) and the phenomenological equations (32)-(37) we have

$$
\begin{aligned}
\sigma^{(in)} = \Bigg(& \varrho L^{(0,0)}_{(M)\alpha\beta} \frac{dm_\beta}{dt} + L^{(0,1)}_{(M)\alpha\beta} B^{(1)}_\beta - \frac{1}{T} L^{(0,q)}_{(M)\alpha\beta} \frac{\partial T}{\partial x_\beta} \\
& + \sum_{k=1}^{n-1} L^{(0,k)}_{(MD)\alpha\beta} X^{(k)}_\beta + \sum_{h=1}^{r} L^{(0,h)}_{(MC)\alpha} A^{(h)} + L^{(0,vi)}_{(M)\alpha\beta\gamma} \frac{d\varepsilon_{\beta\gamma}}{dt} \Bigg) \varrho \frac{dm_\alpha}{dt} \\
& + \Bigg(\varrho L^{(1,0)}_{(M)\alpha\beta} \frac{dm_\beta}{dt} + L^{(1,1)}_{(M)\alpha\beta} B^{(1)}_\beta - \frac{1}{T} L^{(1,q)}_{(M)\alpha\beta} \frac{\partial T}{\partial x_\beta} \\
& + \sum_{k=1}^{n-1} L^{(1,k)}_{(MD)\alpha\beta} X^{(k)}_\beta + \sum_{h=1}^{r} L^{(1,h)}_{(MC)\alpha} A^{(h)} + L^{(1,vi)}_{(M)\alpha\beta\gamma} \frac{d\varepsilon_{\beta\gamma}}{dt} \Bigg) B^{(1)}_\alpha \\
& - \frac{1}{T} \Bigg(\varrho L^{(q,0)}_{(M)\alpha\beta} \frac{dm_\beta}{dt} + L^{(q,1)}_{(M)\alpha\beta} B^{(1)}_\beta - \frac{1}{T} L^{(q,q)}_{\alpha\beta} \frac{\partial T}{\partial x_\beta} + \sum_{k=1}^{n-1} L^{(q,k)}_{(D)\alpha\beta} X^{(k)}_\beta \\
& + \sum_{h=1}^{r} L^{(q,h)}_{(C)\alpha} A^{(h)} + L^{(q,vi)}_{\alpha\beta\gamma} \frac{d\varepsilon_{\beta\gamma}}{dt} \Bigg) \frac{\partial T}{\partial x_\alpha} \\
& + \sum_{j=1}^{n-1} \Bigg(\varrho L^{(j,0)}_{(DM)\alpha\beta} \frac{dm_\beta}{dt} + L^{(j,1)}_{(DM)\alpha\beta} B^{(1)}_\beta - \frac{1}{T} L^{(j,q)}_{(D)\alpha\beta} \frac{\partial T}{\partial x_\beta} \\
& + \sum_{k=1}^{n-1} L^{(j,k)}_{(DD)\alpha\beta} X^{(k)}_\beta + \sum_{h=1}^{r} L^{(j,h)}_{(DC)\alpha} A^{(h)} + L^{(j,vi)}_{(D)\alpha\beta\gamma} \frac{d\varepsilon_{\beta\gamma}}{dt} \Bigg) X^{(j)}_\alpha \\
& + \sum_{l=1}^{r} \Bigg(\varrho L^{(l,0)}_{(CM)\beta} \frac{dm_\beta}{dt} + L^{(l,1)}_{(CM)\beta} B^{(1)}_\beta - \frac{1}{T} L^{(l,q)}_{(C)\beta} \frac{\partial T}{\partial x_\beta} \\
& + \sum_{k=1}^{n-1} L^{(l,k)}_{(CD)\beta} X^{(k)}_\beta + \sum_{h=1}^{r} L^{(l,h)}_{(CC)} A^{(h)} + L^{(l,vi)}_{(C)\beta\gamma} \frac{d\varepsilon_{\beta\gamma}}{dt} \Bigg) A^{(l)} \\
& + \Bigg(\varrho L^{(vi,0)}_{(M)\alpha\beta\gamma} \frac{dm_\gamma}{dt} + L^{(vi,1)}_{(M)\alpha\beta\gamma} B^{(1)}_\gamma - \frac{1}{T} L^{(vi,q)}_{\alpha\beta\gamma} \frac{\partial T}{\partial x_\gamma} \\
& + \sum_{k=1}^{n-1} L^{(vi,k)}_{(D)\alpha\beta\gamma} X^{(k)}_\gamma + \sum_{h=1}^{r} L^{(vi,h)}_{(C)\alpha\beta} A^{(h)} + L^{(vi,vi)}_{\alpha\beta\gamma\delta} \frac{d\varepsilon_{\gamma\delta}}{dt} \Bigg) \frac{d\varepsilon_{\alpha\beta}}{dt}.
\end{aligned}
\tag{51}
$$

Using Onsager-Casimir relations we obtain

$$
\sigma^{(in)} = \Bigg(\varrho L^{(0,0)}_{(M)\alpha\beta} \frac{dm_\beta}{dt} - \frac{1}{T} L^{(0,q)}_{(M)\alpha\beta} \frac{\partial T}{\partial x_\beta} + \sum_{k=1}^{n-1} L^{(0,k)}_{(MD)\alpha\beta} X^{(k)}_\beta
$$

$$+\sum_{h=1}^{r} L^{(0,h)}_{(MC)\alpha} A^{(h)}\Big) \varrho \frac{dm_\alpha}{dt} + \left(L^{(1,1)}_{(M)\alpha\beta} B^{(1)}_\beta + L^{(1,vi)}_{(M)\alpha\beta\gamma} \frac{d\varepsilon_{\beta\gamma}}{dt}\right) B^{(1)}_\alpha$$

$$\frac{1}{T}\Big(\varrho L^{(q,0)}_{(M)\alpha\beta}\frac{dm_\beta}{dt} - \frac{1}{T} L^{(q,q)}_{\alpha\beta}\frac{\partial T}{\partial x_\beta} + \sum_{k=1}^{n-1} L^{(q,k)}_{(D)\alpha\beta} X^{(k)}_\beta$$

$$+\sum_{h=1}^{r} L^{(q,h)}_{(C)\alpha} A^{(h)}\Big)\frac{\partial T}{\partial x_\alpha} + \sum_{j=1}^{n-1}\Big(\varrho L^{(j,0)}_{(DM)\alpha\beta}\frac{dm_\beta}{dt}$$

$$-\frac{1}{T}L^{(j,q)}_{(D)\alpha\beta}\frac{\partial T}{\partial x_\beta} + \sum_{k=1}^{n-1} L^{(j,k)}_{(DD)\alpha\beta}X^{(k)}_\beta + \sum_{h=1}^{r} L^{(j,h)}_{(DC)\alpha}A^{(h)}\Big) X^{(j)}_\alpha$$

$$+\sum_{l=1}^{r}\Big(\varrho L^{(l,0)}_{(CM)\beta}\frac{dm_\beta}{dt} - \frac{1}{T} L^{(l,q)}_{(C)\beta}\frac{\partial T}{\partial x_\beta} + \sum_{k=1}^{n-1} L^{(l,k)}_{(CD)\beta}X^{(k)}_\beta$$

$$+\sum_{h=1}^{r} L^{(l,h)}_{(CC)}A^{(h)}\Big)A^{(l)} + \left(L^{(vi,1)}_{(M)\alpha\beta\gamma}B^{(1)}_\gamma + L^{(vi,vi)}_{\alpha\beta\gamma\delta}\frac{d\varepsilon_{\gamma\delta}}{dt}\right)\frac{d\varepsilon_{\alpha\beta}}{dt}, \tag{52}$$

and finally

$$\sigma^{(in)} = \varrho^2 L^{(0,0)}_{(M)\alpha\beta}\frac{dm_\beta}{dt}\frac{dm_\alpha}{dt} - 2\varrho\frac{1}{T}L^{(0,q)}_{(M)\alpha\beta}\frac{\partial T}{\partial x_\beta}\frac{dm_\alpha}{dt}$$

$$+2\varrho\sum_{k=1}^{n-1} L^{(0,k)}_{(MD)\alpha\beta}X^{(k)}_\beta\frac{dm_\alpha}{dt} + 2\varrho\sum_{h=1}^{r} L^{(0,h)}_{(MC)\alpha}A^{(h)}\frac{dm_\alpha}{dt} + L^{(1,1)}_{(M)\alpha\beta}B^{(1)}_\beta B^{(1)}_\alpha$$

$$+2L^{(1,vi)}_{(M)\alpha\beta\gamma}\frac{d\varepsilon_{\beta\gamma}}{dt}B^{(1)}_\alpha - \frac{2}{T}\sum_{k=1}^{n-1} L^{(q,k)}_{(D)\alpha\beta}X^{(k)}_\beta\frac{\partial T}{\partial x_\alpha} - \frac{2}{T}\sum_{h=1}^{r} L^{(q,h)}_{(C)\alpha}A^{(h)}\frac{\partial T}{\partial x_\alpha}$$

$$+\frac{1}{T^2}L^{(q,q)}_{\alpha\beta}\frac{\partial T}{\partial x_\alpha}\frac{\partial T}{\partial x_\beta} + 2\sum_{j=1}^{n-1}\sum_{h=1}^{r} L^{(j,h)}_{(DC)\alpha}A^{(h)}X^{(j)}_\alpha + \sum_{j,k=1}^{n-1} L^{(j,k)}_{(DD)\alpha\beta}X^{(k)}_\beta X^{(j)}_\alpha$$

$$+\sum_{l,h=1}^{r} L^{(l,h)}_{(CC)}A^{(h)}A^{(l)} + L^{(vi,vi)}_{\alpha\beta\gamma\delta}\frac{d\varepsilon_{\gamma\delta}}{dt}\frac{d\varepsilon_{\alpha\beta}}{dt}. \tag{53}$$

