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ON $\mathcal{I}$-DEFERRED STATISTICAL CONVERGENCE OF ORDER α FOR COMPLEX UNCERTAIN SEQUENCES*

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Abstract

In this paper, we introduce the concepts of $\mathcal{I}$-deferred statistical convergence almost surely of order α, $\mathcal{I}$-deferred statistical convergence in measure of order α, $\mathcal{I}$-deferred statistical convergence in mean of order α, $\mathcal{I}$-deferred statistical convergence in distribution of order α, $\mathcal{I}$-deferred statistical convergence in uniformly almost surely of order α and some relationships among them are discussed.

Keywords: uncertainty theory, complex uncertain variable, deferred statistical convergence, $\mathcal{I}$-convergence.

MSC: 60B10, 40A35, 40G15.

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1 Introduction

Uncertainty theory is unpreventable to quantify the future when no data is available, to evaluate the future when an emergency like war, flood, earthquake arises, or the past when counting precise observations or performing measures is nearly impossible. The uncertainty theory and uncertain

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measure was defined and developed by Liu [11] to describe the subjective uncertain phenomena. Liu established the notion of uncertain variables as a function from measurable space to real numbers. Many researchers have also done a lot of theoretical work based on complex uncertain variables, such as Tripathy and Nath [16], Das et al. [4], Roy et al. [13], Chen et al. [2], Kişi [7], and many others. Nowadays, the research on uncertainty theory became quite famous, viz., uncertain risk analysis, uncertain reliability analysis, uncertain logic, and many more.

On the other hand, in 1951, the concept of statistical convergence was introduced independently by Fast [5] and Steinhaus [15]. Later on, it has been studied from different aspects by Fridy [6] and many others. Statistical convergence has many generalizations with the aim of providing deeper insights into the summability theory.

In 2001, the idea of $\mathcal{I}$-convergence was developed by Kostyrko et al. [9] mainly as an extension of statistical convergence and usual convergence. They also showed that many other known notions of convergence were a particular type of $\mathcal{I}$-convergence by considering particular ideals. Consequently, this direction gradually got more attention from the researchers and became one of the most active areas of research. Several investigations and extensions of $\mathcal{I}$-convergence can be found in the works of Mohiuddine and Hazarika [12], and many others.

In 1932, Agnew [1] introduced deferred Cesàro mean as a generalization of Cesàro mean of a real (or complex) valued sequence $x = (x_k)$ by

$$(D_{p,q}x)_n = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k, \ n \in \mathbb{N},$$

where $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ are the sequences of non-negative integers satisfying $p(n) < q(n)$ and $q(n) \to \infty$ as $n \to \infty$. In 2016, Küçükaslan and Yilmaztürk [10] utilized this and introduced deferred statistical convergence as follows:

A sequence $x = (x_k)$ is said to be deferred statistically convergent to $l \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} | \{k : p(n) < k \leq q(n), | x_k - l |\geq \varepsilon\} |= 0,$$

holds and it is denoted by $\lim_{n\to\infty} x_n = l(DS[p,q])$. It is obvious that if $q(n) = n$ and $p(n) = 0$, then the above definition reduces to the definition of statistical convergence and if $q(n) = k_n, p(n) = k_{n-1}$ (for any lacunary sequence of non-negative integers with $k_n - k_{n-1} \to \infty$ as $n \to \infty$) then the above definition turns to the definition of lacunary statistical convergence. Further if $q(n) = \lambda_n$ (where λ_n is a strictly increasing sequence of natural numbers such that $\lambda_n \to \infty$ as $n \to \infty$) and $p(n) = 0$, then the above definition reduces to $\lambda-$statistical convergence of sequences. For more details on deferred statistical convergence and its subsequent developments, [8] can be addressed where many more references can be found.