Relation (53) shows that the entropy production is a quadratic form in the components of the time derivative of the total specific magnetization axial vector $\frac{d\mathbf{m}}{dt}$, the components of the thermodynamic force conjugate to the partial specific magnetization axial vector $\mathbf{B}^{(1)}$, the k-th thermodynamic force $\mathbf{X}^{(k)}$ conjugate to the k-th diffusion flux $\mathbf{J}^{(k)}_{(diff)}$ $(k = 1, \ldots, n-1)$, the chemical affinity of h-th reaction A^h $(h = 1, \ldots, r)$, the components of the temperature gradient, and the components of the time derivative of the strain tensor. The entropy production is a positive definite quadratic form, i.e.

$$\sigma^{(s)} \geq 0. \tag{54}$$

From the positive definite character of the entropy production several inequalities may be derived for the components of the phenomenological coefficients, resulting from the fact that all the elements of the main diagonal of the matrix associated to the quadratic form (53) must be non-negative and all principal minors of this matrix must be non-negative (see [38], [39] and [35], as examples in the case of three-dimensional isotropic and anisotropic rigid media and isotropic magnetizable media). For instance, we have

$$\begin{aligned} &L^{(0,0)}_{(M)\alpha\alpha} \geq 0, \quad L^{(1,1)}_{(M)\alpha\alpha} \geq 0, \\ &L^{(q,q)}_{\alpha\alpha} \geq 0, \quad L^{(k,k)}_{(DD)\alpha\alpha} \geq 0 \quad (k = 1, \dots, n-1), \quad L^{(vi,vi)}_{\alpha\beta\alpha\beta} \geq 0. \end{aligned} \tag{55}$$

Also, from the fifth of the inequalities we obtain, by virtue of symmetry and Onsager-Casimir relations,

$$L^{(vi,vi)}_{\alpha\beta\beta\alpha} \geq 0, \quad L^{(vi,vi)}_{\beta\alpha\alpha\beta} \geq 0, \quad L^{(vi,vi)}_{\beta\alpha\beta\alpha} \geq 0. \tag{56}$$

6 Linear equations of state for anisotropic reacting fluid mixtures with magnetic relaxation

In this Section we give a review of the results obtained in the paper [7]. In order to obtain the linear equations of state we use the specific free energy f as thermodynamic potential, defined by

$$f = u - Ts. \tag{57}$$

Using the Gibbs relation (19), we obtain the following expression for the differential of f,

$$df = -sdT + v\tau^{(eq)}_{\alpha\beta} d\varepsilon_{\alpha\beta} + \mathbf{B}^{(eq)} \cdot d\mathbf{m} - \mathbf{B}^{(1)} \cdot d\mathbf{m}^{(1)} + \sum_{k=1}^{n} \mu^{(k)} dc^{(k)}. \tag{58}$$

Therefore, the following definitions are valid

$$s = -\frac{\partial}{\partial T} f\left(T, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}, c^{(1)}, \dots, c^{(n)}\right), \tag{59}$$

$$\tau^{(eq)}_{\alpha\beta} = \varrho \frac{\partial}{\partial \varepsilon_{\alpha\beta}} f\left(T, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right), \tag{60}$$

$$\mathbf{B}^{(eq)} = \frac{\partial}{\partial \mathbf{m}} f\left(T, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right), \tag{61}$$

$$\mathbf{B}^{(1)} = -\frac{\partial}{\partial \mathbf{m}^{(1)}} f\left(T, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right), \tag{62}$$

$$\mu^{(k)} = \frac{\partial}{\partial c^{(k)}} f\left(T, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right) \quad (k = 1, \ldots, n). \tag{63}$$

Let us consider a reference state of the medium (indicated by the symbol "$_{(0)}$") and we also require that this reference state is a state of thermodynamic equilibrium. We assume that in this state there is an uniform temperature, that has an arbitrary (but fixed) value $T_{(0)}$, the concentrations $c^{(k)}$ $(k = 1, \ldots, n)$ of the n fluid components have fixed values $c^{(k)}_{(0)}$ and the values of the strain tensor $\varepsilon_{(0)\alpha\beta}$, the $\mathbf{m}_{(0)}$ and $\mathbf{m}^{(1)}_{(0)}$ are null. Then, we assume that the mechanical stress tensor $\tau_{\alpha\beta}$, the magnetic field $\mathbf{B}$, $\mathbf{B}^{(1)}$ and the $\mu^{(k)}$ vanish in this state, i.e. we assume that

$$\tau^{(eq)}_{\alpha\beta}\left(T_{(0)}, \varepsilon_{(0)\alpha\beta}, \mathbf{m}_{(0)}, \mathbf{m}^{(1)}_{(0)}, c^{(1)}_{(0)}, \ldots, c^{(n)}_{(0)}\right) = 0, \tag{64}$$

$$\mathbf{B}^{(eq)}\left(T_{(0)}, \varepsilon_{(0)\alpha\beta}, \mathbf{m}_{(0)}, \mathbf{m}^{(1)}_{(0)}, c^{(1)}_{(0)}, \ldots, c^{(n)}_{(0)}\right) = \mathbf{0}, \tag{65}$$

$$\mathbf{B}^{(1)}\left(T_{(0)}, \varepsilon_{(0)\alpha\beta}, \mathbf{m}_{(0)}, \mathbf{m}^{(1)}_{(0)}, c^{(1)}_{(0)}, \ldots, c^{(n)}_{(0)}\right) = \mathbf{0}, \tag{66}$$

$$\mu^{(k)}\left(T_{(0)}, \varepsilon_{(0)\alpha\beta}, \mathbf{m}_{(0)}, \mathbf{m}^{(1)}_{(0)}, c^{(1)}_{(0)}, \ldots, c^{(n)}_{(0)}\right) = 0 \quad (k = 1, \ldots, n). \tag{67}$$

if

$$\begin{gathered} T = T_{(0)}, \quad c^{(k)} = c^{(k)}_{(0)} \quad (k = 1, \ldots, n), \\ \varepsilon_{(0)\alpha\beta} = 0, \quad \mathbf{m}_{(0)} = \mathbf{0}, \quad \mathbf{m}^{(1)}_{(0)} = \mathbf{0}. \end{gathered} \tag{68}$$

Let us expand the free energy f into Taylor's series with respect to the considered reference state and we consider very small deviations with respect to this state.

We postulate the following form for the specific free energy f

$$f = f^{(1)} + f^{(2)}, \tag{69}$$

where

$$f^{(1)} = v_{(0)} \left[\frac{1}{2} a_{\alpha\beta\gamma\zeta} \varepsilon_{\alpha\beta} \varepsilon_{\gamma\zeta} + a_{\alpha\beta} \varepsilon_{\alpha\beta} (T - T_{(0)}) \right.$$

$$+\sum_{k=1}^{n} b^{(k)}\left(c^{(k)}-c^{(k)}_{(0)}\right)(T-T_0)+\sum_{k=1}^{n} b^{(k)}_{\alpha\beta}\left(c^{(k)}-c^{(k)}_{(0)}\right)\varepsilon_{\alpha\beta}$$

$$\left.+\frac{1}{2}\sum_{i,k=1}^{n} b^{(i,k)}\left(c^{(i)}-c^{(i)}_{(0)}\right)\left(c^{(k)}-c^{(k)}_{(0)}\right)\right]-\varphi(T) \tag{70}$$

and

$$f^{(2)}=\frac{1}{2}\varrho_{(0)}\left[a^{(0,0)}_{(M)\alpha\beta}m_\alpha\left(m_\beta-2m^{(1)}_\beta\right)+a^{(1,1)}_{(M)\alpha\beta}\,m^{(1)}_\alpha m^{(1)}_\beta\right]+$$

$$+\left(a^{(0)}_{(M)\alpha}m_\alpha-a^{(1)}_{(M)\alpha}m^{(1)}_\alpha\right)\left(T-T_{(0)}\right)$$

$$+\sum_{k=1}^{n}\left(b^{(0,k)}_{(M)\alpha}m_\alpha+b^{(1,k)}_{(M)\alpha}m^{(1)}_\alpha\right)\left(c^{(k)}-c^{(k)}_{(0)}\right). \tag{71}$$