In 2010, Çolak [3] extended the notion of statistical convergence to statistical convergence of order α $(0 < \alpha \leq 1)$ using $\alpha-$density. Generalizing a few of the above-mentioned convergence concepts in 2019, Şengül et al. [14] introduced the notion of $\mathcal{I}$-deferred statistical convergence of order α, which is the main motivation for us to investigate the notion in complex uncertain space settings.

2 Definitions and preliminaries

Definition 1. *[11] Let $\mathfrak{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:*
Axiom 1 (Normality): $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2 (Duality): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathfrak{L}$;
Axiom 3 (Subadditivity): For every countable sequence of $\{\Lambda_j\} \in \mathfrak{L}$, we have

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathfrak{L}, \mathcal{M})$ is called an uncertainty space and each element Λ in $\mathfrak{L}$ is called an event. In order to obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu [11] as:

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}.$$

Definition 2. *[11] An uncertain variable is a function ζ from an uncertainty space $(\Gamma, \mathfrak{L}, \mathcal{M})$ to the set of real numbers such that $\{\zeta \in \mathfrak{B}\} = \{\gamma \in \Gamma : \zeta(\gamma) \in \mathfrak{B}\}$ is an event for any Borel set $\mathfrak{B}$ of real numbers.*
An uncertainty distribution Φ of an uncertain variable ζ is defined by

$$\Phi(x) = \mathcal{M}\{\zeta \leq x\}, \quad \forall x \in \mathbb{R}.$$

Definition 3. *[11] Let ζ be an uncertain variable. The expected value of ζ is defined by*

$$E[\zeta] = \int_{0}^{+\infty} \mathcal{M}\{\zeta \geq y\}dy - \int_{-\infty}^{0} \mathcal{M}\{\zeta \leq y\}dy,$$

provided that at least one of the above two integrals is finite.

Definition 4. *[11] An uncertain sequence (ζ_n) is said to be convergent almost surely (a.s.) to ζ if for every $\varepsilon > 0$ there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{n\to\infty} \mid \zeta_n(\gamma) - \zeta(\gamma) \mid = 0,$$

for every $\gamma \in \Lambda$.

Definition 5. *[11] An uncertain sequence (ζ_n) is said to be convergent in measure to ζ if*

$$\lim_{n\to\infty} \mathcal{M}(\mid \zeta_n - \zeta \mid \geq \varepsilon) = 0,$$

for every $\varepsilon > 0$.

Definition 6. *[11] An uncertain sequence (ζ_n) is said to be convergent in mean to ζ if*

$$\lim_{n\to\infty} E[\mid \zeta_n - \zeta \mid] = 0.$$

Definition 7. *[11] Let $\Phi, \Phi_1, \Phi_2, ...$ be the uncertainty distributions of uncertain variables $\zeta, \zeta_1, \zeta_2, ...$ respectively. Then, the sequence (ζ_n) is said to be convergent in distribution to ζ if*

$$\lim_{n\to\infty} \mid \Phi_n(x) - \Phi(x) \mid = 0,$$

for all x at which $\Phi(x)$ is continuous.

Definition 8. *[9] A non-void class $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if $\mathcal{I}$ is additive (i.e., $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$) and hereditary (i.e., $A \in \mathcal{I}$ and $B \subseteq A \implies B \in \mathcal{I}$).*
An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq 2^{\mathbb{N}}$. A non-trivial ideal $\mathcal{I}$ is said to be admissible if $\mathcal{I}$ contains every finite subset of $\mathbb{N}$.

Example 1. *(i) The set $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$ is an admissible ideal in $\mathbb{N}$. Here $\mathbb{N}$ denotes the set of all natural numbers.*
(ii) The set $\mathcal{I}_d$ of all subsets of natural numbers having natural density 0 is an admissible ideal in $\mathbb{N}$.

Definition 9. *[9] A sequence $x = (x_n)$ is said to be $\mathcal{I}$ convergent if there exists $L \in \mathbb{R}$ such that for all $\varepsilon > 0$, the set $\{n \in \mathbb{N} :\mid x_n - L \mid\geq \varepsilon\} \in \mathcal{I}$. The usual convergence of sequences is a special case of $\mathcal{I}$-convergence ($\mathcal{I}$=$\mathcal{I}_f$-the ideal of all finite subsets of $\mathbb{N}$). The statistical convergence of sequences is also a special case of $\mathcal{I}$-convergence. In this case, $\mathcal{I}$=$\mathcal{I}_d = \{A \subseteq \mathbb{N} : \lim\limits_{n\to\infty} \frac{|A\cap\{1,2,\ldots,n\}|}{n} = 0\}$, where $\mid B \mid$ being the cardinality of the set B.*

3 Main results

Definition 10. *Let α be any real number such that $0 < \alpha \leq 1$. A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-deferred statistically convergent almost surely of order α to ζ if for every $\varepsilon, \kappa > 0$ there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\Big\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^\alpha} \mid \{k : p(n) < k \leq q(n),$$
$$\qquad\qquad\qquad \| \zeta_k(\gamma) - \zeta(\gamma) \| \geq \varepsilon\} \mid\geq \kappa\Big\} \in \mathcal{I},$$

for every $\gamma \in \Lambda$. In that case, we write $DS^\alpha_{p,q}(\mathcal{I}) - \lim \zeta_n = \zeta$.

Remark 1. *(i) If $q(n) = n, p(n) = 0, \alpha = 1, \mathcal{I} = \mathcal{I}_f$, then the above definition coincides with convergence of complex uncertain sequences.*
(ii) If $q(n) = n, p(n) = 0, \alpha = 1, \mathcal{I} = \mathcal{I}_d$, then the above definition coincides with the statistical convergence of complex uncertain sequences.
(ii) If $q(n) = n, p(n) = 0, \mathcal{I} = \mathcal{I}_d$, then the above definition reduces to the definition of statistical convergence of order α of complex uncertain sequences.
(iii) If $\alpha = 1, \mathcal{I} = \mathcal{I}_d$, then the above definition turns to the definition of deferred statistical convergence of complex uncertain sequences.

Definition 11. *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-deferred statistically convergent in measure of order α to ζ if*

$$\Big\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^\alpha} \mid \{k : p(n) < k \leq q(n),$$
$$\qquad\qquad\qquad \mathcal{M}(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} \mid\geq \kappa\Big\} \in \mathcal{I},$$

for every $\varepsilon, \delta, \kappa > 0$.

Definition 12. *A complex uncertain sequence* (ζ_n) *is said to be* $\mathcal{I}$*-deferred statistically convergent in mean of order* α *to* ζ *if*

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \mid \{k : p(n) < k \leq q(n), E(\| \zeta_k - \zeta \|) \geq \varepsilon\} \mid \geq \kappa\right\} \in \mathcal{I},$$

for every $\varepsilon, \kappa > 0$.

Definition 13. *Let* $\Phi, \Phi_1, \Phi_2, \ldots..$ *be the complex uncertainty distributions of complex uncertain variables* $\zeta, \zeta_1, \zeta_2, \ldots.,$ *respectively. Then the sequence* (ζ_n) *is said to be* $\mathcal{I}$*-deferred statistically convergent in distribution of order* α *to* ζ *if for every* $\varepsilon, \kappa > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \mid \{k : p(n) < k \leq q(n), \right.$$
$$\left. \| \Phi_k(x) - \Phi(x) \| \geq \varepsilon\} \mid \geq \kappa\right\} \in \mathcal{I},$$

for all x *at which* $\Phi(x)$ *is continuous.*

Theorem 1. *If a complex uncertain sequence* (ζ_n) *is* $\mathcal{I}$*-deferred statistically convergent in mean of order* α *to* ζ*, then it is* $\mathcal{I}$*-deferred statistically convergent in measure of order* α *to* ζ*.*