In (70) $\varphi(T)$ is some function of the temperature, $v_{(0)}$ is the specific volume in the reference state, defined by $v_{(0)}=\dfrac{1}{\varrho_{(0)}}$, that in the following will be replaced by $v=\dfrac{1}{\varrho}$, supposed constant. Furthermore, in (70) the scalars $b^{(k)}$ and the tensors $a_{\alpha\beta\gamma\zeta}$, $a_{\alpha\beta}$, $b^{(k)}_{\alpha\beta}$, $b^{(i,k)}$ are constant and satisfy the following symmetry relations

$$a_{\alpha\beta\gamma\zeta}=a_{\beta\alpha\gamma\zeta}=a_{\alpha\beta\zeta\gamma}=a_{\beta\alpha\zeta\gamma}=a_{\gamma\zeta\alpha\beta}=a_{\gamma\zeta\beta\alpha}=a_{\zeta\gamma\alpha\beta}=a_{\zeta\gamma\beta\alpha},$$

$$a_{\alpha\beta}=a_{\beta\alpha},\quad b^{(k)}_{\alpha\beta}=b^{(k)}_{\beta\alpha},\quad b^{(i,k)}=b^{(k,i)}\quad (i,k=1,\dots,n). \tag{72}$$

In (71) the tensors $a^{(0,0)}_{(M)\alpha\beta}$, $a^{(1,1)}_{(M)\alpha\beta}$ are constant and satisfy the following symmetry relations

$$a^{(0,0)}_{(M)\alpha\beta}=a^{(0,0)}_{(M)\beta\alpha},\quad a^{(1,1)}_{(M)\alpha\beta}=a^{(1,1)}_{(M)\beta\alpha} \tag{73}$$

and the vector components $a^{(0)}_{(M)\alpha}$, $a^{(1)}_{(M)\alpha}$, $b^{(0,k)}_{(M)\alpha}$, $b^{(1,k)}_{(M)\alpha}$ $(k=1,\dots,n)$ are constant. The symmetry properties of the above phenomenological coefficients come from the physical interpretation of the second partial derivatives of the free energy with respect to the considered independent variables, with their properties of invariance respect to the priority of derivation with respect to the considered variables. Furthermore, the symmetry of the small strain tensor $\varepsilon_{\alpha\beta}$ is taken into consideration. All these constants are determined by the physical properties of the medium in the reference state.

From (60) and (69)-(71) we have the following state equation for the equilibrium stress tensor $\tau^{(eq)}_{\alpha\beta}$

$$\tau^{(eq)}_{\alpha\beta} = a_{\alpha\beta\gamma\zeta}\varepsilon_{\gamma\zeta} + a_{\alpha\beta}(T - T_{(0)}) + \sum_{k=1}^{n} b^{(k)}_{\alpha\beta}\left(c^{(k)} - c^{(k)}_{(0)}\right). \tag{74}$$

Finally, from (61), (62), (63) and (69)-(71) we have the following equations of state

$$B^{(eq)}_{\alpha} = a^{(0,0)}_{(M)\alpha\beta}\left(M_\beta - M^{(1)}_\beta\right) + a^{(0)}_{(M)\alpha}(T - T_{(0)}) + \sum_{k=1}^{n} b^{(0,k)}_{(M)\alpha}\left(c^{(k)} - c^{(k)}_{(0)}\right), \tag{75}$$

$$B^{(1)}_{\alpha} = a^{(0,0)}_{(M)\alpha\beta}M_\beta - a^{(1,1)}_{(M)\alpha\beta}M^{(1)}_\beta + a^{(1)}_{(M)\alpha}(T - T_{(0)}) + \sum_{k=1}^{n} b^{(1,k)}_{(M)\alpha}\left(c^{(k)} - c^{(k)}_{(0)}\right), \tag{76}$$

$$\mu^{(k)} = v\Bigg[b^{(k)}(T - T_0) + b^{(k)}_{\alpha\beta}\varepsilon_{\alpha\beta} + \sum_{i=1}^{n} b^{(i,k)}\left(c^{(i)} - c^{(i)}_{(0)}\right) + b^{(0,k)}_{(M)\alpha}M_\alpha - b^{(1,k)}_{(M)\alpha}M^{(1)}_\alpha\Bigg], \tag{77}$$

where we have defined the fields $\mathbf{M}^{(0)}$ and $\mathbf{M}^{(1)}$ as

$$\mathbf{M}^{(0)} = \varrho\mathbf{m}^{(0)}, \quad \mathbf{M}^{(1)} = \varrho\mathbf{m}^{(1)}. \tag{78}$$

If in equations (32)-(37) all cross effects are neglected, except for possible interactions among the different types of magnetic relaxation phenomena, taking into consideration the state equations, we obtain the following equations for the irreversible magnetic relaxation phenomena, the stress tensor, the electric flux and the heat flux, respectively

$$B_\alpha = B^{(eq)}_\alpha + L^{(0,0)}_{(M)\alpha\beta}\frac{dM_\beta}{dt} + L^{(0,1)}_{(M)\alpha\beta}B^{(1)}_\beta, \tag{79}$$

$$\frac{dM^{(1)}_\alpha}{dt} = L^{(1,0)}_{(M)\alpha\beta}\frac{dM_\beta}{dt} + L^{(1,1)}_{(M)\alpha\beta}B^{(1)}_\beta, \tag{80}$$

$$\tau_{\alpha\beta} = a_{\alpha\beta\gamma\zeta}\varepsilon_{\gamma\zeta} + a_{\alpha\beta}(T - T_{(0)}) + \sum_{k=1}^{n} b^{(k)}_{\alpha\beta}\left(c^{(k)} - c^{(k)}_{(0)}\right) + L^{(vi,vi)}_{\alpha\beta\gamma\zeta}\frac{d\varepsilon_{\gamma\zeta}}{dt} \tag{81}$$

(being $\tau^{(vi)}_{\alpha\beta} = \tau_{\alpha\beta} - \tau^{(eq)}_{\alpha\beta}$),

$$J^{(q)}_\alpha = -\frac{1}{T}L^{(q,q)}_{\alpha\beta}\frac{\partial T}{\partial x_\beta}, \quad J^{(j)}_{(diff)\alpha} = \sum_{k=1}^{n-1} L^{(j,k)}_{(DD)\alpha\beta}X^{(k)}_\beta \quad (j, k = 1, \ldots, n-1),$$

$$J^{(l)}_{(chem)} = \sum_{h=1}^{r} L^{(l,h)}_{(CC)} A^{(h)} \quad (l = 1, \ldots, r). \tag{82}$$

Taking into account (75), (76), equations (79) and (80) may be written, respectively, in the form (see [7])

$$c^{(1)}_{\alpha\beta} M^{(1)}_{\beta} = Q^{(1)}_{(0,0)\alpha}, \tag{83}$$

where

$$c^{(1)}_{\alpha\beta} = a^{(0,0)}_{(M)\alpha\beta} + L^{(0,1)}_{(M)_{\alpha\gamma}} a^{(1,1)}_{(M)\gamma\beta}, \tag{84}$$

$$\begin{aligned} Q^{(1)}_{(0,0)\alpha} = &\left(a^{(0,0)}_{(M)\alpha\beta} + L^{(0,1)}_{(M)\alpha\gamma} a^{(0,0)}_{(M)\gamma\beta}\right) M_{\beta} + L^{(0,0)}_{(M)\alpha\beta} \frac{dM_{\beta}}{dt} - B_{\alpha} \\ &+ \left(a^{(0)}_{(M)\alpha} + L^{(0,1)}_{(M)\alpha\beta} a^{(1)}_{(M)\beta}\right)(T - T_{(0)}) \\ &+ \sum_{k=1}^{n} \left(b^{(0,k)}_{(M)\alpha} + L^{(0,1)}_{(M)\alpha\beta} b^{(1,k)}_{(M)\beta}\right)\left(c^{(k)} - c^{(k)}_{(0)}\right) \end{aligned} \tag{85}$$

and

$$\frac{dM^{(1)}_{\beta}}{dt} + h_{\beta\gamma} M^{(1)}_{\gamma} = Q_{(1,0)\beta}, \tag{86}$$

where

$$h_{\beta\gamma} = L^{(1,1)}_{(M)\beta\eta} a^{(1,1)}_{(M)\eta\gamma} \tag{87}$$

and

$$\begin{aligned} Q_{(1,0)\beta} = &L^{(1,1)}_{(M)\beta\eta} a^{(0,0)}_{(M)\eta\gamma} M_{\gamma} + L^{(1,0)}_{(M)\beta\gamma} \frac{dM_{\gamma}}{dt} \\ &+ L^{(1,1)}_{(M)\beta\gamma} a^{(1)}_{(M)\gamma}(T - T_{(0)}) + \sum_{k=1}^{n} L^{(1,1)}_{(M)\beta\gamma} b^{(1,k)}_{(M)\gamma}\left(c^{(k)} - c^{(k)}_{(0)}\right). \end{aligned} \tag{88}$$

Assuming that it is possible to define the inverse matrix $\left(c^{(1)}_{\alpha\beta}\right)^{-1}$, such that

$$\left(c^{(1)}_{\alpha\beta}\right)^{-1} c^{(1)}_{\beta\gamma} = c^{(1)}_{\alpha\beta}\left(c^{(1)}_{\beta\gamma}\right)^{-1} = \delta_{\alpha\gamma}, \tag{89}$$

the partial magnetization field $\mathbf{M}^{(1)}$ is given by (see (84)) and (85)

$$M^{(1)}_{\alpha} = \left(c^{(1)}_{\alpha\beta}\right)^{-1} Q^{(1)}_{(0,0)\beta}. \tag{90}$$

In [7] from (83) and (86), using (90), the magnetic relaxation equation for anisotropic reacting fluid mixtures under consideration was obtained, having the form of a linear relation among the components of the magnetic field, the components of the total magnetization, the temperature, the concentrations of the n fluid components, the first time derivative of the components of the magnetic field, of the temperature, of the concentrations of the n fluid components, of the total magnetization field, and the second derivative with respect to time of this last axial vector.