Proof. Using Markov inequality we can see that for any given $\varepsilon, \delta, \kappa > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \mid \{k : p(n) < k \leq q(n), \right.$$
$$\left. \mathcal{M}(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} \mid \geq \kappa\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \mid \{k : p(n) < k \leq q(n), \right.$$
$$\left. \left(\frac{E(\| \zeta_k - \zeta \|)}{\varepsilon}\right) \geq \delta\} \mid \geq \kappa\right\} \in \mathcal{I}.$$

Thus (ζ_n) is $\mathcal{I}$-deferred statistically convergent in measure of order α to ζ and the theorem is thus proved. □

Remark 2. *The converse of the above theorem is not true. i.e.,* $\mathcal{I}$*-deferred statistical convergence of order* α *in measure does not imply* $\mathcal{I}$*-deferred statistical convergence of order* α *in mean.*

Example 2. *Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with*

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{1}{(2n+1)}, \ if \ \sup\limits_{\gamma_n \in \Lambda} \frac{1}{(2n+1)} < 0.5 \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{1}{(2n+1)}, \ \ if \ \sup\limits_{\gamma_n \in \Lambda^c} \frac{1}{(2n+1)} < 0.5 \\ 0.5, \ \ otherwise, \end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_n(\gamma) = \begin{cases} \sqrt{n^3} i, \ if \ \gamma = \gamma_n; \\ 0, \ \ \ otherwise; \end{cases}$$

for $n \in \mathbb{N}$ and $\zeta \equiv 0$. Take $p(n) = n$ and $q(n) = 0$ and $\mathcal{I} = \mathcal{I}_d$. For small number $\varepsilon, \delta > 0$ and $n \geq 2$, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n^\alpha} \mid \{k \leq n : \mathcal{M}(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} \mid \\ &= \lim_{n\to\infty} \frac{1}{n^\alpha} \mid \{k \leq n : \mathcal{M}(\gamma : \| \zeta_k(\gamma) - \zeta(\gamma) \| \geq \varepsilon) \geq \delta\} \mid \\ &= \lim_{n\to\infty} \frac{1}{n^\alpha} \mid \{n \in \mathbb{N} : \mathcal{M}\{\gamma_n\} \geq \delta\} \mid = 0, \end{aligned}$$

$$i.e., \left\{ n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^\alpha} \mid \{k : p(n) < k \leq q(n), \right. \\ \left. \mathcal{M}(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} \mid \geq \kappa \right\} \in \mathcal{I}.$$

Thus, the sequence (ζ_n) is $\mathcal{I}$-deferred statistically convergent of order α in measure to ζ. However, for each $n \geq 2$, we have the uncertainty distribution of uncertain variable (ζ_n) is

$$\Phi_n(x) = \begin{cases} 0, \ if \ x < 0, \\ 1 - \frac{1}{2n+1}, \ if \ 0 \leq x < \sqrt{n^3}, \\ 1, \ \ if \ x \geq \sqrt{n^3}. \end{cases}$$

So, for each $n \geq 2$ and $\alpha \in (0, \frac{1}{2}]$, we have

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \mid \{k \leq n : E(\| \zeta_k - \zeta \|) \geq \varepsilon\} \mid \\ = \lim_{n\to\infty} \frac{1}{n^\alpha} \left[\int_0^{\sqrt{n^3}} 1 - \left(1 - \frac{1}{2n+1}\right) dx \right] \neq 0,$$

$$i.e., \left\{ n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \mid \{k : p(n) < k \leq q(n), \right.$$
$$\left. E(\| \zeta_k - \zeta \|) \geq \varepsilon\} \mid \geq \kappa \right\} \notin \mathcal{I}.$$

Therefore, the sequence (ζ_n) is not $\mathcal{I}$-deferred statistically convergent of order α in mean to ζ.