7 Derivation of the field $\mathbf{B}^{(1)}$

In this Section we work out the expression for the field $\mathbf{B}^{(1)}$, conjugate to the internal variable $\mathbf{m}^{(1)}$.

Using (90) and (85) we see that $\mathbf{B}^{(1)}$ can be written as follows

$$\begin{aligned} B_\alpha^{(1)} = {} & a^{(0,0)}_{(M)\alpha\beta} M_\beta - a^{(1,1)}_{(M)\alpha\beta} \left(c^{(1)}_{\beta\gamma}\right)^{-1} \Bigg[\left(a^{(0,0)}_{(M)\gamma\zeta} + L^{(0,1)}_{(M)\gamma\eta} a^{(0,0)}_{(M)\eta\zeta}\right) M_\zeta \\ & + L^{(0,0)}_{(M)\gamma\zeta} \frac{dM_\zeta}{dt} - B_\gamma + \left(a^{(0)}_{(M)\gamma} + L^{(0,1)}_{(M)\gamma\zeta} a^{(1)}_{(M)\zeta}\right)(T - T_0) \\ & + \sum_{k=1}^{n} \left(b^{(0,k)}_{(M)\gamma} + L^{(0,1)}_{(M)\gamma\zeta} b^{(1,k)}_{(M)\zeta}\right)\left(c^{(k)} - c^{(k)}_{(0)}\right) \Bigg] \\ & + a^{(1)}_{(M)\alpha}(T - T_0) + \sum_{k=1}^{n} b^{(1,k)}_{(M)\alpha}\left(c^{(k)} - c^{(k)}_{(0)}\right), \end{aligned} \tag{91}$$

where $c^{(0)}_{\alpha\beta}$ is defined by

$$c^{(0)}_{\alpha\beta} = a^{(0,0)}_{(M)\alpha\beta} + L^{(0,1)}_{(M)\alpha\gamma} a^{(0,0)}_{(M)\gamma\beta} \tag{92}$$

and $c^{(1)}_{\alpha\beta}$ is given by (84). We have

$$\begin{aligned} B_\alpha^{(1)} = {} & a^{(0,0)}_{(M)\alpha\beta} \left(c^{(1)}_{\beta\gamma}\right)^{-1} c^{(1)}_{\gamma\zeta} M_\zeta - a^{(1,1)}_{(M)\alpha\beta} \left(c^{(1)}_{\beta\gamma}\right)^{-1} \left(a^{(0,0)}_{(M)\gamma\zeta} + L^{(0,1)}_{(M)\gamma\eta} a^{(0,0)}_{(M)\eta\zeta}\right) M_\zeta \\ & - a^{(1,1)}_{(M)\alpha\beta} \left(c^{(1)}_{\beta\gamma}\right)^{-1} L^{(0,0)}_{(M)\gamma\zeta} \frac{dM_\zeta}{dt} + a^{(1,1)}_{(M)\alpha\beta} \left(c^{(1)}_{\beta\gamma}\right)^{-1} B_\gamma \\ & - a^{(1,1)}_{(M)\alpha\beta} \left(c^{(1)}_{\beta\gamma}\right)^{-1} \left(a^{(0)}_{(M)\gamma} + L^{(0,1)}_{(M)\gamma\eta} a^{(1)}_{(M)\eta}\right)(T - T_0) + a^{(1)}_{(M)\alpha}(T - T_0) \\ & - a^{(1,1)}_{(M)\alpha\beta} \left(c^{(1)}_{\beta\gamma}\right)^{-1} \sum_{k=1}^{n} \left(b^{(0,k)}_{(M)\gamma} + L^{(0,1)}_{(M)\gamma\eta} b^{(1,k)}_{(M)\eta}\right)\left(c^{(k)} - c^{(k)}_{(0)}\right) \\ & + \sum_{k=1}^{n} b^{(1,k)}_{(M)\alpha}\left(c^{(k)} - c^{(k)}_{(0)}\right), \end{aligned} \tag{93}$$

i.e.

$$B_\alpha^{(1)} = D^{(1)}_{(M)\alpha\zeta} M_\zeta + D^{(2)}_{(M)\alpha\zeta} \frac{dM_\zeta}{dt} + D^{(3)}_{(M)\alpha\zeta} B_\zeta$$

$$+D^{(4)}_{(M)\alpha}(T-T_0)+\sum_{k=1}^{n} D^{(5,k)}_{(M)\alpha}\left(c^{(k)}-c^{(k)}_{(0)}\right), \tag{94}$$

where

$$D^{(1)}_{(M)\alpha\zeta}=\left(c^{(1)}_{\beta\gamma}\right)^{-1}\left[a^{(0,0)}_{(M)\alpha\beta}c^{(1)}_{\gamma\zeta}-a^{(1,1)}_{(M)\alpha\beta}\left(a^{(0,0)}_{(M)\gamma\zeta}+L^{(0,1)}_{(M)\gamma\eta}a^{(0,0)}_{(M)\eta\zeta}\right)\right], \tag{95}$$

$$D^{(2)}_{(M)\alpha\zeta}=-\left(c^{(1)}_{\beta\gamma}\right)^{-1}a^{(1,1)}_{(M)\alpha\beta}L^{(0,0)}_{(M)\gamma\zeta}, \tag{96}$$

$$D^{(3)}_{(M)\alpha\zeta}=\left(c^{(1)}_{\beta\zeta}\right)^{-1}a^{(1,1)}_{(M)\alpha\beta}. \tag{97}$$

$$D^{(4)}_{(M)\alpha}=-\left(c^{(1)}_{\beta\gamma}\right)^{-1}a^{(1,1)}_{(M)\alpha\beta}\left(a^{(0)}_{(M)\gamma}+L^{(0,1)}_{(M)\gamma\eta}a^{(1)}_{(M)\eta}\right)+a^{(1)}_{(M)\alpha}, \tag{98}$$

$$D^{(5,k)}_{(M)\alpha}=-\left(c^{(1)}_{\beta\gamma}\right)^{-1}a^{(1,1)}_{(M)\alpha\beta}\left(b^{(0,k)}_{(M)\gamma}+L^{(0,1)}_{(M)\gamma\eta}b^{(1,k)}_{(M)\eta}\right)+b^{(1,k)}_{(M)\alpha}. \tag{99}$$

8 First law of thermodynamics

In this section we work out the first law of thermodynamics for the anisotropic reacting fluid mixtures under consideration.