Theorem 2. *Assume complex uncertain sequence (ζ_n) with real part (ξ_n) and imaginary part (η_n) are $\mathcal{I}$-deferred statistically convergent of order α in measure to ξ and η, respectively. Then, the complex uncertain sequence (ζ_n) is $\mathcal{I}$-deferred statistically convergent of order α in measure to $\zeta = \xi + i\eta$.*

Proof. From the definition of $\mathcal{I}$-deferred statistical convergence of order α in measure of uncertain sequence, it follows that for any small $\varepsilon, \delta, \kappa > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \mid \{k : p(n) < k \leq q(n), \right.$$
$$\left. \mathcal{M}(\| \xi_k - \zeta \| \geq \varepsilon) \geq \delta\} \mid \geq \kappa \right\} \in \mathcal{I},$$

$$\text{and } \left\{ n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \mid \{k : p(n) < k \leq q(n), \right.$$
$$\left. \mathcal{M}(\| \eta_k - \zeta \| \geq \varepsilon) \geq \delta\} \mid \geq \kappa \right\} \in \mathcal{I}.$$

We have, $\| \zeta_n - \zeta \| = \sqrt{\mid \xi_n - \xi \mid^2 + \mid \eta_n - \eta \mid^2}$. Also, we have

$$\{\| \zeta_n - \zeta \| \geq \varepsilon\} \subset \left\{ \| \xi_n - \xi \| \geq \frac{\varepsilon}{\sqrt{2}} \right\} \cup \left\{ \| \eta_n - \eta \| \geq \frac{\varepsilon}{\sqrt{2}} \right\}.$$

Using the subadditivity axiom of uncertain measure, we obtain

$$\mathcal{M}\{\| \zeta_n - \zeta \| \geq \varepsilon\} \leq \mathcal{M}\left\{ \| \xi_n - \xi \| \geq \frac{\varepsilon}{\sqrt{2}} \right\} + \mathcal{M}\left\{ \| \eta_n - \eta \| \geq \frac{\varepsilon}{\sqrt{2}} \right\}.$$

Then,

$$\left\{ n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \mid \{k : p(n) < k \leq q(n), \right.$$
$$\left. \mathcal{M}(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} \mid \geq \kappa \right\} \in \mathcal{I}.$$

□

Proposition 1. *Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. Then, (ζ_n) is $\mathcal{I}$-deferred statistically convergent a.s. of order α to ζ if and only if for any $\varepsilon, \delta, \kappa > 0$, we have*

$$\Bigg\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}}\Bigg|\Bigg\{k : p(n) < k \leq q(n), \mathcal{M}\Bigg(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} \| \zeta_k - \zeta \| \geq \varepsilon\Bigg) \geq \delta\Bigg\}\Bigg| \geq \kappa\Bigg\} \in \mathcal{I}.$$

Proof. From the definition of $\mathcal{I}$-deferred statistical convergence a.s. of order α, we have there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that, for every $\varepsilon, \kappa > 0$ we have

$$\Bigg\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \mid \{k : p(n) < k \leq q(n), \| \zeta_k(\gamma) - \zeta(\gamma) \| \geq \varepsilon\} \mid \geq \kappa\Bigg\} \in \mathcal{I}.$$

Then for any $\varepsilon > 0$, there exists k such that $\| \zeta_n - \zeta \| < \varepsilon$, where $n > k$ and for any $\gamma \in \Lambda$, that is equivalent to

$$\Bigg\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}}\Bigg|\Bigg\{k : p(n) < k \leq q(n), \mathcal{M}\Bigg(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} \| \zeta_k - \zeta \| < \varepsilon\Bigg) \geq 1\Bigg\}\Bigg| \geq \kappa\Bigg\} \in \mathcal{I}.$$

It follows from the duality axiom of uncertain measure that

$$\Bigg\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}}\Bigg|\Bigg\{k : p(n) < k \leq q(n), \mathcal{M}\Bigg(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} \| \zeta_k - \zeta \| \geq \varepsilon\Bigg) \geq \delta\Bigg\}\Bigg| \geq \kappa\Bigg\} \in \mathcal{I}.$$

$\square$

Definition 14. *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-deferred statistically convergent uniformly almost surely of order α to ζ if for every $\varepsilon, \kappa, \exists \delta > 0$ and a sequence (X_k) of events such that*

$$\Bigg\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \mid \{k : p(n) < k \leq q(n), \mid \mathcal{M}(X_k) - 0 \mid \geq \varepsilon\} \mid \geq \kappa\Bigg\} \in \mathcal{I},$$

equivalently,

$$\Bigg\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \mid \{k : p(n) < k \leq q(n), \| \zeta_k(\gamma) - \zeta(\gamma) \| \geq \delta\} \mid \geq \kappa\Bigg\} \in \mathcal{I}.$$

Proposition 2. *Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. Then, (ζ_n) is $\mathcal{I}$-deferred statistically convergent uniformly a.s. of order α to ζ if and only if for any $\varepsilon, \delta, \kappa > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}}\left|\left\{k : p(n) < k \leq q(n), \mathcal{M}\left(\bigcup_{n=k}^{\infty} \| \zeta_k - \zeta \| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \kappa\right\} \in \mathcal{I}.$$

Proof. If (ζ_n) is $\mathcal{I}$-deferred statistically convergent uniformly a.s. of order α to ζ, then for any $\delta > 0$ there exists B such that $\mathcal{M}\{B\} < \delta$ and (ζ_n) is $\mathcal{I}$-deferred statistically uniformly convergent to ζ on $\Gamma - B$. Thus, for any $\varepsilon > 0$, there exists $k > 0$ such that $\| \zeta_n - \zeta \| < \varepsilon$, where $n \geq K$ and $\gamma \in \Gamma - B$. That is

$$\bigcup_{n=k}^{\infty} \{\| \zeta_k - \zeta \| \geq \varepsilon\} \subset B.$$

It follows from the subadditivity axiom of uncertain measure that

$$\mathcal{M}\left(\bigcup_{n=k}^{\infty} \| \zeta_k - \zeta \| \geq \varepsilon\right) \leq \mathcal{M}\{B\} < \delta.$$

Then,

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}}\left|\left\{k : p(n) < k \leq q(n), \mathcal{M}\left(\bigcup_{n=k}^{\infty} \| \zeta_k - \zeta \| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \kappa\right\} \in \mathcal{I}.$$

Conversely,

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}}\left|\left\{k : p(n) < k \leq q(n), \mathcal{M}\left(\bigcup_{n=k}^{\infty} \| \zeta_k - \zeta \| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \kappa\right\} \in \mathcal{I}.$$

Then for any $\varepsilon, \delta, \kappa > 0$ and $m \geq 1$, there exists m_k such that

$$\delta\left(\mathcal{M}\left(\bigcup_{n=m_k}^{\infty}\left\{ \| \zeta_n - \zeta \| \geq \frac{1}{m}\right\}\right)\right) < \frac{\delta}{2^m}.$$

Let
$$B=\bigcup_{m=1}^{\infty}\bigcup_{n=m_k}^{\infty}\{\|\zeta_k-\zeta\|\geq\frac{1}{m}\}.$$

Then,
$$\delta(\mathcal{M}\{B\})\leq\sum_{m=1}^{\infty}\delta\Big(\mathcal{M}\Big(\bigcup_{n=m_k}^{\infty}\Big\{\|\zeta_n-\zeta\|\geq\frac{1}{m}\Big\}\Big)\Big)\leq\sum_{m=1}^{\infty}\frac{\delta}{2^m}.$$

Furthermore, we have
$$\mathcal{I}-\sup_{\gamma\in\Gamma-B}\|\zeta_n-\zeta\|<\frac{1}{m}$$
for any $m=1,2,3,....$ and $n>m_k$. The proposition is thus proved. □