From (57) and (69)-(71) we have

$$u=f+Ts=f^{(1)}+f^{(2)}+Ts. \tag{100}$$

From (59) we have the following form for the specific entropy s

$$s=-\left[a^{(0)}_{(M)\alpha}m_\alpha-a^{(1)}_{(M)\alpha}m^{(1)}_\alpha+\sum_{k=1}^{n}b^{(k)}\left(c^{(k)}-c^{(k)}_{(0)}\right)\right]-va_{\alpha\beta}\varepsilon_{\alpha\beta}+\frac{d\varphi}{dT}. \tag{101}$$

Thus, from (100) we obtain

$$\begin{aligned}u=v\Bigg[&\frac{1}{2}a_{\alpha\beta\gamma\zeta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\zeta}-T_{(0)}a_{\alpha\beta}\varepsilon_{\alpha\beta}-T_{(0)}\sum_{k=1}^{n}b^{(k)}\left(c^{(k)}-c^{(k)}_{(0)}\right)\\ &+\sum_{k=1}^{n}b^{(k)}_{\alpha\beta}\left(c^{(k)}-c^{(k)}_{(0)}\right)\varepsilon_{\alpha\beta}+\frac{1}{2}\sum_{i,k=1}^{n}b^{(i,k)}\left(c^{(i)}-c^{(i)}_{(0)}\right)\left(c^{(k)}-c^{(k)}_{(0)}\right)\Bigg]\\ &+\frac{1}{2}\varrho\left[a^{(0,0)}_{(M)\alpha\beta}\,m_\alpha\left(m_\beta-2m^{(1)}_\beta\right)+a^{(1,1)}_{(M)\alpha\beta}\,m^{(1)}_\alpha m^{(1)}_\beta\right]\\ &-\varphi(T)-T_{(0)}\left(a^{(0)}_{(M)\alpha}\,m_\alpha-a^{(1)}_{(M)\alpha}m^{(1)}_\alpha\right)+T\frac{d\varphi}{dT}\\ &\qquad+\sum_{k=1}^{n}\left(c^{(k)}-c^{(k)}_{(0)}\right)\left(b^{(0,k)}_{(M)\alpha}m_\alpha+b^{(1,k)}_{(M)\alpha}m^{(1)}_\alpha\right),\end{aligned} \tag{102}$$

i.e.,

$$\begin{aligned}
u = \frac{1}{2}\varrho & \left[a^{(0,0)}_{(M)\alpha\beta}\, m_\alpha \left(m_\beta - 2m^{(1)}_\beta\right) + a^{(1,1)}_{(M)\alpha\beta}\, m^{(1)}_\alpha m^{(1)}_\beta\right] \\
& - T_{(0)} \left(a^{(0)}_{(M)\alpha}\, m_\alpha - a^{(1)}_{(M)\alpha} m^{(1)}_\alpha\right) \\
& + v \left(\frac{1}{2} a_{\alpha\beta\gamma\zeta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\zeta} - T_{(0)} a_{\alpha\beta}\varepsilon_{\alpha\beta}\right) + T\frac{d\varphi}{dT} - \varphi(T) \\
& - vT_{(0)} \sum_{k=1}^{n} b^{(k)} \left(c^{(k)} - c^{(k)}_{(0)}\right) + v_{(0)} \sum_{k=1}^{n} b^{(k)}_{\alpha\beta} \left(c^{(k)} - c^{(k)}_{(0)}\right) \varepsilon_{\alpha\beta} \\
& + \frac{v}{2} \sum_{i,k=1}^{n} b^{(i,k)} \left(c^{(i)} - c^{(i)}_{(0)}\right) \left(c^{(k)} - c^{(k)}_{(0)}\right) \\
& \qquad + \sum_{k=1}^{n} \left(c^{(k)} - c^{(k)}_{(0)}\right) \left(b^{(0,k)}_{(M)\alpha} m_\alpha + b^{(1,k)}_{(M)\alpha} m^{(1)}_\alpha\right). \qquad (103)
\end{aligned}$$

The specific heat at constant deformation $c(\varepsilon)$ can be defined by

$$c(\varepsilon) = \frac{\partial u}{\partial T} \left(T, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right), \qquad (104)$$

$$c(\varepsilon) = \frac{d}{dT}\left(T\frac{d\varphi}{dT} - \varphi(T)\right) = \frac{d}{dT}\left(T\frac{d\varphi}{dT}\right) - \frac{d\varphi}{dT}, \qquad (105)$$

i.e.

$$c(\varepsilon) = T\frac{d^2\varphi}{dT^2}.$$

If $c(\varepsilon)$ is constant one obtains the result

$$\varphi = c(\varepsilon) T \log\frac{T}{T_{(0)}} + s_{(0)} T - c(\varepsilon)(T - T_{(0)}) - u_{(0)}, \qquad (106)$$

where $s_{(0)} = \left(\frac{d\varphi}{dT}\right)_{T=T_{(0)}}$ (see (101)) and $u_{(0)} = T_{(0)} \left(\frac{d\varphi}{dT}\right)_{T=T_{(0)}} - \varphi(T_{(0)})$ (see (103)) are the specific entropy and the specific internal energy in the reference state respectively.

Thus, $\varphi(T_{(0)}) = T_{(0)} s_{(0)} - u_{(0)}$.

The result (106) was derived in [40] but without demonstration.

Indeed, from equation $(105)_1$, one has

$$T\frac{d\varphi}{dT} - \varphi - c(\varepsilon)T = A_{(0)} \qquad (107)$$

and taking into account that $\dfrac{d}{dT}\left(\dfrac{\varphi}{T}\right)=\dfrac{1}{T}\left(\dfrac{d\varphi}{dT}\right)-\dfrac{\varphi}{T^2}$,

$$\frac{d\varphi}{dT}=T\frac{d}{dT}\left(\frac{\varphi}{T}\right)+\frac{\varphi}{T}. \tag{108}$$

From (107) one gets

$$\frac{d\varphi}{dT}-\frac{\varphi}{T}-c(\varepsilon)=\frac{A_{(0)}}{T}, \tag{109}$$

from which, by virtue of (108), one has

$$T\frac{d}{dT}\left(\frac{\varphi}{T}\right)-c(\varepsilon)=\frac{A_{(0)}}{T}, \tag{110}$$

$$\frac{d}{dT}\left(\frac{\varphi}{T}\right)-\frac{c(\varepsilon)}{T}=\frac{A_{(0)}}{T^2},\quad \frac{\varphi}{T}=c(\varepsilon)\log T-\frac{A_{(0)}}{T}+B_{(0)}, \tag{111}$$

$$\varphi=Tc(\varepsilon)\log T-A_{(0)}+B_{(0)}T, \tag{112}$$

with $A_{(0)}$ and $B_{(0)}$ integration constants calculated in the reference state for $T=T_{(0)}$. From (109) and $(111)_2$ $A_{(0)}=u_{(0)}-c(\varepsilon)T_{(0)}$,
$B_{(0)}=s_{(0)}-c(\varepsilon)\log T_{(0)}-c(\varepsilon)T_{(0)}$ and, thus, from (112) the result (106) is obtained.

Then, the first law of thermodynamics (16) becomes

$$\begin{aligned}
&\frac{1}{2}a^{(0,0)}_{(M)\alpha\beta}\frac{d}{dt}\left[M_\alpha\left(M_\beta-2M^{(1)}_\beta\right)\right]+a^{(1,1)}_{(M)\alpha\beta}M^{(1)}_\alpha\frac{d}{dt}M^{(1)}_\beta\\
&\quad-T_{(0)}\left(a^{(0)}_{(M)\alpha}\frac{d}{dt}M_\alpha-a^{(1)}_{(M)\alpha}\frac{d}{dt}M^{(1)}_\alpha\right)\\
&\quad+\frac{1}{2}a_{\alpha\beta\gamma\zeta}\frac{d}{dt}(\varepsilon_{\alpha\beta}\varepsilon_{\gamma\zeta})-T_{(0)}a_{\alpha\beta}\frac{d}{dt}\varepsilon_{\alpha\beta}+\varrho c_{(\varepsilon)}\frac{d}{dt}T\\
&\quad-T_{(0)}\sum_{k=1}^{n}b^{(k)}\frac{d}{dt}c^{(k)}+\sum_{k=1}^{n}b^{(k)}_{\alpha\beta}\frac{d}{dt}\left[\left(c^{(k)}-c^{(k)}_{(0)}\right)\varepsilon_{\alpha\beta}\right]\\
&\quad+\frac{1}{2}\sum_{i,k=1}^{n}b^{(i,k)}\frac{d}{dt}\left[\left(c^{(i)}-c^{(i)}_{(0)}\right)\left(c^{(k)}-c^{(k)}_{(0)}\right)\right]\\
&\quad+\sum_{k=1}^{n}\frac{d}{dt}\left[\left(c^{(k)}-c^{(k)}_{(0)}\right)\left(b^{(0,k)}_{(M)\alpha}M_\alpha+b^{(1,k)}_{(M)\alpha}M^{(1)}_\alpha\right)\right]=\\
&=\frac{\partial}{\partial x_\alpha}\left(T^{-1}L^{(q,q)}_{\alpha\beta}\frac{\partial T}{\partial x_\beta}\right)+\tau^{(vi)}_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt}+J^{(el)}_\alpha E_\alpha+B_\alpha\frac{dM_\alpha}{dt},
\end{aligned}$$

where we have supposed that $J^{(el)}_\alpha$, $J^{(q)}_\alpha$, $\tau^{(eq)}_{\alpha\beta}$ are given by (81) and (82).

9 Heat conduction equation and the heat dissipation function

In this Section we work out the heat conduction equation and the heat dissipation function for the media under consideration.

From (101) we have

$$\varrho\frac{ds}{dt} = -a^{(0)}_{(M)\alpha}\frac{dM_\alpha}{dt} + a^{(1)}_{(M)\alpha}\frac{dM^{(1)}_\alpha}{dt}$$

$$-\sum_{k=1}^{n} b^{(k)}\varrho\left(c^{(k)} - c^{(k)}_{(0)}\right) - a_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt} + \varrho\frac{d^2\varphi}{dT^2}\frac{dT}{dt} \qquad (113)$$

By using (105) we obtain

$$\varrho T\frac{ds}{dt} = T\left[-a^{(0)}_{(M)\alpha}\frac{dM_\alpha}{dt} + a^{(1)}_{(M)\alpha}\frac{dM^{(1)}_\alpha}{dt}\right.$$

$$\left.-\sum_{k=1}^{n} b^{(k)}\varrho\left(c^{(k)} - c^{(k)}_{(0)}\right) - a_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt}\right] + \varrho c_{(\varepsilon)}\frac{dT}{dt}. \qquad (114)$$

From the balance equation (26) we have

$$\varrho T\frac{ds}{dt} = -T\nabla\cdot\mathbf{J}^{(s)} + T\sigma^{(s)} =$$

$$-\nabla\cdot\mathbf{J}^{(q)} + \frac{1}{T}\mathbf{J}^{(q)}\cdot\nabla T + T\nabla\cdot\left(\frac{1}{T}\sum_{k=1}^{n}\mu^{(k)}\mathbf{J}^{(k)}_{(diff)}\right) + T\sigma^{(s)}, \qquad (115)$$

being $\mathbf{J}^{(s)}$ and $\sigma^{(s)}$ given by (22) and (27).

Taking into consideration equation (82), in components (115) takes the following form

$$\varrho T\frac{ds}{dt} = \frac{\partial}{\partial x_\alpha}\left(T^{-1}L^{(q,q)}_{\alpha\beta}\frac{\partial T}{\partial x_\beta}\right) - T^{-2}L^{(q,q)}_{\alpha\beta}\frac{\partial T}{\partial x_\alpha}\frac{\partial T}{\partial x_\beta}$$

$$+T\frac{\partial}{\partial x_\alpha}\left(T^{-1}\sum_{k=1}^{n}\mu^{(k)}J^{(k)}_{(diff)\alpha}\right) + T\sigma^{(s)}. \qquad (116)$$

Thus, comparing (114) and (116) we have

$$\varrho c(\varepsilon)\frac{dT}{dt} = T\left[a^{(0)}_{(M)\alpha}\frac{dM_\alpha}{dt} - a^{(1)}_{(M)\alpha}\frac{dM^{(1)}_\alpha}{dt} + a_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt} + \sum_{k=1}^{n} b^{(k)}\varrho\left(c^{(k)} - c^{(k)}_{(0)}\right)\right]$$

$$+\frac{\partial}{\partial x_\alpha}\left(T^{-1}L^{(q,q)}_{\alpha\beta}\frac{\partial T}{\partial x_\beta}\right)+T\frac{\partial}{\partial x_\alpha}\left(T^{-1}\sum_{k=1}^{n}\mu^{(k)}J^{(k)}_{(diff)\alpha}\right)+\tau^{(vi)}_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt}$$

$$+B^{(ir)}_\alpha\frac{dM_\alpha}{dt}+B^{(1)}_\alpha\frac{dM^{(1)}_\alpha}{dt}+\sum_{h=1}^{r}A^{(h)}J^{(h)}+\sum_{k=1}^{n-1}\mathbf{J}^{(k)}_{(diff)}\cdot\mathbf{X}^{(k)}, \tag{117}$$

i.e.,

$$\varrho c(\varepsilon)\frac{dT}{dt}=T\left[a^{(0)}_{(M)\alpha}\frac{dM_\alpha}{dt}-a^{(1)}_{(M)\alpha}\frac{dM^{(1)}_\alpha}{dt}+a_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt}+\sum_{k=1}^{n}b^{(k)}\varrho_{(0)}\left(c^{(k)}-c^{(k)}_{(0)}\right)\right]$$

$$+\frac{\partial}{\partial x_\alpha}\left(T^{-1}L^{(q,q)}_{\alpha\beta}\frac{\partial T}{\partial x_\beta}\right)+T\frac{\partial}{\partial x_\alpha}\left(T^{-1}\sum_{k=1}^{n}\mu^{(k)}J^{(k)}_{(diff)\alpha}\right)+\sigma^{(h)}, \tag{118}$$

where

$$\sigma^{(h)}=\tau^{(vi)}_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt}+\sum_{h=1}^{r}A^{(h)}J^{(h)}+\sum_{k=1}^{n-1}\mathbf{J}^{(k)}_{(diff)}\cdot\mathbf{X}^{(k)}+B^{(1)}_\alpha\frac{dM^{(1)}_\alpha}{dt}+B^{(ir)}_\alpha\frac{dM_\alpha}{dt}$$

$$=\sigma^{(R)}+\sigma^{(chem)}+\sigma^{(diff)}+\sigma^{(M)}, \tag{119}$$

with

$$\sigma^{(R)}=\tau^{(vi)}_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt}=L^{(vi,vi)}_{\alpha\beta\gamma\zeta}\frac{d\varepsilon_{\alpha\beta}}{dt}\frac{d\varepsilon_{\gamma\zeta}}{dt}, \tag{120}$$

$$\sigma^{(chem)}=\sum_{h=1}^{r}A^{(h)}J^{(h)},\quad \sigma^{(diff)}=\sum_{k=1}^{n-1}\mathbf{J}^{(k)}_{(diff)}\cdot\mathbf{X}^{(k)}, \tag{121}$$

$$\sigma^{(M)}=B^{(ir)}_\alpha\frac{dM_\alpha}{dt}+B^{(1)}_\alpha\frac{dM^{(1)}_\alpha}{dt}$$

$$=\left(L^{(0,0)}_{(M)\alpha\beta}\frac{dM_\beta}{dt}+L^{(0,1)}_{(M)\alpha\beta}B^{(1)}_\beta\right)\frac{dM_\alpha}{dt}+B^{(1)}_\alpha\left(L^{(1,0)}_{(M)\alpha\beta}\frac{dM_\beta}{dt}+L^{(1,1)}_{(M)\alpha\beta}B^{(1)}_\beta\right),$$

where we have used (79) and (80).

Hence, using the Onsager-Casimir relation $L^{(0,1)}_{(M)\alpha\beta}=-L^{(1,0)}_{(M)\beta\alpha}$, we have

$$\sigma^{(M)}=L^{(0,0)}_{(M)\alpha\beta}\frac{dM_\alpha}{dt}\frac{dM_\beta}{dt}+L^{(1,1)}_{(M)\alpha\beta}B^{(1)}_\alpha B^{(1)}_\beta, \tag{122}$$

from which, taking into account equations (94) and (95)-(99), we obtain

$$\sigma^{(M)}=L^{(0,0)}_{(M)\alpha\beta}\frac{dM_\alpha}{dt}\frac{dM_\beta}{dt}+L^{(1,1)}_{(M)\alpha\beta}\left[D^{(1)}_{(M)\alpha\zeta}M_\zeta+D^{(2)}_{(M)\alpha\zeta}\frac{dM_\zeta}{dt}+D^{(3)}_{(M)\alpha\zeta}B_\zeta\right.$$

$$\left.+D^{(4)}_{(M)\alpha}(T-T_0)+\sum_{k=1}^{n}D^{(5,k)}_{(M)\beta}\left(c^{(k)}-c^{(k)}_{(0)}\right)\right]\left[D^{(1)}_{(M)\beta\eta}M_\eta+D^{(2)}_{(M)\beta\eta}\frac{dM_\eta}{dt}\right.$$

$$+D^{(3)}_{(M)\beta\eta}B_{\eta}+D^{(4)}_{(M)\beta}(T-T_0)+\sum_{j=1}^{n}D^{(5,j)}_{(M)\beta}\left(c^{(j)}-c^{(j)}_{(0)}\right)\Bigg],\quad \text{or} \tag{123}$$

$$\sigma^{(M)}=L^{(0,0)}_{(M)\alpha\beta}\frac{dM_\alpha}{dt}\frac{dM_\beta}{dt}+L^{(1,1)}_{(M)\gamma\lambda}\Bigg[D^{(1)}_{(M)\gamma\alpha}M_\alpha+D^{(2)}_{(M)\gamma\alpha}\frac{dM_\alpha}{dt}+D^{(3)}_{(M)\gamma\alpha}B_\alpha$$

$$+D^{(4)}_{(M)\gamma}(T-T_0)+\sum_{k=1}^{n}D^{(5,k)}_{(M)\gamma}\left(c^{(k)}-c^{(k)}_{(0)}\right)\Bigg]\Bigg[D^{(1)}_{(M)\lambda\beta}M_\beta+D^{(2)}_{(M)\lambda\beta}\frac{dM_\beta}{dt}$$

$$+D^{(3)}_{(M)\lambda\beta}B_\beta+D^{(4)}_{(M)\lambda}(T-T_0)+\sum_{j=1}^{n}D^{(5,j)}_{(M)\lambda}\left(c^{(j)}-c^{(j)}_{(0)}\right)\Bigg]. \tag{124}$$