Proposition 3. *If the complex uncertain sequence (ζ_n) is $\mathcal{I}$-deferred statistically convergent uniformly a.s. to ζ, then (ζ_n) is $\mathcal{I}$-deferred statistically convergent a.s. to ζ.*

Proof. We know from the above proposition that if (ζ_n) is $\mathcal{I}$-deferred statistically convergent uniformly a.s. to ζ, then
$$\Big\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))^{\alpha}}\Big|\Big\{k:p(n)<k\leq q(n),\ \mathcal{M}\Big(\bigcup_{n=k}^{\infty}\|\zeta_n-\zeta\|\geq\varepsilon\Big)\geq\delta\Big\}\Big|\geq\kappa\Big\}\in\mathcal{I}.$$

Since the inequation
$$\delta\Big(\mathcal{M}\Big(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\Big\{\|\zeta_n-\zeta\|\geq\varepsilon\Big\}\Big)\Big)\leq\delta\Big(\mathcal{M}\Big(\bigcup_{n=k}^{\infty}\Big\{\|\zeta_n-\zeta\|\geq\varepsilon\Big\}\Big)\Big)$$
holds, so letting $n\to\infty$ on both sides, we have
$$\delta\Big(\mathcal{M}\Big(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\Big\{\|\zeta_n-\zeta\|\geq\varepsilon\Big\}\Big)\Big)=0.$$

By Proposition (1), $\{\zeta_n\}$ $\mathcal{I}$-deferred statistically converges a.s. to ζ. □

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References

[1] R.P. Agnew, On deferred Cesàro means, *Ann. Math.* 33 (1932), 413-421.

[2] X. Chen, Y. Ning and X. Wang, Convergence of complex uncertain sequences, *J. Intell. Fuzzy Syst.* 30 (2016), 3357-3366.

[3] R. Çolak, *Statistical convergence of order* α, Modern Methods in Analysis and Its Applications, Anamaya Pub, New Delhi, India, 2010.

[4] B. Das, B.C. Tripathy, P. Debnath and B. Bhattacharya, Almost convergence of complex uncertain double sequences, *Filomat* 35 (2021), 61-78.

[5] H. Fast, Sur la convergence statistique, *Colloq. Math.* 2 (1951), 241-244.

[6] J.A. Fridy, On statistical convergence, *Analysis* 5 (1985), 301-313.

[7] Ö. Kişi, $S_\lambda(\mathcal{I})$-convergence of complex uncertain sequences, *Mat. Stud.* 51 (2019), 183-194.

[8] C. Koşar, M. Küçükaslan and M. Et, On asymptotically deferred statistical equivalence of sequences, *Filomat* 31 (2017), 5139-5150.

[9] P. Kostyrko, T. Šalát and W. Wilczyński, $\mathcal{I}$-convergence, *Real Anal. Exch.* 26 (2000-2001), 669-686.

[10] M. Küçükaslan and M. Yilmaztürk, On deferred statistical convergence of sequences, *Kyungpook Math. J.* 56 (2016), 357-366.

[11] B. Liu, *Uncertainty Theory*, Springer–Verlag, Berlin, 2015.

[12] S. A. Mohiuddine and B. Hazarika, Some classes of ideal convergent sequences and generalized difference matrix operator, *Filomat* 31 (2017), 1827-1834.

[13] S. Roy, B.C. Tripathy and S. Saha, Some results on p-distance and sequence of complex uncertain variables, *Commun. Korean. Math. Soc.* 35 (2020), 907-916.

[14] H. Şengül, M. Et and M. Işık, On $\mathcal{I}$-deferred statistical convergence of order α, *Filomat* 33 (2019), 2833-2840.

[15] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.* 2 (1951), 73-74.

[16] B.C. Tripathy and P.K. Nath, Statistical convergence of complex uncertain sequences, *New Math. Nat. Comput.* 13 (2017), 359-374.