Thus, using the Onsager-Casimir relation $L^{(1,1)}_{(M)\alpha\beta}=L^{(1,1)}_{(M)\beta\alpha}$, we obtain

$$\sigma^{(M)}=A^{(1)}_{\alpha\beta}\frac{dM_\alpha}{dt}\frac{dM_\beta}{dt}+A^{(2)}_{\alpha\beta}M_\alpha M_\beta+A^{(3)}_{\alpha\beta}M_\alpha\frac{dM_\beta}{dt}+A^{(4)}_{\alpha\beta}\ M_\alpha B_\beta$$

$$+A^{(5)}_{\alpha\beta}\frac{dM_\alpha}{dt}B_\beta+A^{(6)}_{\alpha\beta}B_\alpha B_\beta+A^{(7)}_{\alpha}(T-T_0)M_\alpha+A^{(8)}_{\alpha}(T-T_0)\frac{dM_\alpha}{dt}$$

$$+A^{(9)}_{\alpha}(T-T_0)B_\alpha+A^{(10)}(T-T_0)^2+\sum_{k=1}^{n}D^{(5,k)}_{(M)\lambda}\left(c^{(k)}-c^{(k)}_{0}\right)A^{(11)}_{\lambda\alpha}M_\alpha$$

$$+\left[A^{(12)}_{\lambda\alpha}\frac{dM_\alpha}{dt}+A^{(13)}_{\lambda\alpha}B_\alpha+A^{(14)}_{\lambda}(T-T_0)\right]\sum_{k=1}^{n}D^{(5,k)}_{(M)\lambda}\left(c^{(k)}-c^{(k)}_{(0)}\right)$$

$$+\sum_{j,k=1}^{n}L^{(1,1)}_{(M)\alpha\beta}D^{(5,j)}_{(M)\alpha}D^{(5,k)}_{(M)\beta}\left(c^{(j)}-c^{(j)}_{(0)}\right)\left(c^{(k)}-c^{(k)}_{(0)}\right), \tag{125}$$

where

$$A^{(1)}_{\alpha\beta}=L^{(0,0)}_{(M)\alpha\beta}+L^{(1,1)}_{(M)\gamma\lambda}D^{(2)}_{(M)\gamma\alpha}D^{(2)}_{(M)\lambda\beta},\quad A^{(2)}_{\alpha\beta}=L^{(1,1)}_{(M)\gamma\lambda}D^{(1)}_{(M)\gamma\alpha}D^{(1)}_{(M)\lambda\beta},$$

$$A^{(3)}_{\alpha\beta}=2L^{(1,1)}_{(M)\gamma\lambda}D^{(1)}_{(M)\gamma\alpha}D^{(2)}_{(M)\lambda\beta},\quad A^{(4)}_{\alpha\beta}=2L^{(1,1)}_{(M)\gamma\lambda}D^{(1)}_{(M)\gamma\alpha}D^{(3)}_{(M)\lambda\beta},$$

$$A^{(5)}_{\alpha\beta}=2L^{(1,1)}_{(M)\gamma\lambda}D^{(2)}_{(M)\gamma\alpha}D^{(3)}_{(M)\lambda\beta},\quad A^{(6)}_{\alpha\beta}=L^{(1,1)}_{(M)\gamma\lambda}D^{(3)}_{(M)\gamma\alpha}D^{(3)}_{(M)\lambda\beta},$$

$$A^{(7)}_{\alpha}=2L^{(1,1)}_{(M)\gamma\lambda}D^{(1)}_{(M)\gamma\alpha}D^{(4)}_{(M)\lambda},\quad A^{(8)}_{\alpha}=2L^{(1,1)}_{(M)\gamma\lambda}D^{(2)}_{(M)\gamma\alpha}D^{(4)}_{(M)\lambda},$$

$$A^{(9)}_{\gamma}=2L^{(1,1)}_{(M)\gamma\lambda}\ D^{(3)}_{(M)\gamma\alpha}D^{(4)}_{(M)\lambda},\quad A^{(10)}_{\gamma}=L^{(1,1)}_{(M)\alpha\beta}D^{(4)}_{(M)\alpha}D^{(4)}_{(M)\beta},$$

$$A^{(11)}_{\gamma}=2L^{(1,1)}_{\gamma\lambda}D^{(5,k)}_{(M)\gamma}D^{(1)}_{(M)\lambda\alpha},\quad A^{(12)}_{\lambda\alpha}=2L^{(1,1)}_{(M)\gamma\lambda}D^{(2)}_{(M)\gamma\alpha},$$

$$A_{\lambda}^{(13)} = 2L_{(M)\gamma\lambda}^{(1,1)} D_{(M)\gamma\alpha}^{(3)}, \quad A_{\lambda}^{(14)} = 2L_{(M)\gamma\lambda}^{(1,1)} D_{(M)\gamma}^{(4)}. \tag{126}$$

The physical quantity $\sigma^{(h)}$ is called heat dissipation function. It is due to the chemical reactions, the magnetic relaxation, the electric conduction, the viscous, magnetic, temperature fields and the diffusion and the concentrations of the n fluid components. $\sigma^{(R)}$ is a generalized Rayleigh dissipation function.

10 Heat dissipation function in the Snoek case, where $\mathbf{m}^{(0)}$ is reversible

In this Section we work out the heat conduction equation and the heat dissipation function in the Snoek case.

In this special case $\mathbf{B} = \mathbf{B}^{(eq)}$, $\mathbf{B}^{(ir)}$ vanishes (see $(23)_2$) and from (79), (80) and (29) one gets

$$L_{(M)\alpha\beta}^{(0,0)} = 0 \quad \text{and} \quad L_{(M)\alpha\beta}^{(0,1)} = -L_{(M)\beta\alpha}^{(1,0)} = 0. \tag{127}$$

From equations (84) and (87) in this case we have

$$c_{\alpha\gamma}^{(1)} = a_{(M)\alpha\gamma}^{(0,0)}, \quad h_{\gamma\zeta} = L_{(M)\gamma\eta}^{(1,1)} a_{(M)\eta\zeta}^{(1,1)}, \quad \left(c_{\zeta\beta}^{(1)}\right)^{-1} = \left(a_{(M)\zeta\beta}^{(0,0)}\right)^{-1}. \tag{128}$$

Taking into account the expression (94) for $\mathbf{B}^{(1)}$ in the Snoek case, from (95)-(99) and $(128)_1$, we have the following form for the coefficients $D_{(M)\alpha\zeta}^{(1)}$, $D_{(M)\alpha\zeta}^{(2)}$, $D_{(M)\alpha\zeta}^{(3)}$, $D_{(M)\alpha}^{(4)}$ and $D_{(M)\alpha}^{(5,k)}$

$$D_{(M)\alpha\zeta}^{(1)} = a_{(M)\alpha\zeta}^{(0,0)} - a_{(M)\alpha\zeta}^{(1,1)}, \quad D_{(M)\alpha\zeta}^{(2)} = 0, \quad D_{(M)\alpha\zeta}^{(3)} = \left(a_{(M)\beta\zeta}^{(0,0)}\right)^{-1} a_{(M)\alpha\beta}^{(1,1)},$$

$$D_{(M)\alpha}^{(4)} = -\left(a_{(M)\beta\gamma}^{(0,0)}\right)^{-1} a_{(M)\alpha\beta}^{(1,1)} a_{(M)\gamma}^{(0)} + a_{(M)\alpha}^{(1)},$$

$$D_{(M)\alpha}^{(5,k)} = -\left(a_{(M)\beta\gamma}^{(0,0)}\right)^{-1} a_{(M)\alpha\beta}^{(1,1)} b_{(M)\gamma}^{(0,k)} + b_{(M)\alpha}^{(1,k)}. \tag{129}$$

In the Snoek case the heat conduction equation (118) takes the form

$$\varrho c(\varepsilon)\frac{dT}{dt} = T\Bigg[a_{(M)\alpha}^{(0)}\frac{dM_\alpha}{dt} - a_{(M)\alpha}^{(1)} L_{(M)\alpha\beta}^{(1,1)} B_{\beta}^{(1)}$$

$$+a_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt} + \sum_{k=1}^{n} b^{(k)}\varrho\left(c^{(k)} - c_{(0)}^{(k)}\right)\Bigg]$$

$$+\frac{\partial}{\partial x_\alpha}\left(T^{-1}L^{(q,q)}_{\alpha\beta}\frac{\partial T}{\partial x_\beta}\right)+T\frac{\partial}{\partial x_\alpha}\left(T^{-1}\sum_{k=1}^{n}\mu^{(k)}J^{(k)}_{(diff)\alpha}\right)+\sigma^{(h)}, \tag{130}$$

where we have used equation (80) and in $\sigma^{(h)}$ given by (119), $\sigma^{(R)}$, $\sigma^{(chem)}$, $\sigma^{(diff)}$ keep the same form and $\sigma^{(M)}$, from (125), (127) and (129), becomes

$$\begin{aligned}\sigma^{(M)}_S &= A^{(2)}_{\alpha\beta}M_\alpha M_\beta+A^{(4)}_{\alpha\beta}M_\alpha B_\beta+A^{(6)}_{\alpha\beta}B_\alpha B_\beta+A^{(7)}_{\alpha}(T-T_0)M_\alpha\\ &+A^{(9)}_{\alpha}(T-T_0)B_\alpha+A^{(10)}(T-T_0)^2+\sum_{k=1}^{n}D^{(5,k)}_{(M)\lambda}\left(c^{(k)}-c^{(k)}_0\right)A^{(11)}_{\lambda\alpha}M_\alpha\\ &+\left[A^{(13)}_{\lambda\alpha}B_\alpha+A^{(14)}_{\lambda}(T-T_0)\right]\sum_{k=1}^{n}D^{(5,k)}_{(M)\lambda}\left(c^{(k)}-c^{(k)}_{(0)}\right)\\ &+\sum_{j,k=1}^{n}L^{(1,1)}_{(M)\alpha\beta}D^{(5,j)}_{(M)\alpha}D^{(5,k)}_{(M)\beta}\left(c^{(j)}-c^{(j)}_{(0)}\right)\left(c^{(k)}-c^{(k)}_{(0)}\right),\end{aligned} \tag{131}$$

where

$$\begin{gathered}A^{(1)}_{\alpha\beta}=A^{(3)}_{\alpha\beta}=A^{(5)}_{\alpha\beta}=A^{(12)}_{\alpha\lambda}=0,\ A^{(8)}_{\alpha}=0,\\ A^{(2)}_{\alpha\beta}=L^{(1,1)}_{(M)\gamma\lambda}D^{(1)}_{(M)\gamma\alpha}D^{(1)}_{(M)\lambda\beta},\quad A^{(4)}_{\alpha\beta}=2L^{(1,1)}_{(M)\gamma\lambda}D^{(1)}_{(M)\gamma\alpha}D^{(3)}_{(M)\lambda\beta},\\ A^{(6)}_{\alpha\beta}=L^{(1,1)}_{(M)\gamma\lambda}D^{(3)}_{(M)\gamma\alpha}D^{(3)}_{(M)\lambda\beta},\quad A^{(7)}_{\alpha}=2L^{(1,1)}_{(M)\gamma\lambda}D^{(1)}_{(M)\gamma\alpha}D^{(4)}_{(M)\lambda},\\ A^{(9)}_{\alpha}=2L^{(1,1)}_{(M)\gamma\lambda}D^{(3)}_{(M)\gamma\alpha}D^{(4)}_{(M)\lambda},\quad A^{(10)}=L^{(1,1)}_{(M)\alpha\beta}D^{(4)}_{(M)\alpha}D^{(4)}_{(M)\beta},\\ A^{(11)}_{\alpha}=2L^{(1,1)}_{\gamma\lambda}D^{(5,k)}_{(M)\gamma}D^{(1)}_{(M)\lambda\alpha},\ A^{(13)}_{\alpha\lambda}=2L^{(1,1)}_{(M)\gamma\lambda}D^{(3)}_{(M)\gamma\alpha},\ A^{(14)}_{\alpha}=2L^{(1,1)}_{(M)\gamma\alpha}D^{(4)}_{(M)\gamma}.\end{gathered} \tag{132}$$

For the magnetic relaxation equation in this Snoek case see [7].

11 Heat dissipation function in De Groot - Mazur case, where $\mathbf{m}^{(1)}$ is null

In this Section we work out the heat conduction equation and the heat dissipation function in De Groot - Mazur case.

In this special case

$$L^{(1,1)}_{(M)\alpha\beta}=0,\quad\text{and}\quad L^{(1,0)}_{(M)\alpha\beta}=-L^{(0,1)}_{(M)\alpha\beta}=0, \tag{133}$$

equations (79) and (80) become

$$B_{\alpha}^{(ir)} = L_{(M)\alpha\beta}^{(0,0)} \frac{dM_{\beta}}{dt} \quad \text{and} \quad \frac{dM_{\alpha}^{(1)}}{dt} = 0. \tag{134}$$

It is seen that $\mathbf{M}^{(1)}$ is constant and it can be supposed that $\mathbf{M}^{(1)} = \mathbf{0}$ (i.e. there is no internal variable). From equations (84) and (87) in this case we have $c_{\alpha\gamma}^{(1)} = a_{(M)\alpha\gamma}^{(0,0)}, \quad h_{\gamma\zeta} = 0$.

Taking into account the expression (94) for $\mathbf{B}^{(1)}$ in the de Groot-Mazur case, from (95)-(99) and $(128)_1$, we have the following form for the coefficients $D_{(M)\alpha\zeta}^{(1)}$, $D_{(M)\alpha\zeta}^{(2)}$, $D_{(M)\alpha\zeta}^{(3)}$, $D_{(M)\alpha}^{(4)}$, $D_{(M)\alpha}^{(5,k)}$:

$$D_{(M)\alpha\zeta}^{(1)} = a_{(M)\alpha\zeta}^{(0,0)} - a_{(M)\alpha\zeta}^{(1,1)}, \quad D_{(M)\alpha\zeta}^{(2)} = -\left(a_{(M)\beta\gamma}^{(0,0)}\right)^{-1} a_{(M)\alpha\beta}^{(1,1)} L_{(M)\gamma\zeta}^{(0,0)},$$

$$D_{(M)\alpha\zeta}^{(3)} = \left(a_{(M)\beta\zeta}^{(0,0)}\right)^{-1} a_{(M)\alpha\beta}^{(1,1)},$$

$$D_{(M)\alpha}^{(4)} = -\left(a_{(M)\beta\gamma}^{(0,0)}\right)^{-1} a_{(M)\alpha\beta}^{(1,1)} a_{(M)\gamma}^{(0)} + a_{(M)\alpha}^{(1)},$$

$$D_{(M)\alpha}^{(5,k)} = -\left(a_{(m)\beta\gamma}^{(0,0)}\right)^{-1} a_{(M)\alpha\beta}^{(1,1)} b_{(M)\gamma}^{(0,k)} + b_{(M)\alpha}^{(1,k)}.$$

In the de Groot-Mazur case the heat conduction equation (118) takes the form

$$\varrho c(\varepsilon) \frac{dT}{dt} = T\left[a_{(M)\alpha}^{(0)} \frac{dM_{\alpha}}{dt} + \sum_{k=1}^{n} b^{(k)} \varrho \left(c^{(k)} - c_{(0)}^{(k)}\right) + a_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt}\right]$$

$$+\frac{\partial}{\partial x_{\alpha}}\left(T^{-1} L_{\alpha\beta}^{(q,q)} \frac{\partial T}{\partial x_{\beta}}\right) + T \frac{\partial}{\partial x_{\alpha}}\left(T^{-1} \sum_{k=1}^{n} \mu^{(k)} J_{(diff)\alpha}^{(k)}\right) + \sigma^{(h)}, \tag{135}$$

where we have used equation (80) and in $\sigma^{(h)}$ given by (119), $\sigma^{(R)}$, $\sigma^{(chem)}$, $\sigma^{(diff)}$ keep the same form and $\sigma^{(M)}$ becomes

$$\sigma_{dG}^{(M)} = L_{(M)\alpha\beta}^{(0,0)} \frac{dM_{\alpha}}{dt} \frac{dM_{\beta}}{dt}. \tag{136}$$

In this case from (126) we have

$$A_{\alpha\beta}^{(1)} = L_{(M)\alpha\beta}^{(0,0)}, \quad A_{\alpha\beta}^{(2)} = A_{\alpha\beta}^{(3)} = A_{\alpha\beta}^{(4)} = A_{\alpha\beta}^{(7)} = 0, \; A_{\alpha}^{(5)} = A_{\alpha}^{(6)} = 0,$$

$$A_{\alpha}^{(8)} = A_{\alpha}^{(9)} = 0, \; A_{\alpha\beta}^{(10)} = A_{\alpha\lambda}^{(11)} = A_{\alpha\lambda}^{(12)} = 0, \; A_{\alpha}^{(13)} = 0, \; A^{(14)} = 0.$$

12 Conclusions

In this paper anisotropic magnetizable reacting fluid mixtures, where different types of irreversible microscopic phenomena give rise to magnetic relaxation, are taken into consideration, using a model devoped in [7]. Also a model for magnetizable media with relaxation was given by Maugin, that described the internal mechanism in these media using particular internal variables. In this paper the total specific magnetization is the sum of two irreversible parts and, linearizing the theory, the heat conduction equation and the heat dissipation function for these magnetizable anisotropic media are derived. The Snoek special case, where changes in $\mathbf{m}^{(0)}$ become reversible, and the De Groot-Mazur special case, where there is no internal variable $\mathbf{m}^{(1)}$, are studied. The obtained results can be applied in several physical situations, in nuclear magnetic resonance in medicine and biology and other different fields of applied sciences, where complex media are used.

In [19], [20] and [41] the heat conduction equation and the heat dissipation function were derived for anisotropic and isotropic polarizable media with dielectric relaxation and for anisotropic mechanical media with relaxation, respectively, within the same thermodynamic framework and using the same standard procedures of irreversible thermodynamics with internal variables.

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